

Two Experiments to Teach Modulus of Elasticity and Modulus of Rigidity

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Abstract

The relationship between loads and deformation in a structure is a difficult concept mechanics students often must master with little prior exposure to materials science concepts such as Hooke's law for elastic modulus. Two hands on experiments have been designed to help demonstrate for mechanics students in an introductory strength of materials course the concept of structural stiffness and to help differentiate between the structural stiffness and the modulus of elasticity for a material under applied axial load and the modulus of rigidity for a material under applied shear loading.

In the first experiment two different size wires of the same material are loaded in tension. As the applied load is increased the students record the load and the corresponding deflection of the wires. Using elementary mechanics the students can compute the stiffness of each system from a plot of load versus elongation. Then by applying the fundamental definitions of both stress and strain the data can be recast in the form of a stress-strain diagram for the material and the students can compute the modulus of elasticity for the material of the two wires.

The second experiment looks at the relationship between applied torque and angular displacement. Using a simple apparatus to load both a solid circular rod and a hollow circular rod, the students can record the applied torque and the corresponding angle of twist. Plotting this data the students can compute the torsional stiffness of each system. By manipulating the torque-angular displacement relationship the students can compute the modulus of rigidity for the material of both systems.

In both experiments the students will observe that while the structural stiffness varies with the geometry of the structural element, both the modulus of elasticity and the modulus of rigidity are independent of geometry and thus material properties. The apparatus and application of each of these experiments will be described in detail in this paper.

Introduction

Strength of materials is a second course in solid mechanics building on the first course, statics. The fundamental assumption underlying the static analysis of structures is that all structures and structural elements are rigid and hence the geometry of the structures is unchanged by applied loads. In a strength of materials course we introduce students to the science of deformable bodies. It is easy to convince students that real structures are not in fact rigid but instead that every structure has an inherent deformation response or stiffness in the presence of applied loads. We then go on to teach students fundamental constitutive laws for linear elastic isotropic material behavior. At this juncture it is important to clearly demonstrate for the students the difference between structural stiffness and the material properties E , modulus of elasticity and G , modulus of rigidity. This paper will describe the setup and conduct of two hands-on experiments designed to help teach these important concepts as part of an undergraduate strength of materials course. Instead of watching a technician performing a standard tensile test or a torsion test using electro-mechanical loading frames as in most materials science courses these two experiments give the students hands on experience with measuring the critical dimensions of the test article, they get to apply the load using dead weights and then measure for themselves the resulting deformation of the test article. Both experiments have been designed to be highly interactive, requiring the students to become intimately familiar with each apparatus as well as receive training with some basic engineering instrumentation. The analysis in each case is designed to reinforce fundamental principles of mechanics.

Teaching Modulus of Elasticity

The first experiment involves axial loading of different diameter wires. In this experiment the students will derive Hooke's law for uniaxial tension, $\sigma = E\varepsilon$ and determine E the modulus of elasticity from the measured deformation response of the test wire under applied load. From three different test cases they will be able to see that while the structural stiffness varies on a case by case basis the modulus of elasticity, E is independent of the structural geometry and hence is only a property of the material used in the test wire.

A schematic of the loading apparatus and test wire is shown in Figure 1. The loading apparatus is comprised of vertical tower welded to a heavy metal base, attached to the vertical tower is the support arm to which the test wire is attached. At the opposite end of the test wire, the lever arm is hinged to the vertical tower and fixed to the test wire. Load is applied to the test wire using dead weights suspended in a cradle from the lever arm. Deflection of the test wire is measured using a dial extensometer suspended from the support arm in parallel to the test wire. Three different apparatus are used in this experiment; the first apparatus uses a thin copper wire nominally 0.040" in diameter, the second apparatus uses a thick copper wire nominally 0.065" in diameter, and the third apparatus uses a length of thin wire in combination with a length of thick wire in parallel so that the applied load is shared by the two wires. The third case is particularly interesting since it really represents a statically indeterminate load case. The test wire in each case is copper with a modulus of elasticity, $E = 16 \times 10^6$ psi, assuming the wires have each been properly annealed they should all exhibit the same material properties. Small diameter copper wire is used because with only modest loads the entire linear elastic response of the material can be explored, that and it is of course readily available.

As background for this experiment it is good to review what the students have already learned in physics. In a first course in physics students are exposed to the equation of spring behavior,

$$F = k\delta \quad [1]$$

where F is the force applied to the spring, δ is the deformation of the spring under applied load and k is the spring constant. It is helpful to rework the equation of spring behavior,

$$k = \frac{F}{\delta} \quad [2]$$

to show that the spring constant k in fact describes the stiffness of the spring in terms of force/unit length. In strength of materials we teach that all structural elements behave like a spring, that is they deform under applied load, and that the static analysis of structures (based on rigid body mechanics) is valid so long as the deformation in the system is small. An understanding of structural stiffness then is paramount to grasping more difficult mechanics concepts such as the solution of statically indeterminate structures. However engineering design is seldom done this way, instead engineers work with standard sections and a finite basket of engineering materials. To know the structural stiffness of a structural element requires further development of Equation 1.

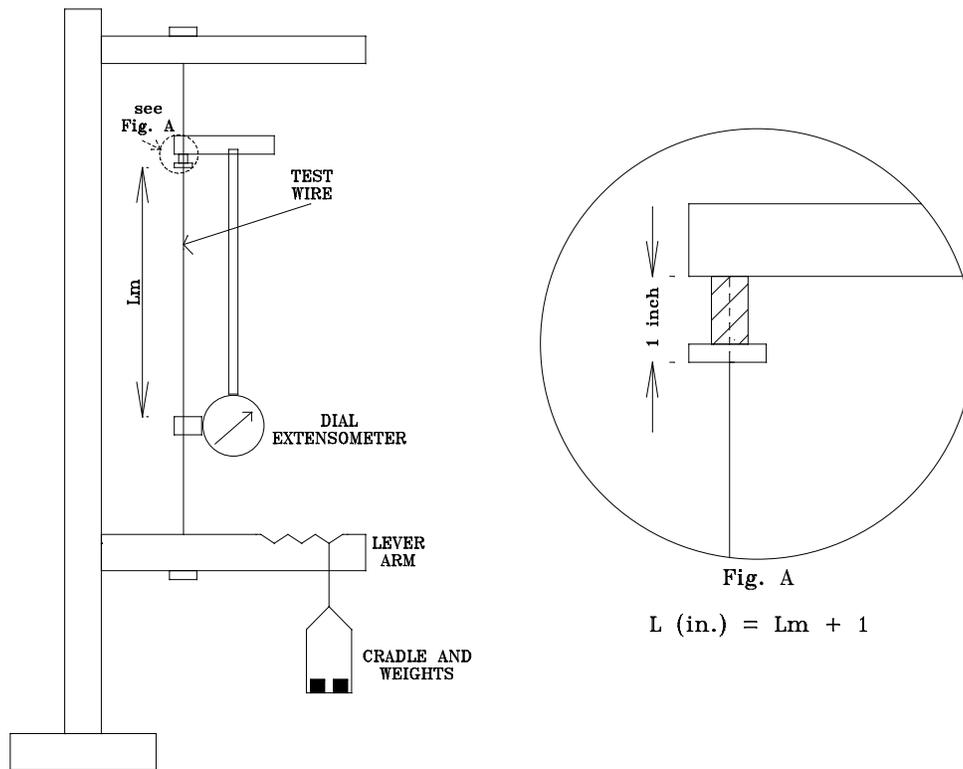


Figure 1 – Schematic of the axial loading apparatus.

Begin by calling on the definition of stress, thus dividing each side of Equation 1 by the area of the structural element, in this case the test wire, hence,

$$\frac{F}{A} = \sigma = \frac{k}{A} \delta.$$

Now use the definition of strain, dividing both sides of the equation by the length of the structural element as follows:

$$\frac{\sigma}{L} = \frac{k}{A} \frac{\delta}{L} = \frac{k}{A} \varepsilon.$$

Rearranging terms we obtain

$$\sigma = \frac{kL}{A} \varepsilon. \quad [3]$$

Comparing Equation 3 with Hooke's law for uniaxial stress-strain we can show that the structural stiffness of an element is a function of the modulus of elasticity of the material, the cross-sectional area of the element, and the length of the element as follows:

$$k = E \frac{A}{L}. \quad [4]$$

Alternatively, in this experiment the modulus of elasticity for the material of each of the test wires can be calculated from the measured stiffness of each wire using:

$$E = k \frac{L}{A} \quad [5]$$

where k is obtained from the experimental observations of the load-deformation behavior in each apparatus.

The experimental procedure for this experiment requires that the students first measure and record the diameter and length of the test wire using a micrometer and tape measure. All three test apparati in this experiment have been constructed to have the same nominal length but it important in the final analysis to know the exact length of each test wire. Correct measurement of the diameter of the test wire in each apparatus is critical to properly evaluating the cross-sectional area of each test wire. Errors in this step can mask the entire objective of this experiment, it is critical that the students receive adequate hands on instruction in the proper usage of a micrometer.

Next the test wire is loaded using one pound weights hung from a loading lever arm, the load applied to the test wire must be adjusted to account for the load ratio of the lever arm. Different load ratios are used for each case to avoid permanent deformation of the test wire. For each load increment (beginning of course with no load-no deformation) the students are to record the applied load and the corresponding deformation of the test wire from readings taken off the dial gage. It is actually instructive to have the students record the load-deformation behavior for loading and unloading to show that the behavior is elastic and repeatable.

When the experiment has been performed on each of the three apparati the students are to graph their data using a spreadsheet/graphing program such as MS Excel. The load-deformation data should be plotted on a single graph to compare the stiffness of each of the three systems. From a least squares fit to the experimental P - δ data of Figure 2, the slope of each plot gives us the stiffness of each system since from Equation 2 we know $k = \frac{\Delta P}{\Delta \delta}$. The resulting stiffness values and fit coefficients are tabulated below together with the computed area for each test wire obtained at the beginning of the experiment.

Test Apparatus	Area (in. ²)	Stiffness (lbs/in.)	Intercept (lbs)	R ²
Thin Wire	0.00129	531.5	0.223	0.976
Thick Wire	0.00332	1435.5	0.0148	0.995
Double Wire	0.00458	1831.5	0.266	0.976

Table 1 – Stiffness results.

As anticipated the greater the cross-sectional area of the test wire the greater the structural stiffness.

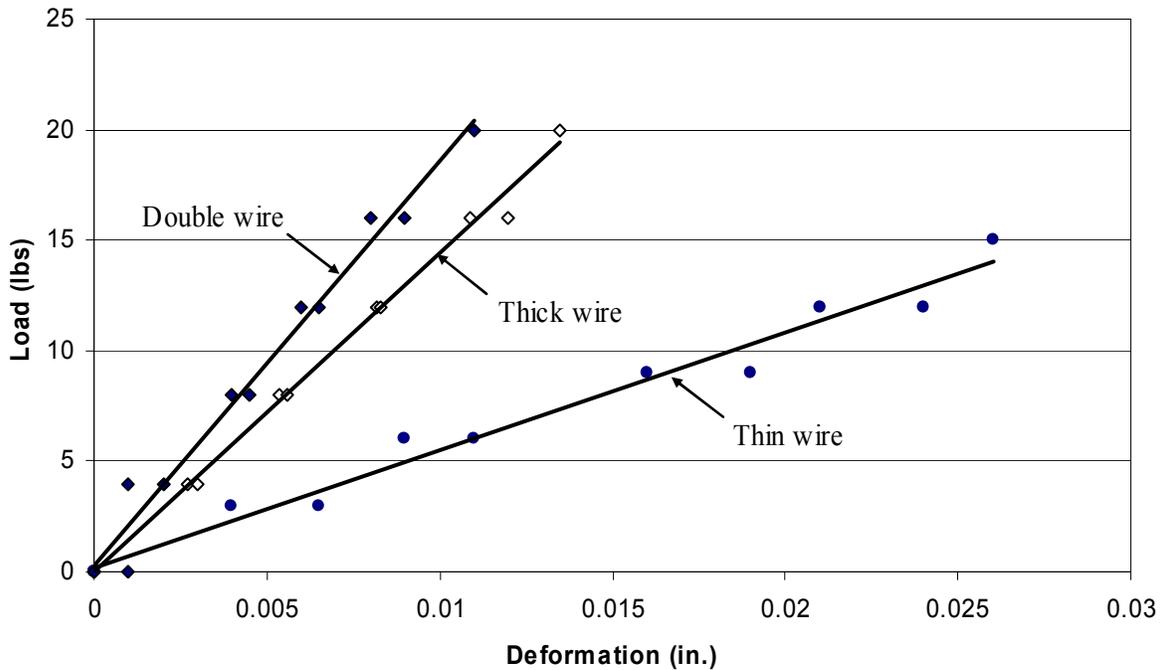


Figure 2 – Load-deformation plot for all three systems.

From the computed stiffness values of Table 1 and the test wire length and area measurements taken prior to loading and unloading the test wire we can now compute the elastic modulus of the test wire for all three systems using Equation 4. It is more interesting however to go through the mechanics of the derivation again, this time with the experimental data. Returning to the raw P - δ data first apply the definition of stress by dividing the load data by the measured cross-sectional area of the test wire, then apply the definition of strain to the deformation data. Plot the resulting stress-strain data, the students should recognize the data in this form. If everything goes according to plan the three curves of Figure 2 should all collapse onto a single line as shown in Figure 3. From Hooke's law for uniaxial stress-strain behavior the modulus of elasticity can be obtained from the slope of the σ - ϵ data. Using linear regression have the students check the modulus of elasticity of each test wire independently.

Test Apparatus	Area (in. ²)	Elastic Modulus (Msi)	% Difference From Theoretical
Thin Wire	0.00129	16.4	2.5
Thick Wire	0.00332	17.8	11.25
Double Wire	0.00458	16.9	5.63

Table 2 – Modulus of Elasticity results.

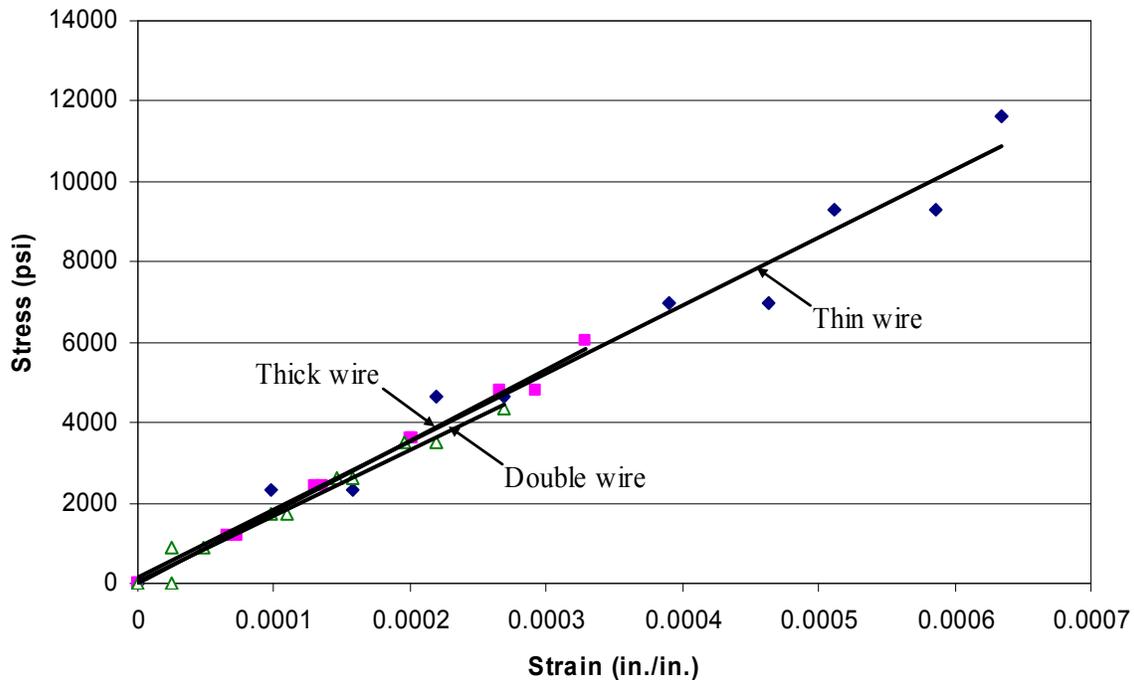


Figure 3 – Stress-strain plot for all three systems.

The elastic modulus values are tabulated in Table 2 together with the computed area for each test wire obtained at the beginning of the experiment. Since both test wires are made of the same copper material it comes as no surprise that when the data is normalized in terms of stress and strain the three curves should collapse into one and the computed modulus of elasticity values should all be in agreement. Checking the experimentally measured values of the modulus of elasticity against the theoretical value of 16 Msi we compute between 2-12% deviation from one test case to another. This result could perhaps be improved upon by fine tuning the experimental apparatus but the data is good enough for the students to compare Figures 2 and 3 and conclude that while the structural stiffness, k is dependent on the cross-sectional geometry of the test wire, the modulus of elasticity, E is dependent only on the material of the test wire. As a secondary exercise, it could be interesting to ask the students to analyze the double wire apparatus further, computing the load in each wire since the two wires are required to share the same deflection.

Teaching Modulus of Rigidity

The second experiment looks at the shear deformation of a cylindrical shaft under applied torsional loading. The objective of this experiment is to demonstrate the difference again between the torsional stiffness of a system and the modulus of rigidity. The students will measure the angular displacement of a cylindrical shaft under an applied torque. The cylindrical shaft in this case behaves like a torsion spring, the students will determine the torsional stiffness of the cylindrical shaft and then mechanics of materials determine the modulus of rigidity for the shaft. The results of this experiment demonstrate the effect geometry has on torsional stiffness and once again the students will see that the modulus of rigidity depends only on the material of the cylindrical shaft. The advantage of using hollow shafts in design will also become clear when this experiment is completed.

The test apparatus for this experiment is shown in Figure 4. The apparatus is comprised of a base to which a cylindrical shaft is fixed on one end and supported by bushings at two locations along the length. Torque is applied to the free end of the cylindrical shaft using a ~six inch long loading lever arm from which is hung a cradle of deadweights. The angular displacement of the shaft in response to torsional loading is measured using protractors two protractors (marked in degrees) located at *B* and *C* along the length of the shaft to eliminate any rigid body motion at the fixed end (*A*). Two different apparatus are used for this experiment; the test piece for the first apparatus is a 0.250" diameter solid cylindrical rod, the test piece for the second apparatus is a hollow rod with an outer diameter of 0.310" and a wall thickness of 0.025". The material of both test pieces is brass with a modulus of rigidity, $G = 5.2 \times 10^6$ psi.

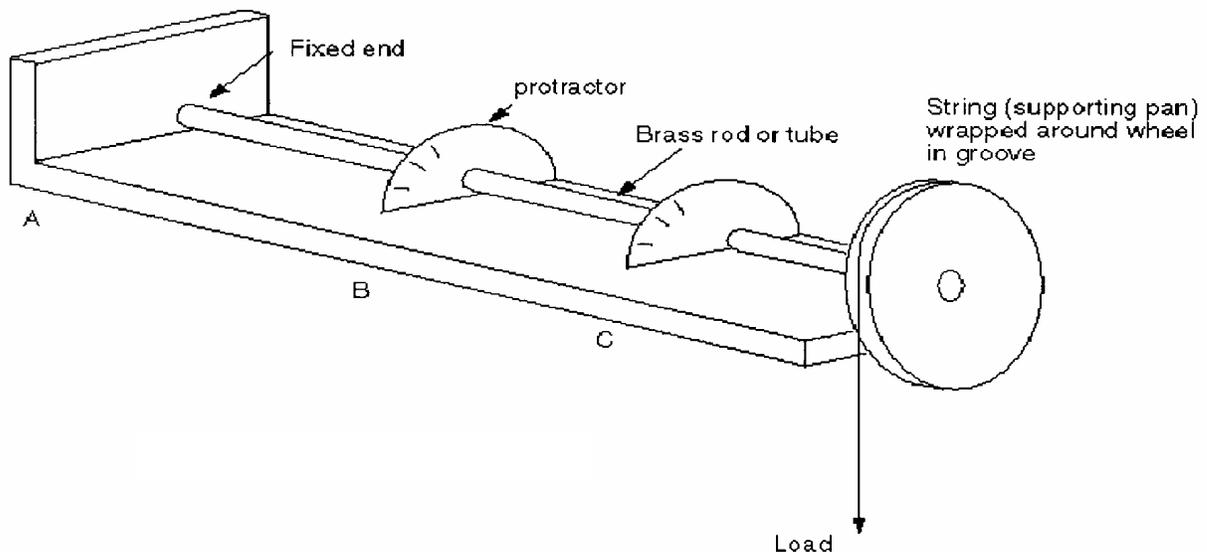


Figure 4 – Schematic of the torsional loading apparatus.

A cylindrical shaft under applied torsional loading will exhibit spring behavior similar to that described by Equation 1, replacing the applied load F by an applied torque T and the spring deflection δ by angular displacement θ we obtain

$$T = kd\theta \quad [6]$$

where k in this case represents the torsional spring stiffness of the cylindrical shaft. Analogous to the first experiment we can solve for the spring stiffness as follows:

$$k = \frac{T}{d\theta} \quad [7]$$

Owing to the non-uniform stress distribution inherent in a shaft under torsional loading and the complex geometry involved in defining the shear strain we begin this exercise from the torque-angular displacement relationship for a uniform circular member under constant applied torque

$$d\theta = \frac{LT}{JG}. \quad [8]$$

where L is the length of the shaft and J is the polar second moment of area for the shaft and J has the form $J = \frac{\pi}{2}c^4$ where c is the outer radius of the shaft. Rearranging Equation 8 to solve for the torque necessary to generate a given angle of twist, $d\theta$ we obtain

$$T = \frac{JG}{L}d\theta. \quad [9]$$

Equilibrating Equations 6 and 9 and solving for the torsional stiffness yields

$$k = G\frac{J}{L}. \quad [10]$$

In this experiment the torsional stiffness k is determined directly from the slope of T - $d\theta$ data and the modulus of rigidity can be calculated using:

$$G = k\frac{L}{J}. \quad [11]$$

To begin this experiment the students must measure and record the diameter of the test piece (the wall thickness of the hollow shaft must be provided) together with the length of the test piece where the angular displacement will be recorded (pts B and C .) The students must also measure the length of loading lever arm. Again this experiment affords an excellent opportunity to train the students in the proper usage of micrometer. The test piece is then loaded using the cradle suspended from the loading lever arm in one pound increments to a maximum torsion of 36 in-lbs and then unloaded. For every load increment the students are to record the corresponding angular displacement at B and C . From the raw data the applied torque and the relative angular displacement (in radians) in segment B - C of the test piece can be computed.

When all of the experimental data has been collected the students are to graph the data using a spreadsheet/graphing program such as MS Excel. The torque-angular displacement data should be plotted on a single graph (as in Figure 5) for comparison. Using linear regression to compute the slope of the T - $d\theta$ data the stiffness of each test piece can be computed from Equation 7. The resulting stiffness values and fit coefficients are tabulated below (Table 3) together with the computed area for each test wire.

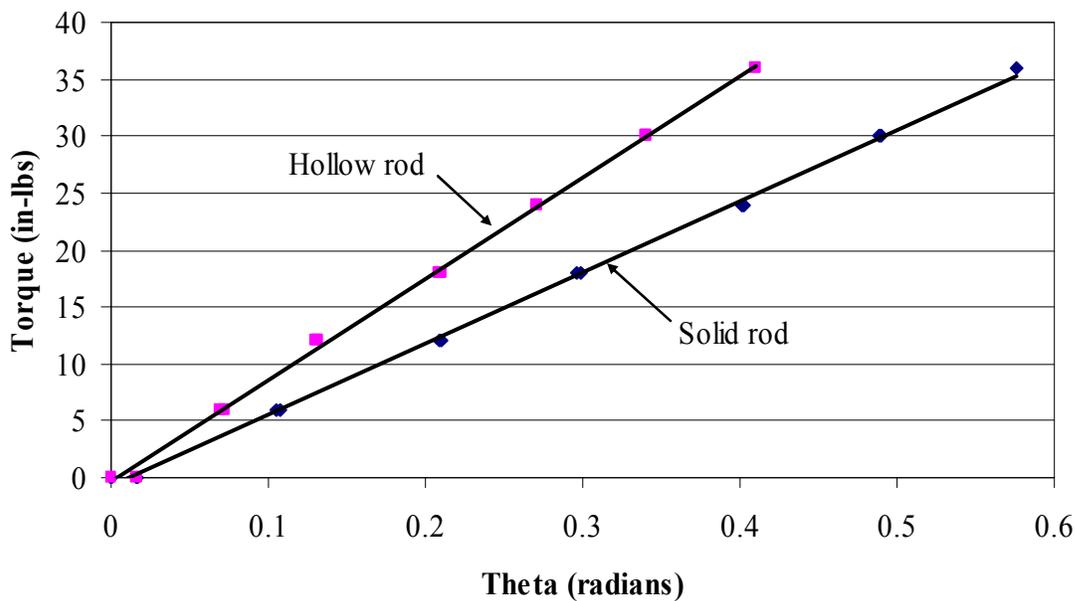


Figure 5 – Torque-angular displacement plot for both systems.

Test Piece	Area (in. ²)	Stiffness (in-lbs/rad.)	Intercept (in-lbs)	R ²
Solid rod	0.0495	63.1	-0.718	0.999
Hollow rod	0.0225	84.2	-0.236	0.999

Table 3 – Stiffness results from torsion experiment.

Contrary to the results of the previous experiment increasing cross-sectional area does not in this instance result in increased torsional stiffness. Instead it is necessary to examine Equation 10 to see that torsional stiffness is dependent on the polar second moment of area, J . Computing the polar second moment of area for both test pieces and collecting the results again in Table 4 we can draw more interesting conclusions. As expected an increase in the polar second moment of area, J results in an increase in the torsional stiffness of the test piece. It is good to point out that since the polar second moment of area, J is proportional to radius to the fourth power, whereas cross-sectional area, $A = \pi r^2$. Thus even though the hollow rod has less cross-sectional area than the solid rod it has a greater polar second moment of area and thus a greater torsional stiffness. This fact explains the efficiency of using hollow shafts for power transmission, especially in weight sensitive applications since the mass of a shaft is proportional to its cross-sectional area.

Finally we can compute the modulus of rigidity for both test pieces using Equation 11. The results should come as no surprise at this point (see Table 5.) Both test pieces are manufactured from the same material and so they should exhibit the same or similar values of G . Checking the experimentally measured values of the modulus of rigidity against the theoretical value of 5.2

Msi we compute approximately 5% error in both cases. These results are remarkably good for such a crude experiment.

Test Piece	Area	Polar Second Moment of Area (<i>J</i>)	Stiffness (<i>k</i>)
	(in. ²)	(in. ⁴)	(in-lbs/rad.)
Solid rod	0.0495	3.897 x 10 ⁻⁴	63.1
Hollow rod	0.0225	4.628 x 10 ⁻⁴	84.2

Table 4 – Torsional stiffness results compared.

Test Piece	Polar Second Moment of Area (<i>J</i>)	Stiffness (<i>k</i>)	Modulus of Rigidity (<i>G</i>)	% Difference From Theoretical
	(in. ⁴)	(in-lbs/rad.)	(Msi)	
Solid rod	3.897 x 10 ⁻⁴	63.1	4.98	4.23
Hollow rod	4.628 x 10 ⁻⁴	84.2	5.50	5.77

Table 5 – Modulus of rigidity results.

Summary

Two hands-on experiments have been designed and demonstrated to augment the conventional lecture only teaching of the concept of structural stiffness and to help differentiate between the structural stiffness and the modulus of elasticity for a material under applied axial load and the modulus of rigidity for a material under applied shear loading. Details of both apparatus and the methodology for each of these experiments have been described in full. Both experiments serve to demonstrate that while the structural stiffness varies with the geometry of the structural element, both the modulus of elasticity and the modulus of rigidity are independent of geometry and thus material properties. This can be a challenging concept for students taking an introductory mechanics course, particularly when mechanics of materials precedes any sort of materials science course. Through hands-on experimentation and deliberate data manipulation the students acquire a better understanding of the difference between structural properties such as stiffness and material properties such as modulus of elasticity and modulus of rigidity.

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