Session 1368

Learning About Stress and Strain Transformations by Comparing Theoretical, Experimental, and Finite Element Results

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One way of teaching a new physical concept effectively to students is to arrive at that physical concept via different approaches. Stress and stain transformations, together with combined loading and von Mises failure criterion for ductile materials, are among those subject matters in solid mechanics in which students have difficulty to visualize and understand. The objective of this paper is to help students to understand and reinforce their comprehension of these fundamental concepts of solid mechanics by introducing them to the 3 different approaches outlined and discussed here.

An L-shaped high strength aluminum beam, E = 10.4E6 psi, cantilevered at one end and subject to a concentrated load P at the free end (Figure 1) is used to teach these 3 fundamental concepts.

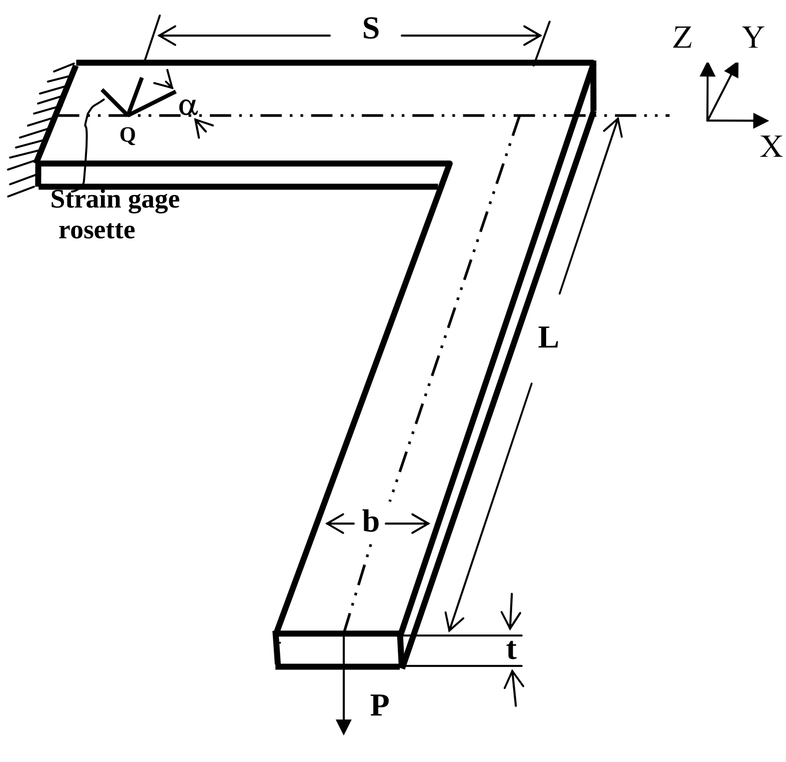

Figure 1. Schematic Diagram of the Setup

First the biaxial state of stress at point Q, on the longitudinal axis of the beam, is calculated using the classical equations of solid mechanics[1, 2]. The principal stresses together with the von Mises equivalent stress at point Q are then evaluated[1, 2] (Approach I). A rectangular strain gage rosette is mounted on the beam at point Q such that the axis of the closest gage to the longitudinal axis of the beam is at angle α with respect to the longitudinal axis of the beam. Measurement Group™ P-3500 strain indicator unit together with a switch and balance unit measures the strains along the axes of the rosette[3, 4]. Strain transformation equations are then used to obtain the state of strain and principal strains at Q[1, 2]. Equations of isotropic linear elasticity for biaxial stress situation[5, 6, 7, 8] are then used to evaluate the principal and von Mises stresses based on measured strains (Approach II). Finally a finite element model of the beam is made using Algor finite element package to arrive at the von Mises stress at the point of interest, Q (Approach III). The results of these 3 approaches are then compared to enforce the theory behind all these important concepts. Students get to utilize a lot of theoretical equations along with experimental and computational tools to thoroughly understand and verify fundamental concepts of linear elasticity. In the remainder of this article a detailed study of these approaches along with comparison of the results and potential usage of this simple L-shaped beam in other subject matters is presented.

Approach I

In this approach the internal load at point Q is first found by sectioning the beam through Q by a plane normal to the axis of the beam. The internal load at Q is:

$$V = P \qquad , \qquad T = PL \qquad , \qquad M = PS$$

Where V is the shear, T is the torsion and M is the bending moment at point Q. P, L, and S, are shown in Figure 1. The biaxial state of stress at Q is then found as:

$$\tau = \frac{T}{C_1 b t^2} \qquad \text{, and} \qquad \sigma = \frac{M \frac{t}{2}}{I}$$

Where C_1 is a geometrical constant, b is the width of the beam, t is the thickness of the beam and I is the moment of inertia of the beam. I and C_1 depend on b and t and are given as:

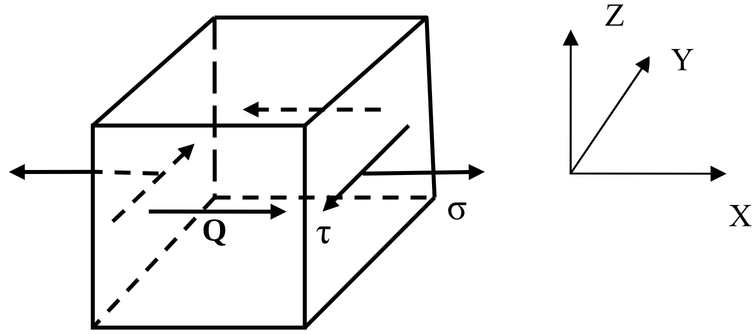

Figure 2. State of Stress at Q

$$I = \frac{1}{12} b t^3 \qquad\qquad C_1 = \frac{1}{3}\left(1 - 0.630\frac{t}{b}\right) \quad \text{for } b/t>5$$

The principal and von Mises stresses are then found as:

$$\sigma_p = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \qquad\qquad \sigma_q = \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \quad ,$$

$$\sigma_{vonMises} = \left(\sigma_p^{\ 2} + \sigma_q^{\ 2} - \sigma_p \sigma_q\right)^{1/2} \qquad (1)$$

Approach II

In this approach the rectangular strain gage rosette mounted at point Q on the beam measures normal strains along its 3 axes. The state of strain at Q, along the longitudinal (X-direction) and transversal (Y-direction) axes of the beam are then found using the following equations:

$$\varepsilon_1 = \varepsilon_x \cos^2\alpha + \varepsilon_y \sin^2\alpha + \gamma_{xy}\sin\alpha\cos\alpha$$

$$\varepsilon_2 = \varepsilon_x \cos^2(\alpha + \frac{\pi}{4}) + \varepsilon_y \sin^2(\alpha + \frac{\pi}{4}) + \gamma_{xy}\sin(\alpha + \frac{\pi}{4})\cos(\alpha + \frac{\pi}{4})$$

$$\varepsilon_3 = \varepsilon_x \cos^2(\alpha + \frac{\pi}{2}) + \varepsilon_y \sin^2(\alpha + \frac{\pi}{2}) + \gamma_{xy}\sin(\alpha + \frac{\pi}{2})\cos(\alpha + \frac{\pi}{2})$$

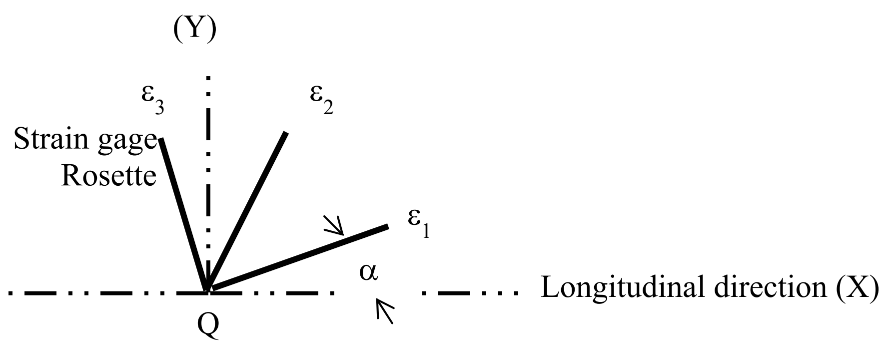

Figure 3. Rosette Orientation at Q

Where ε_x and ε_y are the normal strains along the longitudinal and transversal axes of the beam at point Q respectively, and γ_{xy} is the shear stress at that point. ε_1 , ε_2, and ε_3 are normal strains measured by the rosette at point Q and along the axes which make 10^o , 55^o , and 100^o angles with longitudinal axis of the beam respectively. Once the state of strain is found at Q, the principal strains, ε_p(maximum) and ε_q , are readily calculated as:

$$\varepsilon_p = \frac{\varepsilon_x + \varepsilon_y}{2} + \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_q = \frac{\varepsilon_x + \varepsilon_y}{2} - \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Finally the isotropic linear elasticity relations between stress and strain render the principal stresses for the measured strains as:

$$\sigma_p = \frac{E}{1-\upsilon^2}\left(\varepsilon_p + \upsilon\varepsilon_q\right)$$

$$\sigma_q = \frac{E}{1-\upsilon^2}\left(\varepsilon_q + \upsilon\varepsilon_p\right)$$

Where E is Young's modulus of elasticity, and υ is the Poisson ratio for aluminum alloy. Von Mises stress is then calculated using equation (1) above.

Approach III

A finite element model of the L-shaped beam is made using Algor plate elements. 48 plate elements were used in this model. The value of von Mises stress for the model at point Q is found for a trial mass of m = 1 kg, using Algor post processor Superview.
Appendix B presents a print out of the result of such a simulation.

Results

A 1 kg trial mass (P = 2.2054 Ibf) was used in the laboratory to do the measurements. For our trial beam (see Figure 1), L = 11.5 in., b = 1 in, t = 1/8 in, $\alpha = 10^{\circ}$, $E = 10.4\times10^6$psi, and $\upsilon = 0.33$. The strains measured were $\varepsilon_1 = 973\mu$, $\varepsilon_2 = 686\ \mu$, and $\varepsilon_3 = 449\ \mu$. Appendix A is a MathCAD file of the calculations for approaches I and II. As it is seen in appendix A von Mises stresses for approaches I and II are calculated as:

$$\sigma_{\text{von Mises, approach I}} = 12470 \text{ psi}$$

$$\sigma_{\text{von Mises, approach II}} = 11930 \text{ psi}$$

Appendix B indicates that the Algor file renders:

$$\sigma_{\text{von Mises, approach III}} = 11874 \text{ psi}$$

There is a 4.3% discrepancy between approaches I and II, and a 4.7% discrepancy between approaches I and III.

Implementation

Students are exposed to the theory (approach I) in the regular classroom setting, while approaches II and III are implemented in the laboratory component of the course. Students have to sign up for both lecture and lab in the same semester. This facilitates testing of some of the theoretical concepts that they learn in the lecture in the laboratory while the concept is fresh in their mind.

The lab portion of the course, itself, has two parts. In the first half of the semester, students are introduced to finite element through commercial packages such as Algor. Each student in the lab gets to analyze simple problems such as plane and 3 dimensional trusses, simply supported and cantilever beams using the software. Once they become somewhat proficient in the use of the software, each student then builds the appropriate model for implementing the approach III of this document. In the second half of the semester, students learn about strain gages and work with them in the lab. Students, in small groups, conduct different experiments including the one in the approach II of this document. Each student then compares the results of these 3 different approaches and writes an individual report for the experiment.

Conclusion

One could deduce that the approaches discussed above agree reasonably well in predicting the von Mises stress. Approach I assumes that the principle of superposition holds. Due to the relatively large torque arm and large load P this assumption is not a quite accurate assumption at large loads. It is expected that as the load P decreases the error between approaches 1 and other approaches decrease too. It is worth mentioning also that because of the lack of axisymmetry of the beam, the cross-section of the beam wraps out of its original plane. However these effects are less pronounced when load P is picked as a value less than 1 kg of mass. We also have to mention that the strain values measured in approach II were not modified for the gage transverse sensitivity effect.

Overall the experiment is a success. Students learn a lot from it and get to test their knowledge using theory, experimental methods, and computational techniques. Students who did go through this experiment showed a better comprehension of the stress and strain transformation, by consistently scoring higher in the hourly tests on the subject, than those who did not have such an opportunity in previous years.

The L-shaped beam can be used in vibrations classes to teach students modal analysis and vibration absorber concepts. These subjects will be addressed in a later work.

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Appendix A: Approach I & II Calculations (MathCad File)

Calculation for Approach I

Input Data

$L := 11.5$ in $\quad S := 10$ in $\quad b := 1$ in $\quad t := 0.125$ in $\quad \alpha := 10$ deg $\quad m := 1.$ Kg

$$P := \frac{m \cdot 9.81}{4.4482} \qquad P = 2.2054 \ \text{Ibf}$$

$$\text{(Modulus of Elasticity)} \quad E := 10.4 \cdot 10^{6} \ \text{psi}$$

(Poisson ratio) $\nu := 0.33$

Calculating the Torque and Bending Moment

$$T := P \cdot L \qquad M := P \cdot S$$

Calculating the state of 2-D stress at Q

$$\frac{b}{t} = 8 \qquad \frac{b}{t} > 5 \quad \text{then we have} \quad C_1 := \frac{1}{3} \cdot \left(1 - 0.630 \frac{t}{b}\right)$$

$$\tau := \frac{T}{C_1 \cdot b \cdot t^2} \qquad I := \frac{1}{12} \cdot b \cdot t^3 \qquad \sigma := \frac{M \cdot \left(\frac{t}{2}\right)}{I}$$

Calculating Principal stresses and von Mises Stress

$$\sigma_p := \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \qquad \sigma_q := \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

$$\sigma_p = 1.101 \times 10^{4} \ \text{psi} \qquad \sigma_q = -2.538 \times 10^{3} \ \text{psi}$$

$$\sigma_{von} := \left(\sigma_p{}^2 + \sigma_q{}^2 - \sigma_p \cdot \sigma_q\right)^{\frac{1}{2}}$$

$\sigma_{von} = 1.247 \times 10^{4}$ psi $\quad$ Value of von Mises Stress Using Approach I

Calculation for Approach II

Input Data

$\alpha := 0.175$ rad $\qquad \varepsilon_1 := 9.727 \cdot 10^{-4} \qquad \varepsilon_2 := 6.86 \cdot 10^{-4} \qquad \varepsilon_3 := -4.49 \cdot 10^{-4}$

Evaluating State of Strain at Q

Given

$$\varepsilon_1 = \varepsilon_x \cdot (\cos(\alpha))^2 + \gamma_{xy} \cdot \sin(\alpha) \cdot \cos(\alpha) + \varepsilon_y \cdot (\sin(\alpha))^2$$

$$\varepsilon_2 = \varepsilon_x \cdot \left(\cos\left(\alpha + \frac{\pi}{4}\right)\right)^2 + \gamma_{xy} \cdot \sin\left(\alpha + \frac{\pi}{4}\right) \cdot \cos\left(\alpha + \frac{\pi}{4}\right) + \varepsilon_y \cdot \left(\sin\left(\alpha + \frac{\pi}{4}\right)\right)^2$$

$$\varepsilon_3 = \varepsilon_x \cdot \left(\cos\left(\alpha + \frac{\pi}{2}\right)\right)^2 + \gamma_{xy} \cdot \sin\left(\alpha + \frac{\pi}{2}\right) \cdot \cos\left(\alpha + \frac{\pi}{2}\right) + \varepsilon_y \cdot \left(\sin\left(\alpha + \frac{\pi}{2}\right)\right)^2$$

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} := \mathrm{Find}(\varepsilon_x, \varepsilon_y, \gamma_{xy}) \rightarrow \begin{pmatrix} 7.84 \cdot 10^{-4} \\ -2.61 \cdot 10^{-4} \\ 1.28 \cdot 10^{-3} \end{pmatrix}$$

Evaluating Principal Strains at Q

$$\varepsilon_p := \frac{\varepsilon_x + \varepsilon_y}{2} + \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_q := \frac{\varepsilon_x + \varepsilon_y}{2} - \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$\varepsilon_p = 1.088 \times 10^{-3}$

$\varepsilon_q = -5.647 \times 10^{-4}$

Evaluating Principal and von Mises Stresses from Principal Strains

$$\sigma_p := \frac{E}{1 - \nu^2} \cdot (\varepsilon_p + \nu \cdot \varepsilon_q)$$

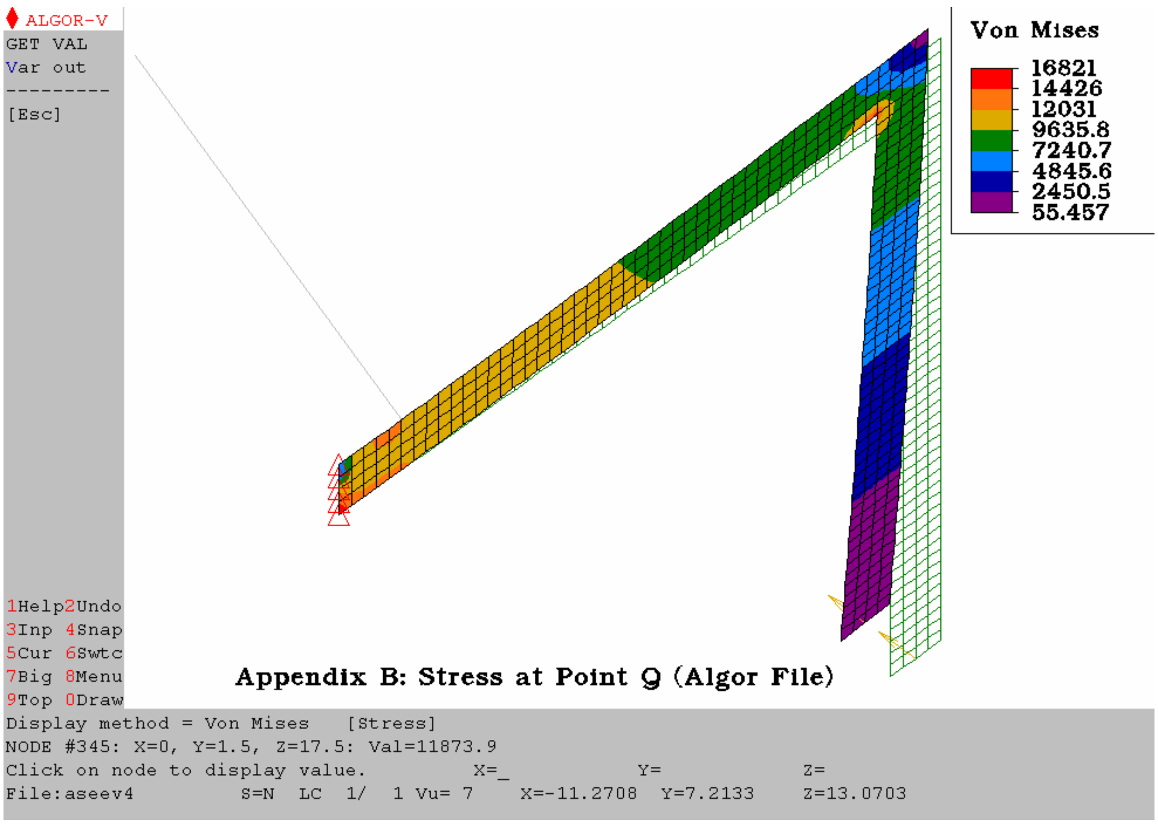
$$\sigma_q := \frac{E}{1 - \nu^2} \cdot (\varepsilon_q + \nu \cdot \varepsilon_p)$$

$\sigma_p = 1.052 \times 10^4$ psi

$\sigma_q = -2.538 \times 10^3$ psi

$$\sigma_{von} := \left(\sigma_p^{\ 2} + \sigma_q^{\ 2} - \sigma_p \cdot \sigma_q\right)^{\frac{1}{2}}$$

$\sigma_{von} = 1.19 \times 10^4$ psi $\quad$ Value of von Mises stress Using Approach II



Appendix B: Stress at Point Q (Algor File)