

**A Spreadsheet Approach to Teaching Numerical Methods
For Steady and Unsteady Heat Transfer**

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A unified method for teaching steady (using Gauss-Seidel) and unsteady (using Explicit) numerical methods for conduction heat transfer using a spreadsheet (Excel™) is described. This method makes use of the similarity between nodes in numerical heat transfer and cells in a spreadsheet. Extensive use is made of the electrical analogy for heat transfer in the difference equations. Students program in the difference equations and the boundary conditions in a template provided to them. Students prepare their own macros for the iteration schemes. The versatility of the spreadsheet approach allows spatially and temporally dependent boundary conditions which in turn permits almost immediate applications to engineering design including optimization. Steady heat transfer applications have included fins for electronic and engine cooling. Unsteady heat transfer applications have included the design of automobile and aircraft disk brakes. The advantage of this approach is that students prepare their own numerical model, which enhances understanding of the numerical methods and boundary conditions, explore convergence and stability criteria, and move quickly into thermal design studies using the numerical tool that they have developed.

Introduction and Background

It is important that students studying heat transfer are introduced to numerical methods early since the majority of heat transfer analysis and design in industry is done numerically. It is also important that they understand the nature of heat transfer, the range and complexity of boundary conditions and obtain a sense of the physics involved in engineering applications such as fins and disk brakes. One of the shortcomings of using software that solves the conduction equation directly is that often students do not understand what the software is actually doing. Furthermore, it is difficult to introduce applications with spatially and temporally dependent boundary conditions, which often occur in real-world applications.

The conventional approach (described in reference 1) is illustrated in Figure 1 for steady conduction heat transfer and in Figure 2 for unsteady heat transfer.

For steady conduction heat transfer, Laplace's equation is discretized (as shown in Figure 1) and a set of simultaneous equations for the temperatures of the nodes or elements is obtained. These equations can be solved in a variety of ways, the most common being a Matrix inversion or a relaxation method such as the Gauss-Seidel method. The Gauss-Seidel method essentially involves repeatedly solving for the temperature at each element or node in terms of the surrounding temperatures. This iterative process converges on the steady solution when an appropriate convergence criterion is satisfied. These same difference equations can be obtained by an energy balance on each node¹. Using this approach the difference equations can be determined for different types of boundary nodes¹. This approach does not lend itself easily to problems in which it is desirable to vary the grid size, nor to problems in which the surface boundary conditions vary with position and/or time.

For unsteady conduction heat transfer, the Heat Equation is also discretized (as shown in Fig. 2). In the discretization approach shown in Figure 2, the explicit method is used; solving for the temperature at the next time step (one time step ahead of the present time) using only the values of temperature for all nodes at the present time. The equation for the temperature at the next time step contains a numerical Fourier number which is a non-dimensional time step. In the explicit method, the time-marching temperatures become unstable if the time step (Fourier number) choice is too large. The stability criterion for the Fourier number (and corresponding time step) is that it be less than $\frac{1}{2}$ for 1-D solutions and less than $\frac{1}{4}$ for 2-D solutions.

The solution of these difference equations for steady and unsteady heat transfer can be accomplished using a variety of software including spreadsheets. The spreadsheet approach was advocated by Becker² as early as 1986. A method for using spreadsheets to perform a steady analysis of one and two dimensional conductions is described by Eid³. O'Neal⁴ describes a steady analysis of a fin and an implicit and explicit method for one-dimensional conduction using spreadsheets. Thomson⁵ performs a spreadsheet analysis of steady two-dimensional shapes and a one dimensional transient solution. In 1992 the author and one of his students described the use of Spreadsheets for the numerical solution of one and two dimensional unsteady conduction⁶. Since these early uses of spreadsheets, their use for a variety of thermal fluid and heat transfer problems has become wide-spread. McDaniel⁷ has developed a Thermal Fluid Toolbox which allows properties to be obtained in individual cells. This allows applications such as a combined cycle power plant⁸ and many other applications with iterative design solutions⁹. McDaniel recently prepared a spreadsheet for the Computational Aero course at the USAF Academy that solves the one dimensional heat equation using both explicit and implicit methods. There are also spreadsheet programs available for heat transfer analysis such as EZ-Thermal⁹. The textbook by Holman¹⁰ contains an Appendix titled "Use of Microsoft ExcelTM for Solution of Heat-Transfer Problems". The present paper presents a unified approach to teaching numerical methods for steady and unsteady heat transfer emphasizing the use of the electrical analogy.

In the next section it is shown that the equations used in the steady numerical solution (Gauss-Seidel relaxation method) and the equations used for an (explicit) unsteady numerical solution

can be derived and put in the form of the electrical analogy. In the following sections it is shown that this simplifies the numerical solutions and makes it possible to handle a variety of issues including uneven grids, spatially dependent boundary conditions, and time-dependent boundary conditions using a Spreadsheet template.

Numerical Solutions Using the Electrical Analogy

Steady Solutions

In the early Aerospace days, numerical heat transfer was often done using the electrical analogy. The author learned of this approach at North American Aviation in the early 60's. This approach is useful for setting up the difference equations and boundary conditions for use in a Spreadsheet. An electrical analogy for heat transfer in a 1-D wall broken into nodes is shown in Figure 3. In this analogy, boundary nodes and surface nodes do not have a means for storing energy whereas internal nodes have a mass associated with them and therefore a thermal capacitance. The thermal capacitance represents the amount of energy per unit temperature change and is the product of the specific heat and mass of a node. The resistances between nodes can be analogous to convection, conduction and even radiation.

In Figure 4 an energy balance is done on a node for 1-D steady heat transfer using the electrical analogy and L'Hospital's rule assuming no volumetric heat generation. If volumetric heat generation is included the equation becomes:

$$T_i = \frac{\dot{Q}_i + \sum_j \frac{T_j}{R_{ij}}}{\sum_j \frac{1}{R_{ij}}} \quad (1)$$

In this equation T_i is the node temperature being calculated, \dot{Q}_i is the volumetric heat generation, T_j is the temperature of a surrounding node and R_{ij} is the thermal resistance between the "i" and "j" nodes. This equation gives the temperature of a node in terms of the surrounding temperatures and thermal resistances and is essentially the Gauss-Seidel equation for a relaxation solution of the steady temperature distribution (in my classes I compare the traditional Gauss-Seidel equation given in Reference 1 to Equation 1 and show that they are essentially the same equation). Although this equation was derived for 1-D, it is also valid for 2-D and 3-D as we will see for the unsteady case. For 2-D and 3-D there are simply more "j" nodes connected to the "i" node. The equation can be used with a grid size that is position dependent and for surface nodes as well as internal nodes. It can also be used for fins, with variable area and boundary conditions by simply including a convection resistance to the surrounding fluid. The resistances can vary with position or temperature allowing for a distributed boundary condition and/or temperature dependent properties.

For 1-D and 2-D solutions, this equation can be iterated using a spreadsheet. Figure 5 is an example of a template set up for a 1-D solution using equation (1). The important columns are the column for the temperature at the current iteration, the column for the temperature at the next iteration (which contains Equation (1)), and the column for the change in temperature during the last iteration (which becomes the convergence criterion). The macros used for iteration are also shown pasted below the template in Figure 5. These are discussed below.

In teaching this Gauss-Seidel approach for the numerical calculation of steady temperatures, the first step is to have students discover for themselves the iterative process involved in using a spreadsheet. This is done by assigning a problem to solve for the surface temperature for a wall with combined conduction, convection, and radiation (see Reference 1, section 1.3.2). Since the surface temperature appears to the first and fourth power in the energy balance equation, this can only be solved iteratively. It involves guessing a surface temperature, then recalculating the surface temperature from the initial guess. One cell is used for the initial guess and another cell contains the equation for the re-calculation of the temperature using this initial guess. The iteration process can be done in ExcelTM using a macro (or program) written in Microsoft Visual Basic. It is not necessary to be proficient in Visual Basic to prepare this macro. Students can use the “record” feature for writing their iteration macro. To help them understand the iterative process, I give them a template that has a macro called “iterate” already imbedded. This macro simply runs the macro called “copy” N times (where the value for N is located in one cell). Students must create their own macro called “copy”. They turn on the macro “record” feature in ExcelTM, copy the cell containing the equation for the temperature being calculated and its current value, then use “paste special” to copy the newly calculated value into the cell containing their initial or previous guess, then turn off the record feature after saving the macro as “copy”. “Paste Special” is required to avoid copying the formula back in to the cell containing the initial or previous guess. I include a “button” they can use to run the “iterate” macro.

After having done an iterative solution for a single temperature, Students are then assigned a problem for a numerical solution for a 1-D wall. That is done on the template shown in Figure 4. Again the macro called “iterate” is provided in the template, students must now add a macro called “CopyColumn”. This is done as before by simply turning on record, then copying the column containing the temperatures calculated using equation (1), then using “paste special as before, then turning off record.

The first 1-D problem that I assign is a wall with fixed temperatures at both sides. Using an initial guess of the average temperature between the two boundary conditions, the temperature distribution in the nodal network is calculated after a fixed number of iterations and plotted. A typical plot is shown in Figure 6. This allows students to see the convergence process at work and the correct linear temperature distribution appears after a sufficient number of iterations (as determined by the convergence criterion).

This is also a good place in the learning process to introduce different boundary conditions. I typically do this for one side of the wall (the right side). The fixed temperature boundary condition has already been done by simply using a fixed temperature for the surface nodes instead of equation (1). The insulated or adiabatic boundary condition is handled by simply setting the temperature at the boundary node of the wall to the temperature of the adjoining

interior node. I explain to the class that this automatically sets the temperature gradient to zero and therefore the heat flux to zero (the insulated or adiabatic boundary condition). The converged results for this case are also shown in Figure 6. Students are often initially surprised to see that the temperature converges to a uniform temperature at the value of the fixed temperature on the opposite (left) wall (see Figure 6). But after a little thought they understand this must be the case. Next, a convective boundary condition is applied to the right wall. This is done by simply adding a convective thermal resistance to the surface node using equation (1). In this case, the right wall temperature converges to a value between the fluid temperature and left wall temperature, dependent on the value of the heat transfer coefficient and wall properties.

At this point, students have a numerical tool (a Spreadsheet template) that they have built themselves that can solve 1-D steady problems including fins with a variable area and even a variable boundary condition along the fin (a heat transfer coefficient that varies with position along the fin). My most recent assignment (at both UC Davis and at the USAF Academy) was an analysis and design study of the fin on a Harley Davidson Softail[®] motorcycle engine. The actual fin is tapered with a thicker base than tip. The heat transfer coefficients along the fin (which are “guesstimated” for cruise conditions) vary with position along the fin and with the gap between fins. I ask students to first ascertain that their numerical solution is working by using a straight fin with a uniform convective boundary condition and to compare their numerical solutions to the exact solutions for both insulated and convective tip boundary conditions. They are then given hypothetical heat transfer coefficient distributions, a variable area (thicker base), and a possible effect of reducing the gap thickness (more fins) and asked to optimize the fin shape. Figure 7 shows some typical results from their numerical solutions. The base heat transfer is also calculated from the temperature gradient at the base and the fin effectiveness and efficiency are then calculated for a range of conditions. The effect of speed on the amount of heat that can be taken by the fins is also considered (different values of heat transfer coefficients). Some inspiration is added to the class by showing video clips of the Harley- Davidson Softail[®] along with some audio clips.

Some other steady heat transfer applications that I have assigned include the cooling fins on a computer chip, a heated pitot tube in a corporate or commercial jet, the anti-ice leading edge on a wing or horizontal stabilizer and the fins on an air-cooled light aircraft engine. In most cases I am able to bring in a piece of the actual hardware we are analyzing which adds to the interest and inspiration of the class.

Two-dimensional solutions have also been done using this approach. In this case, instead of copying a column, a matrix is copied. Two matrices are required, one for the initial guess or current calculation and one for equation (1). These two matrices can both be on the same sheet as the primary template or one can be on a second sheet in the template.

Another advantage of doing this in a spreadsheet is that it is quite easy to plot the calculated results.

Unsteady Solutions

The energy balance on a node with energy storage (thermal capacitance) is shown in Fig. 8. The equation for the thermal capacitance was previously given in Fig. 3. A forward difference is used for the rate of change of temperature, thus when the temperature at a particular node is calculated the equation uses only temperatures at the current time:

$$T_{i,t+\Delta t} = T_{i,t} + \frac{\Delta t}{C_i} \left[\sum_j \frac{T_{j,t}}{R_{ij}} - T_{i,t} \sum_j \frac{1}{R_{ij}} + \dot{Q}_i \right] \quad (2)$$

For a surface node, with no thermal capacitance (see Figure 9) the energy balance gives:

$$T_{i,t+\Delta t} = \frac{\dot{Q}_i + \sum_j \frac{T_{j,t}}{R_{ij}}}{\sum_j \frac{1}{R_{ij}}} \quad (3)$$

This appears to be the same as equation (1) except that now we are determining the temperature at the next time step in terms of the temperature at the current time (whereas in equation (1) we were iterating until we met our convergence criteria for the steady temperature distribution).

Equation (2) was obtained using an explicit method (as explained above) and therefore has a similar stability criterion to that in the traditional approach (a maximum time step in terms of the discrete Fourier number). For equation (2) the criterion is that the maximum time step must not exceed that given by equation (4):

$$\Delta t < RC_{\min} \equiv \left(\frac{C_i}{\sum_j \frac{1}{R_{ij}}} \right)_{\min} \quad (4)$$

For the stability criterion, the RC value for each node with thermal capacitance, as defined by equation (4), must be calculated and the time step must be less than the lowest (minimum) value of the RC in the nodal network. In some cases this might result in redoing the grid in a region of small temperature gradients so as to allow larger time steps. If any of the thermal resistances or thermal capacitances vary with time, the time step may have to be reduced or the initial value chosen such that the stability criterion is not violated at any time during the iterative solution. These issues also exist in the more traditional approach.

These equations, (2), (3) and (4) can be used in 1-D, 2-D and even 3-D solutions. The boundary conditions are handled the same as for the steady solutions, except that it is now possible to include not only spatial dependence in the boundary conditions, but also to allow them to vary with time. This is easily done in a spreadsheet by making the resistances a function of time.

Figure 10 shows a template for a 1-D unsteady solution using equations (2), (3) and (4). The important columns in this case are the column for the temperature at the current time, the column for the calculated temperature at the end of the next time step using equation (2) and (3), and the column for the RC for each node calculated with equation (4). Similar macros to those used for the steady solutions are prepared using Microsoft Visual Basic and the record mode for writing macros in ExcelTM. In this case the temperatures at the next time step are calculated (equation (2)), the time advanced, and the new values pasted (using “paste special”) in the column for the previous time. Again an iteration button is provided to do the series of time steps desired for a particular problem. The maximum time step is calculated using equation (4) and a time step some fraction (usually 0.5 or less) of this is used. The grid size can vary with position. The boundary conditions can also vary with position and time.

As with the steady solutions, the first case assigned to my students is a plane 1-D wall. We start with an initial condition of a uniform temperature in a wall, and set the boundary conditions to insulated on one side (the left side) and convection on the other side (the right side). Students are asked to try a time step larger than the maximum time step calculated with the stability criterion (equation (4)) and to plot the temperature after several time steps. Typical results are shown in Figure 11. The oscillations are observed to become larger with time and to “blow up”. Students then use a time step approximately half the maximum given by equation (4) and determine the temperature after several time intervals. Typical results are shown in Figure 11. These can be compared to the exact solutions to give students confidence in their numerical models. Many applications are then possible.

The most recent application that I have used at UC Davis and at the USAFA Academy for an unsteady solution is the disk brake on a corporate jet (the Citation VII –see Figure 12a) during an aborted takeoff. The temperature transient during the aborted takeoff at maximum gross weight is determined during the braking stage and during the interval after the aircraft has come to a stop. It is assumed that the takeoff is aborted at V1 (below this speed the takeoff can be aborted, above this speed the takeoff is to continue). I chose this particular application because I have served as a pilot in this aircraft and am familiar with the systems and procedures. I also discuss the Boeing 757 (for which I have completed a type course) and use a video for the Boeing 777 flight tests for an aborted takeoff – these show flames coming from the disk brakes after coming to a rest. In this application, there is heat generation on that part of the disk where the brake pads produce friction during braking. There is also convective cooling. After coming to a stop, the conduction in the disk brake continues causing the temperature at the base of the disk, where it is connected to the wheel, to continue to increase. During this time convection is minimal (free convection only). This problem is interesting because the boundary conditions on the disk are not uniform and also change with time. A typical solution is shown in Figure 12b. The original peak in the temperatures is in the vicinity of where the disk brake pads contact the disk. The most interesting part of the solution is when the temperatures continue to increase for several

minutes after coming to a stop at the base of the disk where it connects to the wheel. This causes the tires to heat and can result in an explosion or fire.

Other applications that I have assigned using essentially the same template and solutions include fins on an air-cooled light aircraft engine (rapid temperature changes can increase wear and crack the cylinders), a heated pitot tube and an anti-icing leading edge heater on a jet aircraft (these must be turned on in sufficient time when entering icing conditions), and a disk brake for an Acura NSX racing car. This racing car application was fun and interesting because one of my former PhD students (Dr. Robert Butler) is now a race car instructor and has raced his NSX. He provided details on the disk brake and on racing procedures (I also obtained a NASCAR racing training video to inspire my class). An imaginary triangular shaped race track was selected and my students were asked to analyze the temperature during three laps. The track and conditions (maximum speeds on the straightway, and minimum speeds in the turns) were selected such that after three laps the brakes exceeded their maximum temperature (due to braking into each turn). I asked my students to propose a design modification while choosing from options such as extra cooling (directed air) on the disk, increasing the thickness of the disk brake, or by increasing the diameter of the disk (the most expensive since it requires increasing the size of the wheel well).

In several classes I have chosen the same application for both a steady and unsteady solution. For example, for the Harley Davidson SoftTail we first analyzed the fin for steady cruise operation (a steady solution), and then analyzed and optimized the fin for the case when the Harley Davidson is stopped for a long period of time and can overheat (an unsteady solution). In another class we did steady and unsteady solutions for a fin on an air-cooled aircraft engine. For the leading-edge anti-ice heater on a jet aircraft, we first determined the power requirements with all heaters working and with an inoperative heater for steady operation (a steady solution). Then for the unsteady case we determined the power requirements in order to bring the leading edge up to temperature in a reasonable time period when entering icing conditions.

Two-dimensional unsteady solutions have also been done using this approach with the electrical analogy and a spreadsheet. As with the steady case, this simply requires a matrix be copied instead of a column. One application assigned was that of impinging jet cooling of a surface.

Summary and Conclusions

A set of four equations have been derived to handle the numerical solutions for both steady and unsteady numerical heat conduction solutions. These equations make use of the electrical analogy. Since each node is connected to adjoining nodes and boundaries with a thermal resistance determined by the mode of heat transfer and geometry, these relatively simple equations allow a variety of geometries and boundary conditions including those that vary with position and time. Since students program in the difference equations for a node into a cell in a spreadsheet and also program in the boundary conditions, their perception of the numerical solution method is clear. The use of a spreadsheet makes it easy for students to program the iterative schemes and quickly move on to interesting real-world problems. In the process, they can explore the effects of convergence for steady heat transfer and the stability criterion for unsteady heat transfer. For simple test cases, they can compare their numerical solutions to exact solutions. They can then use the tool they have developed to address real world problems

including fins for electronic and engine cooling, disk brakes for aircraft and automobiles, and anti-icing equipment. The similarity in a spreadsheet “cell” and a “node” or “element” in numerical methods increases the understanding of the numerical methods. Furthermore, the use of the “record” feature in Excel™ for writing the iteration macros requires that students manually perform the steps in their iterative process, thus making it clear what their numerical model is actually doing. The author has found this to be a useful way of introducing students to the basics of numerical methods in heat transfer while considering interesting real-world applications.

Acknowledgement

The author gratefully acknowledges the valuable suggestions by Professor Michael Maixner of the USAF Academy while team teaching this subject at the USAF Academy and his help in the preparation of the figures used in this paper. The author also gratefully acknowledges the support of the Distinguished Visiting Professor program at the USAF Academy administered by Dr. John Sherfese and the research support for my assignment to the USAF Academy by Dr. Thomas McLaughlin.

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Biographical

Professor James W. Baughn is a graduate of the University of California, Berkeley (B.S.) and of Stanford University (M.S. and PhD) in Mechanical Engineering. He spent eight years in the Aerospace Industry and has served as a faculty member at the University of California, Davis since 1973. He is a Fellow of the American Society of Mechanical Engineering, a recipient of the UC Davis Academic Senate Distinguished Teaching Award and the author of numerous publications. He is currently on assignment to the USAF Academy in Colorado Springs as the 2004-2005 Distinguished Visiting Professor of Aeronautics.

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→ Steady Solutions

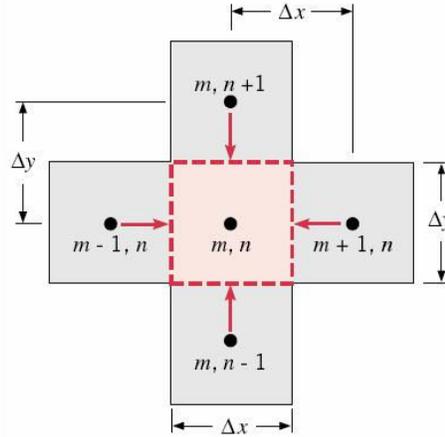
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{\partial^2 T}{\partial x^2} \Big|_{m,n} \approx \frac{\partial T / \partial x \Big|_{m+1/2,n} - \partial T / \partial x \Big|_{m-1/2,n}}{\Delta x}$$

$$\frac{\partial T}{\partial x} \Big|_{m+1/2,n} \approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\frac{\partial T}{\partial x} \Big|_{m-1/2,n} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\frac{\partial^2 T}{\partial x^2} \Big|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$



$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$

Fig. 1 Discretization and Finite-Difference Equations for Steady Heat Transfer (ref. 1)

Unsteady Solutions

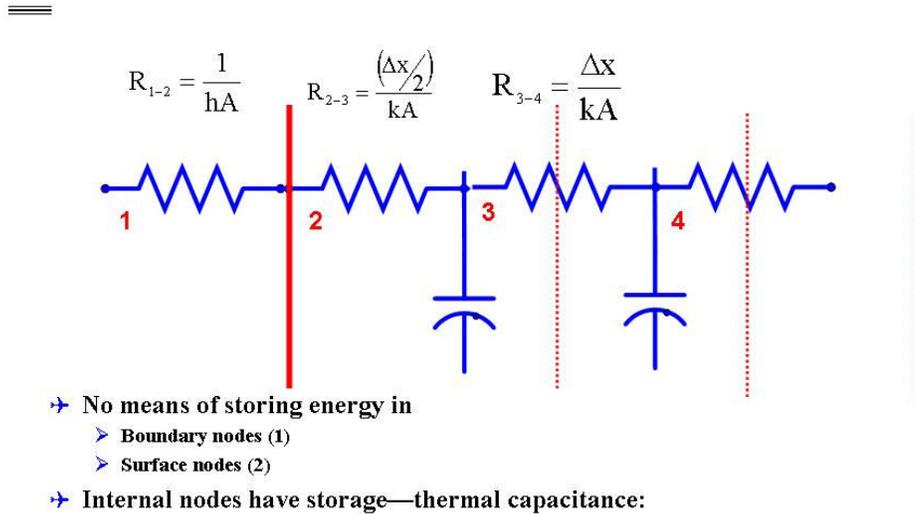
$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \quad \frac{\partial T}{\partial t} \Big|_{m,n} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^p}{\Delta t}$$

$$\frac{1}{\alpha} \frac{T_{m,n}^{p+1} - T_{m,n}^p}{\Delta t} = \frac{T_{m+1,n}^p + T_{m-1,n}^p - 2T_{m,n}^p}{(\Delta x)^2} + \frac{T_{m,n+1}^p + T_{m,n-1}^p - 2T_{m,n}^p}{(\Delta y)^2}$$

$$T_{m,n}^{p+1} = Fo(T_{m+1,n}^p + T_{m-1,n}^p + T_{m,n+1}^p + T_{m,n-1}^p) + (1 - 4Fo)T_{m,n}^p$$

$$Fo = \frac{\alpha \Delta t}{(\Delta x)^2}$$

Fig. 2 Discretization and Finite-Difference Equations for Unsteady Heat Transfer (ref. 1)



$$C_i = \rho c \Delta x A$$

Fig. 3 Thermal Network for a 1-D Wall

Steady – Energy Balance

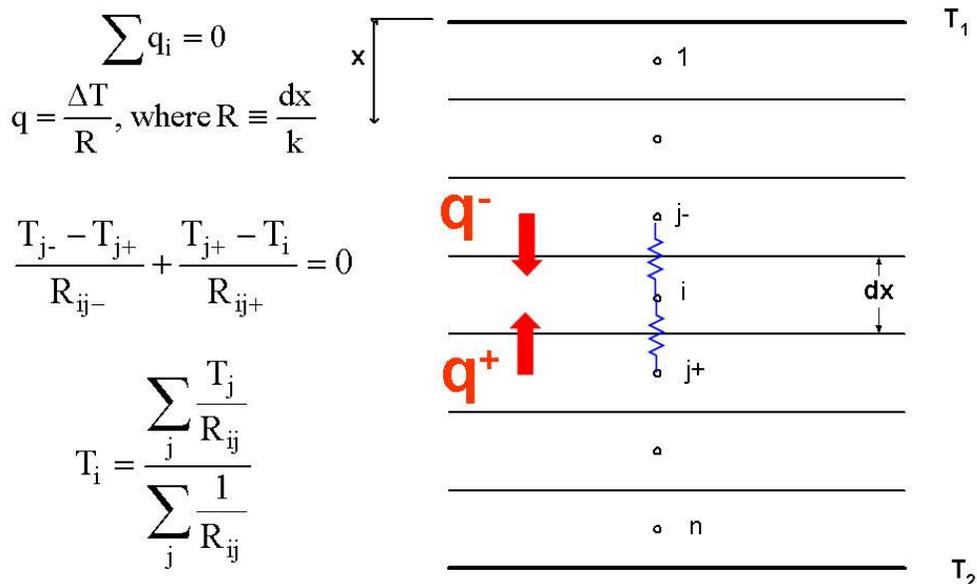


Fig. 4 Energy Balance on a Node (1-D) With No Thermal Capacitance (Steady)

Iteration

1-D Steady State Heat Transfer
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Last Name
First Name

Iteration Input	X_i (m)	$A_c(i)$ (m ²)	T_n (deg C)	T_{n+1} (deg C)	dT (deg C)
$N_i = 1000$	0.0000	0.0035000	378.497	378.517	0.020
thermal conductivity	0.0015	0.0034667	377.757	377.72907	-0.02806
$k = 170$ W/m.K	0.0045	0.0034000	376.282	376.30185	0.02025
h (end)	0.0075	0.0033333	374.930	374.9018	-0.02803
$h_e = 10$ W/m ² .K	0.0105	0.0032667	373.608	373.62823	0.020232
$T_a = 20$ deg C	0.0135	0.0032000	372.416	372.38759	-0.02801
Grid	0.0165	0.0031333	371.259	371.27954	0.020215
$dx = 0.003$ m	0.0195	0.0030667	370.239	370.21106	-0.02798
base thickness	0.0225	0.0030000	369.262	369.2822	0.0202
$t_b = 0.0035$ m	0.0255	0.0029333	368.429	368.40056	-0.02797
tip thickness	0.0285	0.0028667	367.647	367.66672	0.020188
$t_e = 0.0025$ m	0.0315	0.0028000	367.017	366.98897	-0.02795
	0.0345	0.0027333	366.448	366.46854	0.020181
	0.0375	0.0026667	366.042	366.01455	-0.02795
	0.0405	0.0026000	365.709	365.72908	0.020182
	0.0435	0.0025333	365.550	365.52219	-0.02795
	0.0450	0.0025000	365.500	365.520	0.020186

hs W/m.K	t (m)	$A_s(i)$ (m ²)
10	0.0035	0.00000
10	0.0034667	0.0060004
10	0.0034	0.0060004
10	0.0033333	0.0060004
10	0.0032667	0.0060004
10	0.0032	0.0060004
10	0.0031333	0.0060004
10	0.0030667	0.0060004
10	0.003	0.0060004
10	0.0029333	0.0060004
10	0.0028667	0.0060004
10	0.0028	0.0060004
10	0.0027333	0.0060004
10	0.0026667	0.0060004
10	0.0026	0.0060004
10	0.0025333	0.0060004
10	0.0025	0.00000

Iterate Macro

```

Sub Iterate()
  Dim I
  I = 0
  Do Until I = Range("C10").Value
    CopyColumn
    I = I + 1
  Loop
End Sub

Sub Reset()
  Range("G12").Select
  ActiveCell.FormulaR1C1 = "160"
  Selection.Copy
  Range("G12:G27").Select
  Selection.PasteSpecial Paste:=xlValues
  Application.CutCopyMode = False
  Range("A1").Select
End Sub

```

CopyColumn Macro

```

Sub CopyColumn()
  '
  ' CopyColumn Macro
  ' Macro recorded 09.02.2004 by Administrator
  '
  Range("H11:H27").Select
  Selection.Copy
  Range("G11").Select
  Selection.PasteSpecial Paste:=xlPasteValues, Operation:=xlNone, SkipBlanks _
    :=False, Transpose:=False
End Sub

```

Fig. 5 Spreadsheet Template and Macros for 1-D Steady Heat Transfer

Temperature Profile for a Slab After N Iterations

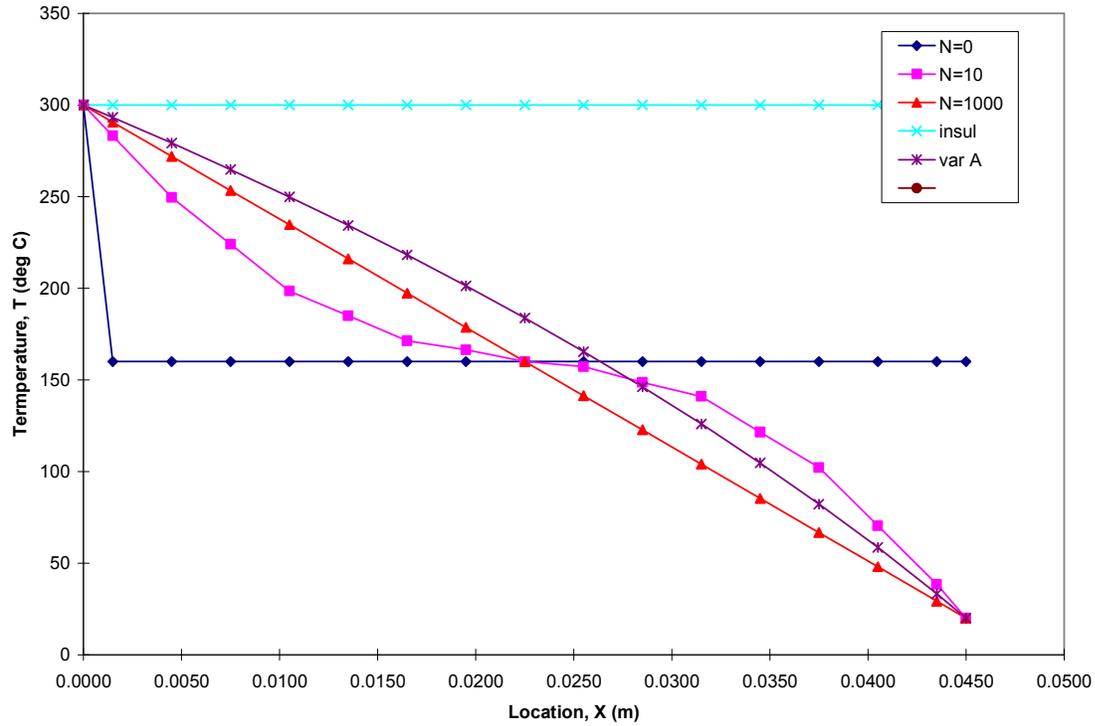
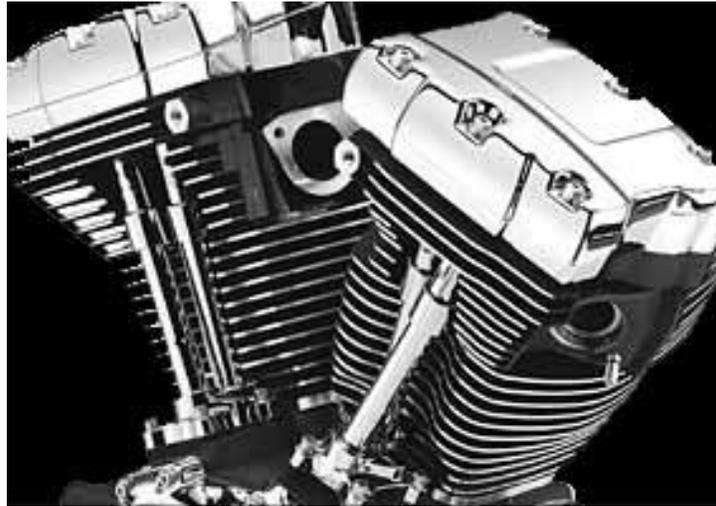


Fig. 6 Convergence Illustration for 1-D Steady Heat Transfer



<http://www.harley-davidson.com/pr/eng/en/engines.asp>

Fin Convection

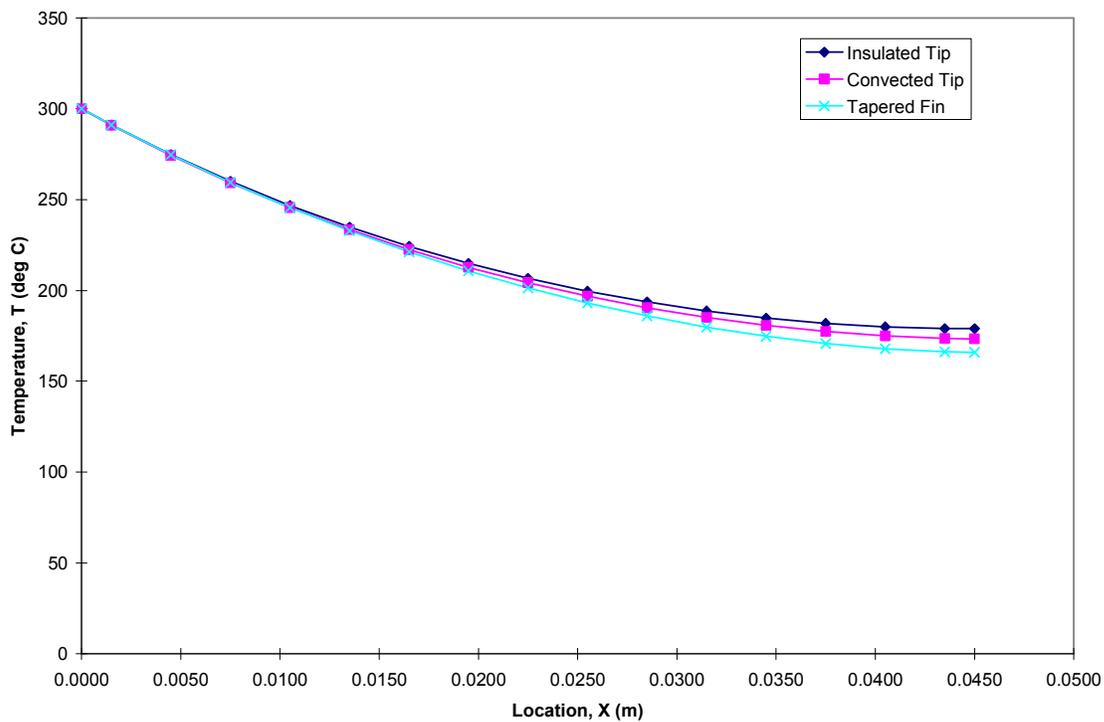


Fig. 7 Numerical Results for 1-D Steady Heat Transfer in a Fin (modeled after a Harley Davidson SOFTAIL® engine)

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→ **Energy Balance on node i:**

$$\begin{aligned} \dot{Q}_{\text{net}} &= C_i \frac{T_{i,t+\Delta t} - T_{i,t}}{\Delta t} \\ &= \dot{Q}_i + \sum_j \frac{T_{j,t} - T_{i,t}}{R_{ij}} \\ &= \dot{Q}_i + \sum_j \frac{T_{j,t}}{R_{ij}} - T_{i,t} \sum_j \frac{1}{R_{ij}} \end{aligned}$$

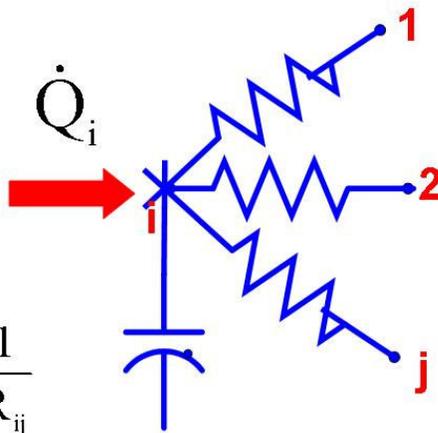
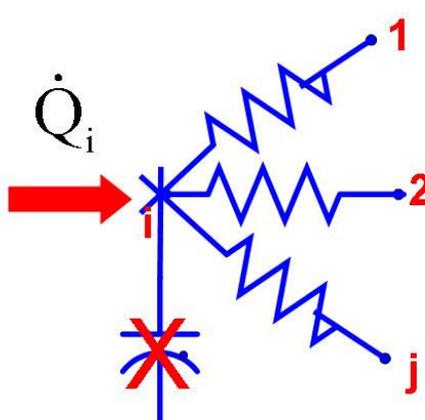
$$T_{i,t+\Delta t} = T_{i,t} + \frac{\Delta t}{C_i} \left[\sum_j \frac{T_{j,t}}{R_{ij}} - T_{i,t} \sum_j \frac{1}{R_{ij}} + \dot{Q}_i \right]$$


Fig. 8 Energy Balance on a Multi-Dimensional Node for Unsteady Heat Transfer

→ **For a surface node (i.e., with no capacitance):**

$$\dot{Q}_{\text{net}} = 0 = \dot{Q}_i + \sum_j \frac{T_{j,t} - T_{i,t+\Delta t}}{R_{ij}}$$


→ **Again, solve for temperature @ next time step using explicit method:**

$$T_{i,t+\Delta t} = \frac{\dot{Q}_i + \sum_j \frac{T_{j,t}}{R_{ij}}}{\sum_j \frac{1}{R_{ij}}}$$

Fig. 9 Energy Balance for a Surface Node (no capacitance) for Unsteady Heat Transfer

Iteration

1-D Unsteady Heat Transfer
 Copyright James W. Baughn 2004

Iteration Input		x_i or r_i	T^i	T^{i+1}	dT	$\dot{q}(i)$	$R(i)$	$R_c(i)$	$h_c(i)$	RC_n	$A_c(i)$	
		(m)	(deg C)	(deg C)	(deg C)	(W)	W/m K	W/m K	(W/m ² K)	(s)	(m ²)	
$t_0 =$	200	s	0.1000	15.0	15.00	0.00	0	N/A	0.0196	N/A	N/A	0
$t_f =$	620	s	0.1025	15.0	15.00	0.00	0	0.0196	0.0379	0.00	1.50	0.006440265
$N =$	420		0.1075	15.0	15.00	0.00	0	0.0379	0.0362	0.00	2.39	0.006754424
Properties			0.1125	15.0	15.00	0.00	0	0.0362	0.0346	0.00	2.39	0.007068583
$k_o =$	20	W/m K	0.1175	15.0	15.00	0.00	0	0.0346	0.0332	0.00	2.39	0.007382743
Input Conditions			0.1225	15.0	15.00	0.00	0	0.0332	0.0318	0.00	2.39	0.007696902
$T_h =$	200	W/m ² K	0.1275	15.0	15.00	0.00	0	0.0318	0.0306	0.00	2.39	0.008011061
$T_c =$	15	deg C	0.1325	15.0	15.00	0.00	0	0.0306	0.0295	0.00	2.39	0.008325221
Grid			0.1375	15.0	15.00	0.00	0	0.0295	0.0284	0.00	2.39	0.008639380
dx or $dr =$	0.005	m	0.1425	15.0	15.00	0.00	0	0.0284	0.0274	0.00	2.39	0.008953539
$w =$	0.01	m	0.1475	15.0	15.00	0.00	0	0.0274	0.0265	0.00	2.39	0.009267698
Unsteady State			0.1525	15.0	15.00	0.00	0	0.0265	0.0257	30.00	2.39	0.009581858
$\rho =$	7980	kg/m ³	0.1575	15.0	15.00	0.00	0	0.0257	0.0249	30.00	2.39	0.009896017
$C_p =$	480	J/kg K	0.1625	15.0	15.00	0.00	0	0.0249	0.0241	30.00	2.39	0.010210176
$RC_{mi} =$	1.59	s	0.1675	15.0	15.00	0.00	0	0.0241	0.0234	30.00	2.39	0.010524335
$dt =$	1	s	0.1725	15.0	15.00	0.00	0	0.0234	0.0115	30.00	1.59	0.010838495
			0.1750	15.0	15.0	0	0	0.0115	N/A	N/A	N/A	0

W=	22,000 LB	\dot{q}	\dot{q}	h_s	h_s	Sub Iterate()	Sub Iterate()
m=	9977.3243 kg	braking	at stop	braking	at stop	Dim I	Dim I
V1=	123 knots	0	0	N/A	N/A	I = 0	I = 0
V1=	63.233808 m/sec	0	0	0	0	Do Until I = Range("C12").Value	Do Until I = Range("C12").Value
ke=	19947238 J	0	0	0	0	CopyColumn	CopyColumn
# rotors	12	0	0	0	0	I = I + 1	I = I + 1
%brakes	80	0	0	0	0	Loop	Loop
qtob/brak	1329815.8 J	0	0	0	0	End Sub	End Sub
t(brake)	20 s	13298.2	0	0	0	Sub CopyColumn()	Sub CopyColumn()
\dot{q}	66490.792	13298.2	0	0	0	Range("G11:G27").Select	Range("G11:G27").Select
nodes	5	13298.2	0	0	0	Selection.Copy	Selection.Copy
\dot{q}	13298.168	13298.2	0	0	0	Range("F11").Select	Range("F11").Select
		0	0	200	30	Selection.PasteSpecial Paste:=xlValu	Selection.PasteSpecial Paste:=xlValues
		0	0	200	30	Application.CutCopyMode = False	Application.CutCopyMode = False
		0	0	200	30	End Sub	End Sub
		0	0	200	30	Sub Reset()	Sub Reset()
		0	0	N/A	N/A	Range("F11").Select	Range("F11").Select
						ActiveCell.FormulaR1C1 = "15"	ActiveCell.FormulaR1C1 = "15"
						Selection.Copy	Selection.Copy
						Range("F12:F27").Select	Range("F12:F27").Select
						Selection.PasteSpecial Paste:=xlValu	Selection.PasteSpecial Paste:=xlValues
						Application.CutCopyMode = False	Application.CutCopyMode = False
						Range("A1").Select	Range("A1").Select
						End Sub	End Sub

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Fig. 10 Spreadsheet Template and Macros for 1-D Unsteady Heat Transfer

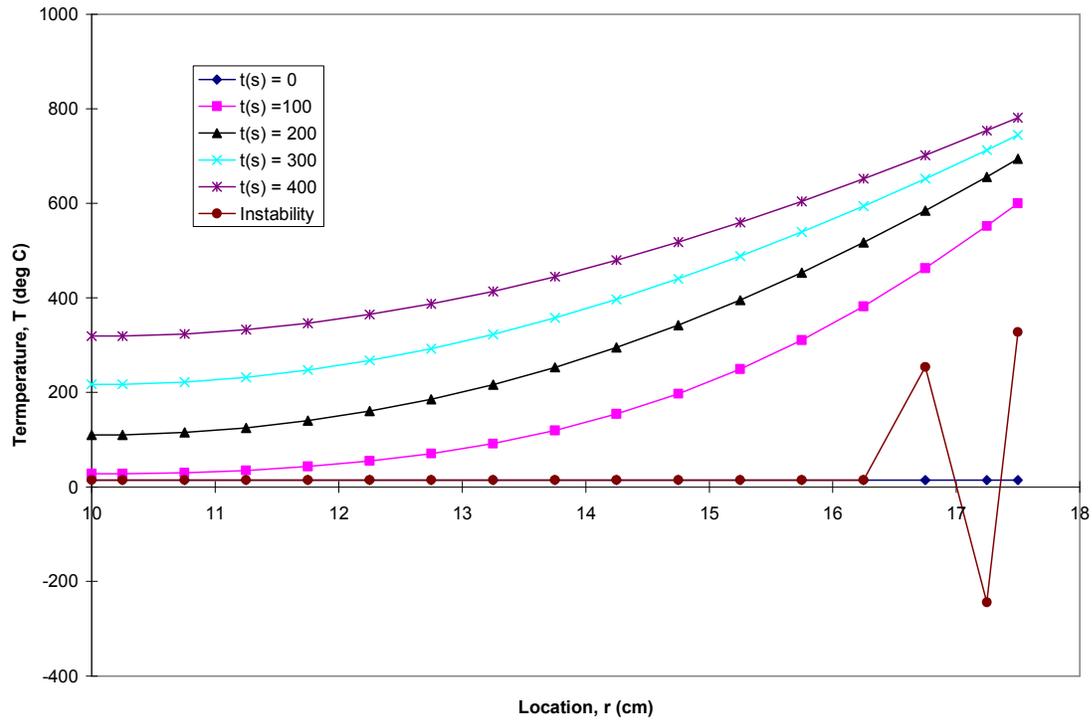


Fig. 11 Demonstration of the Solution and Instability for Unsteady Heat Transfer in a 1-D Wall



Fig. 12a Disk Brakes for Citation VII Corporate Jet

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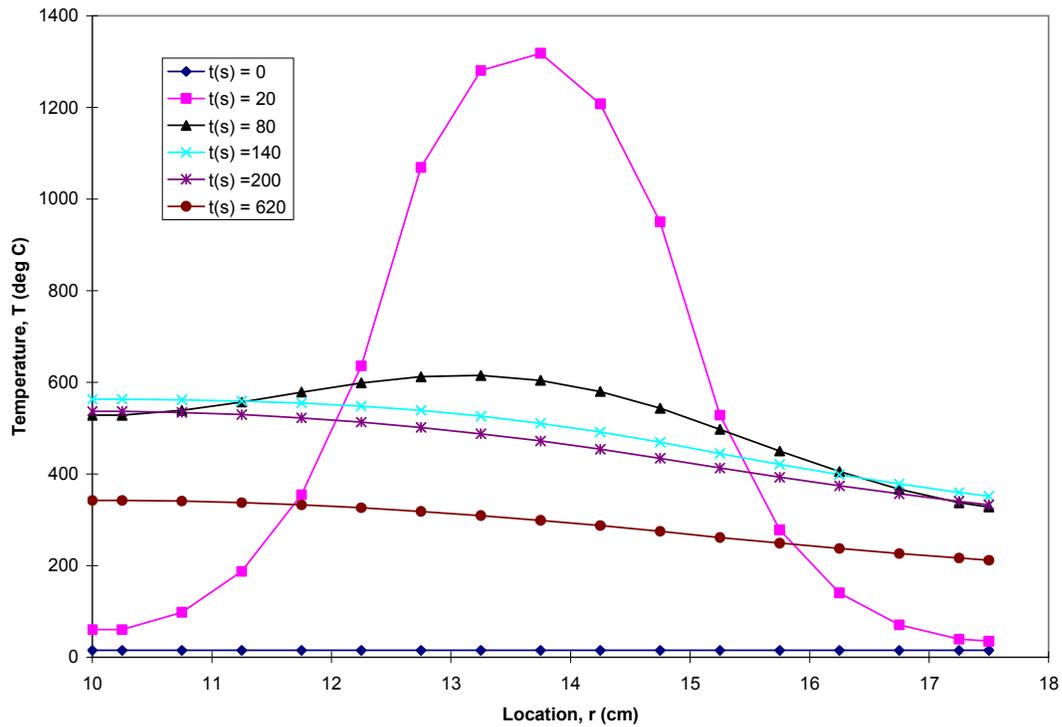


Fig. 12b Numerical Results for 1-D Unsteady Heat Transfer in a Disk Brake during an Aborted Takeoff (V1 Cut)