A Solution to the Dividend Tax Rate Problem

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Abstract

Currently there is a 15 % tax rate on dividends received from stocks. In the latest presidential campaign, Senator John Kerry proposed to increase it to 40 % and President George Bush proposed to eliminate it all together. What difference would these tax rates make to an investor in the long run? More specifically, if one were to make a one-time purchase of shares of stock from a company and then, for each quarter for a certain number of years, reinvest the dividends earned from the stock to buy additional shares of stock (dividends are declared on an annual basis but distributed on a quarterly basis), what would be the accumulations at the different tax rates, given that the value of the stock and dividend increase at certain but perhaps different fixed rates per year?

This question is investigated in this paper, a rather complex formula is derived that can be used to answer the question, and then historical data is used to study stocks of certain companies such as Wal-Mart and Proctor and Gamble to determine future returns for interested investors. In addition, through the examination of different aspects of the formula, some rather surprising results are determined.

Introduction

The paper begins with the derivation of a formula that can be used to determine the accumulation in stock value for a one time purchase of stock using a dividend reinvestment plan. The formula, referred to as the Q-DRIP (Quarterly Dividend Reinvestment) formula, may be used to determine accumulations in the value of a stock over any given number of years and at any given but fixed dividend tax rate.

The formula is applied to a $5000 purchase of Coca Cola stock. Using different but conservative rates of increase as input for both the price and the dividend of the Coca Cola stock, accumulations in the value of the stock are determined at 0%, 15% and 40% dividend tax rates over a 35 year period. Perhaps unexpectedly, the zero percent dividend tax rate generates a substantial difference in total return, particularly when compared to the 40% tax rate.

In case an investor is willing to wait to the end of the year to reinvest the dividend, another formula referred to as the A-DRIP (Annual Dividend Reinvestment) formula is derived. This A-DRIP formula reduces to a formula where a calculator can be used if the rates of increase of the

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stock price and dividend are the same. Even if the two rates are approximately the same, this "calculator" formula is useful in approximating results obtained from the A-DRIP formula.

The Q-DRIP and A-DRIP formulas are compared and it is shown, as one would expect, that the Q-DRIP formula generates more of a return. The affect that the dividend tax rate has on the two formulas is also studied and determined.

Fourteen blue chip stocks ranked number 1 for safety by the *Value Line Investment Survey* are chosen and their accumulations in value over a 20 year period are calculated. Historical data for each of the stocks is used to determine input values for the variables in each of the formulas.

Finally different aspects of the formulas which lead to some rather surprising results are investigated.

Q-DRIP Formula

To derive the Q-DRIP (Quarterly Dividend Reinvestment) formula, the formula used to compute accumulations in stock value, consider an arbitrary stock and let:

C = the initial cost per share of stock,
D = the initial declared dividend per share,
X = the proportion of the dividend to be reinvested to purchase shares of stock,
S = the number of shares initially purchased,
r_D = the rate of increase of the dividend per share ($r_D > 0$),
r_S = the rate of increase of the price per share of stock ($r_S > 0$),
S_B = the number of shares owned at the beginning of the n^{th} quarter,
S_E = the number of shares owned at the end of the n^{th} quarter,
S_P = the number of shares purchased at the end of the n^{th} quarter.

Although the dividend policy can be changed by the board at any time, normally the dividend is declared annually and distributed quarterly. Thus, typically the amount of dividend at the end of each quarter will remain constant throughout year and will not change until the end of first quarter of the following year. This assumption is made in this paper. Thus, the amount of dividend (DIV (n)) generated by one share of stock and used by the investor to purchase additional shares of stock at the end of the n^{th} quarter is:

$$\text{DIV (n)} = \frac{D(1+r_D)^{\left[\frac{n-1}{4}\right]}}{4} \bullet X \ , \qquad (1)$$

where [] denotes the greatest integer function.
Also, the price (PRICE (n)) per share of stock over this same time period is:

$$\text{PRICE (n)} = C(1+r_S)^{\frac{n}{4}} \ . \qquad (2)$$

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Thus the quotient,

$$\text{DIV(n)/PRICE(n)} = \frac{\dfrac{D(1+r_D)^{\left[\frac{n-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{n}{4}}} \tag{3}$$

represents the number of shares of stock purchased by the investor from the dividends of a single share of stock at the end of the n^{th} quarter. This continuing process is illustrated in Table 1.

Table 1

Shares Purchased from the Dividends of One Share of Stock

	Quarter			
Year	1	2	3	4
1	$\frac{\frac{D(1+r_D)^{\left[\frac{1-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{1}{4}}}$	$\frac{\frac{D(1+r_D)^{\left[\frac{2-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{2}{4}}}$	$\frac{\frac{D(1+r_D)^{\left[\frac{3-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{3}{4}}}$	$\frac{\frac{D(1+r_D)^{\left[\frac{4-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{4}{4}=1}}$
2	$\frac{\frac{D(1+r_D)^{\left[\frac{5-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{5}{4}}}$	$\frac{\frac{D(1+r_D)^{\left[\frac{6-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{6}{4}}}$	$\frac{\frac{D(1+r_D)^{\left[\frac{7-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{7}{4}}}$	$\frac{\frac{D(1+r_D)^{\left[\frac{8-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{8}{4}=2}}$
3	$\frac{\frac{D(1+r_D)^{\left[\frac{9-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{9}{4}}}$	$\frac{\frac{D(1+r_D)^{\left[\frac{10-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{10}{4}}}$	$\frac{\frac{D(1+r_D)^{\left[\frac{11-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{11}{4}}}$	$\frac{\frac{D(1+r_D)^{\left[\frac{12-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{12}{4}=3}}$
4	DIV(13)/PRICE(13)	DIV(14)/PRICE(14)	DIV(15)/PRICE(15)	DIV(16)/PRICE(16)
5	DIV(17)/PRICE(17)	DIV(18)/PRICE(18)	DIV(19)/PRICE(19)	DIV(20)/PRICE(20)

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Also note that

$$S_E = S_B + S_P$$

$$= S_B + \text{DIV (n)/PRICE (n)} =$$

$$= S_B + S_B \bullet \left(\frac{\frac{D(1+r_D)^{\left[\frac{n-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{n}{4}}} \right)$$

$$= S_B \left(1 + \frac{\frac{D(1+r_D)^{\left[\frac{n-1}{4}\right]} \bullet X}{4}}{C(1+r_S)^{\frac{n}{4}}} \right)$$

$$= S_B \left(1 + \frac{D}{C} \bullet \frac{X}{4} \bullet \frac{(1+r_D)^{\left[\frac{n-1}{4}\right]}}{(1+r_S)^{\frac{n}{4}}} \right) \qquad (4)$$

Because $\left(1 + \frac{D}{C} \bullet \frac{X}{4} \bullet \frac{(1+r_D)^{\left[\frac{n-1}{4}\right]}}{(1+r_S)^{\frac{n}{4}}} \right)$ occurs as a factor in the above expression for each

value of n, then, by induction, at the end of n^{th} quarters,

$$S_E = S\left[\prod_{i=1}^{n} \left(1 + \frac{D}{C} \bullet \frac{X}{4} \bullet \frac{(1+r_D)^{\left[\frac{i-1}{4}\right]}}{(1+r_S)^{\frac{i}{4}}} \right) \right]. \qquad (5)$$

Therefore, at the end of n quarters the investor will have accumulated a value in stock of A dollars where

$$A = (\text{Cost per share at the end of the quarter}) \bullet S_E$$

$$= C(1+r_S)^{\frac{n}{4}} \bullet S\left[\prod_{i=1}^{n} \left(1 + \frac{D}{C} \bullet \frac{X}{4} \bullet \frac{(1+r_D)^{\left[\frac{i-1}{4}\right]}}{(1+r_S)^{\frac{i}{4}}} \right) \right]$$

$$= CS(1+r_S)^{\frac{n}{4}} \left[\prod_{i=1}^{n} \left(1 + \frac{D}{C} \bullet \frac{X}{4} \bullet \frac{(1+r_D)^{\left[\frac{i-1}{4}\right]}}{(1+r_S)^{\frac{i}{4}}} \right) \right] \quad \text{(Q-DRIP formula)} \qquad (6)$$

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Example Using the Q-DRIP Formula

Assume that an employee of Coca Cola is 25 years of age and begins with 100 shares of stock valued at $ 50.00 per share with a declared dividend of $ 1.00 per share. How much will the employee accumulate in value of stock at the end of a 35 year period if the dividend is taxed at (1) 40 % (2) 15 %, and (3) 0 % given that the value of the stock and the dividend increases at the following annual rates?

a) Stock value and dividend both at 7 %
b) Stock value at 8 % and dividend at 10 %
c) Stock value at 10 % and dividend at 12 %

(It is interesting to note that over the past 15 years, the value of Coca Cola stock has increased at a rate of 10.8 % and its dividend at a rate of 12.5 % per year.)

Table 2
Dividends Reinvested Quarterly using the Q-DRIP formula

a. The dividend is taxed at 40 %.

Stock value rate of increase	Dividend rate of increase	Accumulation
7 %	7 %	$ 79,805.6
8 %	10 %	$128,806.0
10 %	12 %	$242,468.0

b. The dividend is taxed at 15 %.

Stock value rate of increase	Dividend rate of increase	Accumulation
7 %	7 %	$ 94,329.8
8 %	10 %	$162,224.0
10 %	12 %	$304,158.0

c. The dividend is taxed at 0 %.

Stock value rate of increase	Dividend rate of increase	Accumulation
7 %	7 %	$104,274.0
8 %	10 %	$186,270.0
10 %	12 %	$ 348,407.0

A-DRIP Formula

The Q-DRIP formula is some what simplified if the investor is willing to wait until the end of the year to reinvest the dividend. This is especially true if we assume $r_D = r_S$. To derive this annual dividend reinvestment (A-DRIP) formula, let the representations of C, D, X, S, r_D and r_S be as before; but let S_B, S_E, and S_P respectively represent the number of shares owned at the

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beginning of the m^{th} year, the number of shares owned at the end of the m^{th} year and the number of shares purchased at the beginning of the m^{th} year.
Then at the end of the m^{th} year,

$$\text{ADIV (m)} = D(1+r_D)^{m-1} \bullet X \qquad (7)$$

represents the amount of dividend generated by one share of stock and used to purchase additional shares,

$$\text{APRICE (m)} = C(1+r_S)^m \qquad (8)$$

represents the price per share of stock, and

$$\text{ADIV (m)/APRICE(m)} = \frac{D(1+r_D)^{m-1} \bullet X}{C(1+r_S)^m} \qquad (9)$$

represents the number of shares purchased from the dividends of a single share of stock.

We have

$$\begin{aligned} S_E &= S_B + S_P \\ &= S_B + S_B \bullet \left(\frac{D(1+r_D)^{m-1} \bullet X}{C(1+r_S)^m} \right) \\ &= S_B \left(1 + \frac{D(1+r_D)^{m-1} \bullet X}{C(1+r_S)^m} \right). \qquad (10) \end{aligned}$$

Since $1+\dfrac{D(1+r_D)^{m-1} \bullet X}{C(1+r_S)^m}$ occurs for each value of *m*, then at the end of *m* years,

$$S_E = S\left[\prod_{i=1}^{m} \left(1 + \frac{DX}{C} \bullet \frac{(1+r_D)^{i-1}}{(1+r_S)^i} \right) \right]. \qquad (11)$$

Moreover, since the cost per share of stock is $C(1+r_S)^m$, then the accumulation A of value in stock at the end of *m* years is,

$$\begin{aligned} A &= C(1+r_S)^m \bullet S\left[\prod_{i=1}^{m} \left(1 + \frac{DX}{C} \bullet \frac{(1+r_D)^{i-1}}{(1+r_S)^i} \right) \right] \\ &= CS(1+r_S)^m \left[\prod_{i=1}^{m} \left(1 + \frac{DX}{C} \bullet \frac{(1+r_D)^{i-1}}{(1+r_S)^i} \right) \right]. \quad \text{(A-DRIP formula)} \qquad (12) \end{aligned}$$

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The Case Where $r_D \approx r_S$

Now let's assume that $r_S = r_D$, then $\frac{1+r_D}{1+r_S} = 1$, and the accumulation A of value in stock at the end of *m* years is,

$$\text{A} = C(1+r_S)^m \bullet \text{S}\left[\prod_{i=1}^{m}\left(1+\frac{DX}{C}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{i-1}\bullet\frac{1}{1+r_S}\right)\right]$$

$$= CS(1+r_S)^m\left(1+\frac{D}{C}\bullet\frac{1}{1+r_S}\bullet X\right)^m. \text{ (A-DRIP formula when } r_S = r_D\text{)} \qquad (13)$$

This formula can be applied using only a calculator and is a valuable tool in estimating A, particularly if $r_D \approx r_S$.

Comparing the Q-DRIP and A-DRIP Formulas

It is intuitively clear that one would accumulate more value in stock applying the Q-DRIP formula rather than the A-DRIP formula and this is indeed the case. To see this, first note that at the end of any year *j*, the number of shares owned using the Q-DRIP formula is equal to

$$S\prod_{i=1}^{4j}\left(1+\frac{D}{C}\bullet\frac{X}{4}\frac{(1+r_D)^{\left\lceil\frac{i-1}{4}\right\rceil}}{(1+r_S)^{\frac{i}{4}}}\right) =$$

$$\text{S}\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^0\bullet\frac{1}{(1+r_S)^{\frac{1}{4}}}\right)\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^0\bullet\frac{1}{(1+r_S)^{\frac{2}{4}}}\right)$$

$$\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^0\bullet\frac{1}{(1+r_S)^{\frac{3}{4}}}\right)\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^0\bullet\frac{1}{(1+r_S)^{\frac{4}{4}}}\right)$$

$$\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^1\bullet\frac{1}{(1+r_S)^{\frac{1}{4}}}\right)\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^1\bullet\frac{1}{(1+r_S)^{\frac{2}{4}}}\right)$$

$$\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^1\bullet\frac{1}{(1+r_S)^{\frac{3}{4}}}\right)\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^1\bullet\frac{1}{(1+r_S)^{\frac{4}{4}}}\right)$$

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$$\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{j-1}\bullet\frac{1}{(1+r_S)^{\frac{1}{4}}}\right)\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{j-1}\bullet\frac{1}{(1+r_S)^{\frac{2}{4}}}\right)$$

$$\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{j-1}\bullet\frac{1}{(1+r_S)^{\frac{3}{4}}}\right)\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{j-1}\bullet\frac{1}{(1+r_S)^{\frac{4}{4}}}\right)=$$

$$S\prod_{k=1}^{j}\prod_{i=1}^{4}\left(1+\frac{D}{C}\bullet\frac{X}{4}\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{(1+r_S)^{\frac{i}{4}}}\right)$$

Since the price per share of stock at the end of each year *j* is the same in both formulas, it is necessary only to show that:

$$\prod_{i=1}^{4}\left(1+\frac{D}{C}\bullet\frac{X}{4}\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{(1+r_S)^{\frac{i}{4}}}\right)>\left(1+\frac{D}{C}\bullet X\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{1+r_S}\right) \qquad (14)$$

for each value of k.

This is easy to see, for since $0<\frac{1}{1+r_S}<1$, then

$$1>\frac{1}{(1+r_S)^{\frac{1}{4}}}>\frac{1}{(1+r_S)^{\frac{2}{4}}}>\frac{1}{(1+r_S)^{\frac{3}{4}}}>\frac{1}{1+r_S}>0;$$

and by applying the Binomial Theorem for n = 4, it follows that

$$\prod_{i=1}^{4}\left(1+\frac{D}{C}\bullet\frac{X}{4}\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{(1+r_S)^{\frac{i}{4}}}\right)$$

$$>\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{1+r_S}\right)^4$$

$$=1+\left(\frac{D}{C}\bullet X\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{1+r_S}\right)+\ldots\ldots+\left(\frac{D}{C}\bullet X\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{1+r_S}\right)^4$$

$$>\left(1+\frac{D}{C}\bullet X\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{1+r_S}\right).$$

Example Using the A-DRIP Formula

Now let’s take our Coca Cola example and apply the A-DRIP formula. The results are in Table 3.

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Table 3
Dividend Reinvested Annually using the A-DRIP formula

a. The dividend is taxed at 40 %.

Stock value rate of increase	Dividend rate of increase	Accumulation
7 %	7 %	$ 78,872.2
8 %	10 %	$126,371.0
10 %	12 %	$237,112.0

b. The dividend is taxed at 15 %.

Stock value rate of increase	Dividend rate of increase	Accumulation
7 %	7 %	$ 92,683.8
8 %	10 %	$157,612.0
10 %	12 %	$294,178.0

c. The dividend is taxed at 0 %.

Stock value rate of increase	Dividend rate of increase	Accumulation
7 %	7 %	$ 102,070.0
8 %	10 %	$ 179,828.0
10 %	12 %	$ 334,596.0

How the Dividend Tax Rate Affects the Formula

Now, what part does X (the proportion of the dividend to be reinvested to purchase shares of stock) play in the formula?

To answer this question, first consider the A-DRIP formula where $r_S = r_D$. From (13),

$$\begin{aligned} \mathrm{A} &= \mathrm{CS}\left(1+r_S\right)^m \left(1+\frac{D}{C}\bullet\frac{1}{1+r_S}\bullet X\right)^m \\ &= \mathrm{C}\left(1+r_S\right)^m \bullet \mathrm{S}\left(1+\frac{D}{C}\bullet\frac{1}{1+r_S}\bullet X\right)^m, \qquad (15) \end{aligned}$$

where $\mathrm{C}\left(1+r_S\right)^m$ represents the price per share of stock at the end of the m^{th} year and $\mathrm{S}\left(1+\frac{D}{C}\bullet\frac{1}{1+r_S}\bullet X\right)^m$ represents the accumulation of stock at the end of the m^{th} year. Notice that the expression $\left(1+\frac{D}{C}\bullet\frac{1}{1+r_S}\bullet X\right)^m$ has the same form as an ordinary compound interest

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formula. Thus, by considering $\frac{D}{C} \bullet \frac{1}{1+r_S}$ to be a fixed annual rate of increase in the accumulation of shares of stock (which is certainly a legitimate assumption), multiplying it by X has the affect of decreasing it by 1 – X, the dividend tax rate.

Next, assume the dividends are being reinvested annually and $r_S \neq r_D$. Then by (12),

$$A = C(1+r_S)^m \bullet S\prod_{i=1}^{m}\left(1+\frac{D}{C}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{i-1}\bullet\frac{1}{1+r_S}\bullet X\right)$$

where

$$S\prod_{i=1}^{m}\left(1+\frac{D}{C}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{i-1}\bullet\frac{1}{1+r_S}\bullet X\right)$$

represents the accumulation of stock after *m* years. Since $r_S \neq r_D$,

$$\left(\frac{1+r_D}{1+r_S}\right)^{i-1}$$

changes each year. However, if we think of

$$\frac{D}{C}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{i-1}\bullet\frac{1}{1+r_S}$$

as a variable annual rate of increase in the accumulation of shares of stock, then multiplying it by X also has the affect of decreasing it by the dividend tax rate, 1-X.

If the dividends are being reinvested quarterly using the Q-DRIP formula, then by (14), for each year k,

$$\prod_{i=1}^{4}\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{(1+r_S)^{\frac{i}{4}}}\right) > \left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\frac{1}{1+r_S}\right)^4.$$

Since

$$\frac{1}{(1+r_S)^{\frac{1}{4}}} > \frac{1}{(1+r_S)^{\frac{2}{4}}} > \frac{1}{(1+r_S)^{\frac{3}{4}}} > \frac{1}{1+r_S},$$

then it is also true that

$$\prod_{i=1}^{4}\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{(1+r_S)^{\frac{i}{4}}}\right) < \left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\frac{1}{(1+r_S)^{\frac{1}{4}}}\right)^4,$$

and therefore,

$$S\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{1+r_S}\right)^4 < S\prod_{i=1}^{4}\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{(1+r_S)^{\frac{i}{4}}}\right) < S\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\frac{1}{(1+r_S)^{\frac{1}{4}}}\right)^4.$$

Thus, for each year *k*,

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$$S\prod_{i=1}^{4}\left(1+\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{(1+r_S)^{\frac{i}{4}}}\right)$$

is fairly tightly bound between two formulas both of which can be considered to be "*variable quarterly rate of increase*" functions for that particular year *k*. In each case, X has the affect of decreasing their rates of increase by the dividend tax rate, 1-X. For the rest of this paper,

$$\frac{D}{C}\bullet\frac{X}{4}\bullet\left(\frac{1+r_D}{1+r_S}\right)^{k-1}\bullet\frac{1}{(1+r_S)^{\frac{i}{4}}} \qquad (16)$$

will be referred to as the variable rate of increase of the Q-DRIP formula.

Accumulations in Values of Stocks

Tables 4(a) and 4(b) include accumulations in value of one time stock investments of $5000 for 14 companies over a 20 year period. Each company is a blue chip company ranked number 1 for safety by *The Value Line Investment Survey*. In Table 4(a), it is assumed that the dividends are reinvested quarterly and (6) (Q-DRIP formula) is used. In Table 4(b), the dividends are reinvested annually and (12) (A-DRIP formula) is used. In both tables, accumulations are computed at dividend tax rates of 40 %, 15 %, and 0 %. For each stock, the values of r_D and r_S are chosen to be what they have averaged over the past 12 years.

Table 4(a)

Accumulation of Value in Stock for a 20 Year Period Using the Q-DRIP Formula
Dividends Reinvested Quarterly

#	Stocks Name	C	D	S	DIVYLD	r_D	r_S	Accumulation X = .6	Accumulation X = .85	Accumulation X =1
1	Abbot Labs	$ 48.24	1.10	103.65	0.023	0.099	0.098	$ 42,066.70	$ 46,868.50	$ 50,005.30
2	Exxon Mobil Corp.	$ 55.68	1.16	89.80	0.021	0.033	0.087	$ 30,889.10	$ 32,912.70	$ 34,189.10
3	Johnson&Johnson	$ 64.47	1.36	74.11	0.020	0.131	0.133	$ 75,694.30	$ 82,940.70	$ 87,613.10
4	Kimberly-Clark	$ 65.00	1.80	76.92	0.028	0.054	0.069	$ 25,104.50	$ 28,193.30	$ 30,223.60
5	McDonald's Corp	$ 30.89	0.55	161.86	0.018	0.122	0.061	$ 23,604.10	$ 27,497.90	$ 30,131.50
6	McGraw-Hill	$ 43.28	0.66	115.23	0.015	0.064	0.13	$ 63,610.90	$ 66,357.80	$ 68,061.80
7	New Plan	$ 26.46	1.72	188.96	0.065	0.021	0.004	$ 13,276.40	$ 19,643.30	$ 24,605.10
8	Proctor & Gamble	$ 55.73	1.12	89.72	0.020	0.105	0.119	$ 57,858.20	$ 62,872.50	$ 66,084.60
9	SLM Corporation	$ 48.45	0.88	103.20	0.018	0.164	0.161	$ 121,327.00	$ 132,038.00	$ 138,907.00
10	Tootsie Roll Ind.	$ 30.84	0.28	162.13	0.009	0.135	0.109	$ 42,015.00	$ 47,485.50	$ 49,031.20
11	Wal-Mart	$ 47.30	0.60	105.71	0.013	0.174	0.113	$ 54,490.40	$ 60,391.90	$ 64,231.00
12	Sara Lee	$ 20.79	0.79	240.50	0.038	0.086	0.048	$ 24,019.50	$ 31,204.80	$ 36,494.80
13	3 M	$ 77.06	1.68	64.88	0.022	0.047	0.098	$ 38,171.20	$ 40,847.90	$ 42,541.90
14	Wilmington Trust	$ 35.05	1.20	142.65	0.034	0.072	0.107	$ 50,963.20	$ 57,457.40	$ 61,739.20

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Table 4(b)
Accumulation of Value in Stock for a 20 Year Period Using the Q-DRIP Formula
Dividends Reinvested Annually

	Annually							Accumulation	Accumulation	Accumulation
	Stocks Name	**C**	**D**	**S**	**DIVYLD**	r_D	r_S	**X = .6**	**X = .85**	**X =1**
1	Abbot Labs	$ 48.24	1.10	103.65	0.023	0.099	0.098	$ 41,640.40	$ 41,166.00	$ 49,101.70
2	Exxon Mobil Corp.	$ 55.68	1.16	89.80	0.021	0.033	0.087	$ 30,729.60	$ 32,664.00	$ 33,879.40
3	Johnson&Johnson	$ 64.47	1.36	74.11	0.020	0.131	0.133	$ 74,866.90	$ 81,621.70	$ 85,948.30
4	Kimberly-Clark	$ 65.00	1.80	76.92	0.028	0.054	0.069	$ 24,896.50	$ 27,840.80	$ 29,763.00
5	McDonald's Corp	$ 30.89	0.55	161.86	0.018	0.122	0.061	$ 23,351.90	$ 27,042.10	$ 29,514.00
6	McGraw-Hill	$ 43.28	0.66	115.23	0.015	0.064	0.13	$ 63,306.30	$ 65,900.80	$ 67,505.40
7	New Plan	$ 26.46	1.72	188.96	0.065	0.021	0.004	$ 13,254.20	$ 19,023.00	$ 23,554.80
8	Proctor & Gamble	$ 55.73	1.12	89.72	0.020	0.105	0.119	$ 57,338.60	$ 62,049.90	$ 65,050.80
9	SLM Corporation	$ 48.45	0.88	103.20	0.018	0.164	0.161	$ 119,890.00	$ 129,778.00	$ 136,077.00
10	Tootsie Roll Ind.	$ 30.84	0.28	162.13	0.009	0.135	0.109	$ 44,780.60	$ 47,127.50	$ 48,591.20
11	Wal-Mart	$ 47.30	0.60	105.71	0.013	0.174	0.113	$ 53,895.80	$ 59,422.10	$ 62,990.60
12	Sara Lee	$ 20.79	0.79	240.50	0.038	0.086	0.048	$ 23,581.60	$ 30,276.40	$ 35,120.00
13	3 M	$ 77.06	1.68	64.88	0.022	0.047	0.098	$ 37,937.00	$ 40,481.80	$ 42,085.40
14	Wilmington Trust	$ 35.05	1.20	142.65	0.034	0.072	0.107	$ 50,336.70	$ 56,412.10	$ 60,384.50

where C = the initial cost per share of stock,
D = the initial declared dividend per share,
S = the number of shares initially purchased,
DIVYLD = dividend yield = D/C
r_D = the rate of increase of the dividend per share ($r_D > 0$),
r_S = the rate of increase of the price per share of stock ($r_S > 0$),
X = the proportion of the dividend to be reinvested to purchase shares of stock,

Concluding Observations

This paper is concluded with three observations. The first is that the greater the value of the "*variable rate of increase*"

$$\frac{D}{C} \bullet \frac{1}{4} \bullet \left(\frac{1+r_D}{1+r_S}\right)^{k-1} \bullet \frac{1}{(1+r_S)^{\frac{i}{4}}}$$

in the accumulation of shares of stock of a company (see (16)), the more affect X , the proportion of the dividend reinvested to purchase additional shares of stock, (for X<1) has in decreasing the percentage of the accumulation of the shares of stock (from the accumulation when X=1) and therefore reducing the percentage in their cumulative total value.

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To verify this, it is sufficient to show that if S > 0, 0 < X <1, and $r_i < r'_i$ for i = 1, 2, 3, ·····, k, then the percent decrease from the accumulation of S using r_i to the accumulation of S using Xr_i is less than the percent decrease of the accumulation of S using r'_i to the accumulation of S using Xr'_i. Mathematically, this means the following must be true:

$$-\frac{S\prod_{i=1}^{k}(1+Xr_i)-S\prod_{i=1}^{k}(1+r_i)}{S\prod_{i=1}^{k}(1+r_i)} < -\frac{S\prod_{i=1}^{k}(1+Xr'_i)-S\prod_{i=1}^{k}(1+r'_i)}{S\prod_{i=1}^{k}(1+r'_i)}.$$

First observe that

$$\begin{aligned}
r'_i > r_i &\Rightarrow r'_i(1 - X) > r_i(1 - X) \\
&\Rightarrow r'_i - Xr'_i > r_i - Xr_i \\
&\Rightarrow r'_i + Xr_i > r_i + Xr'_i \\
&\Rightarrow r'_i + Xr_i + 1 + Xr_i r'_i > r_i + Xr'_i + 1 + Xr_i r'_i \\
&\Rightarrow (1 + r'_i)(1 + Xr_i) > (1 + r_i)(1 + Xr'_i) \\
&\Rightarrow \frac{1+Xr_i}{1+r_i} > \frac{1+Xr'_i}{1+r'_i}.
\end{aligned}$$

Therefore,

$$\frac{\prod_{i=1}^{k}(1+Xr_i)}{\prod_{i=1}^{k}(1+r_i)} > \frac{\prod_{i=1}^{k}(1+Xr'_i)}{\prod_{i=1}^{k}(1+r'_i)},$$

and the rest easily follows:

$$\frac{\prod_{i=1}^{k}(1+Xr_i)}{\prod_{i=1}^{k}(1+r_i)} > \frac{\prod_{i=1}^{k}(1+Xr'_i)}{\prod_{i=1}^{k}(1+r'_i)} \Rightarrow \frac{\prod_{i=1}^{k}(1+Xr_i)}{\prod_{i=1}^{k}(1+r_i)} - 1 > \frac{\prod_{i=1}^{k}(1+Xr'_i)}{\prod_{i=1}^{k}(1+r'_i)} - 1$$

$$\Rightarrow \frac{S\prod_{i=1}^{k}(1+Xr_i)-S\prod_{i=1}^{k}(1+r_i)}{S\prod_{i=1}^{k}(1+r_i)} > \frac{S\prod_{i=1}^{k}(1+Xr'_i)-S\prod_{i=1}^{k}(1+r'_i)}{S\prod_{i=1}^{k}(1+r'_i)}$$

$$\Rightarrow -\frac{S\prod_{i=1}^{k}(1+Xr_i)-S\prod_{i=1}^{k}(1+r_i)}{S\prod_{i=1}^{k}(1+r_i)} < -\frac{S\prod_{i=1}^{k}(1+Xr'_i)-S\prod_{i=1}^{k}(1+r'_i)}{S\prod_{i=1}^{k}(1+r'_i)}.$$

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This affect of X decreasing the percentage in the accumulation of shares of stock and consequently their cumulative total value can be particularly seen in McDonald's Corporation and in Sara Lee Corporation where changing X from 1 to .60 causes a decrease in the total accumulation of almost 22 % in McDonald's and approximately 34 % in Sara Lee. In both cases, r_D, the rate of increase of the dividend, is considerably larger than r_S, the rate of increase of the price of the stock, which means that the variable rate (16),

$$\frac{D}{C} \bullet \frac{1}{4} \bullet \left(\frac{1+r_D}{1+r_S}\right)^{k-1} \bullet \frac{1}{(1+r_S)^{\frac{i}{4}}},$$

is relatively large and becomes even larger as *k* increases in value. With Sara Lee, the affect is even more pronounced because its dividend yield $\frac{D}{C}$ is so high. This percent decrease would even be more dramatic if we were to extend the time period beyond 20 years. For example, over a 35-year period, McDonald's decrease would be more than 50% and Sara Lee's more than 60%.

Next it is observed that when comparing the accumulation in the value of the shares of stock of McDonald's and Sara Lee Corporations, Sara Lee's accumulation is greater than McDonald's even though its values for r_D and r_S are less than those of McDonald's. The reason for this rather surprising result can be found by examining the variable rate formula (16),

$$\frac{D}{C} \bullet \frac{1}{4} \bullet \left(\frac{1+r_D}{1+r_S}\right)^{k-1} \bullet \frac{1}{(1+r_S)^{\frac{i}{4}}}$$

for the accumulation of shares of stock for both of the companies. Notice that the dividend yield $\frac{D}{C}$ for Sara Lee is more than twice that of McDonald's. Moreover, it turns out that for each stock, the value for $\frac{1+r_D}{1+r_S}$ is only slightly greater than one with McDonald's value exceeding Sara Lee's by only about 2%. Thus, until the value of k becomes quite large, the variable rate for Sara Lee exceeds that of McDonald's. This means that initially the Sara Lee investment will generate more shares of stock; and, it in fact, generates enough additional shares to where their accumulation in value exceeds that of McDonald's. McDonald's cumulative total will eventually pass Sara Lee's but this will not occur with the A-DRIP formula (12) (with X = 1) until n = 188 (47 years)!

Finally, it is observed that if r_1 and r_2 denote two rates of increase and $r_1 > r_2$, the difference between the effective rate of increase derived from r_1 by compounding it quarterly and r_1 is greater than the difference between the effective rate obtained by compounding r_2 quarterly and r_2. That is, if $r_1 > r_2$, then

$$\left[\left(1+\frac{r_1}{4}\right)^4 - 1\right] - r_1 > \left[\left(1+\frac{r_2}{4}\right)^4 - 1\right] - r_2 .$$

(This assertion is easily verified by expanding both $\left[\left(1+\frac{r_1}{4}\right)^4\right]$ and $\left[\left(1+\frac{r_2}{4}\right)^4\right]$.)

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Thus, the greater the variable rate of increase in the accumulation of shares of stock in a company, the greater the difference will be between the amount of stock accumulated on a quarterly basis and on an annual basis. This can be seen to a limited degree when comparing the differences in the accumulated values of stock using the A-DRIP and Q-DRIP formulas with Kimberly-Clark and McDonald's corporation where for X = 1, McDonald's investment accumulates approximately $618 extra and Kimberly-Clark about $461 extra. Observe that although Kimberly-Clark has a higher dividend yield, both companies have approximately the same rate of increase in price per share of stock and McDonald's corporation has a much higher rate of increase in its dividend per share of stock. Because of its higher dividend yield, Kimberly-Clark's variable rate of increase in the accumulation of shares of stock is initially higher than that of McDonald's, but after 6 years, McDonald's variable rate exceeds that of Kimberly-Clark and each year thereafter the difference becomes even grater. If instead of a 20-year period, we were to determine the accumulation for a 35-year period, McDonald's investment would accumulate more than $13,000 extra where as Kimberly-Clark only a little more than $3000 extra.

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