# AC 2007-172: USING ELLIPTIC INTEGRALS AND FUNCTIONS TO STUDY LARGE-AMPLITUDE OSCILLATIONS OF A PENDULUM

**Josue Njock-Libii, Indiana University-Purdue University-Fort Wayne**

Josué Njock Libii is Associate Professor of Mechanical Engineering at Indiana University-Purdue University Fort Wayne, Fort Wayne, Indiana, USA. He earned a B.S.E in Civil Engineering, an M.S.E. in Applied Mechanics, and a Ph.D. in Applied Mechanics (Fluid Mechanics) from the University of Michigan, Ann Arbor, Michigan. His areas of interests are in mechanics, particularly fluid mechanics, applied mathematics, and their applications in engineering, science, and education.

# Using elliptic integrals and functions to study large-amplitude oscillations of a pendulum


Abstract

The solution to the oscillations of a pendulum that includes large amplitudes is presented for the purpose of comparing it to that for small amplitudes. Such a comparison allows for the determination of the limits of applicability of the linearized equation. It is shown that, in both cases, the angle of swing is a periodic function of time but that the nature of the functions involved varies with the amplitude of motion. For small angular displacements, the period of oscillation is a constant and the ensuing angle of swing can be represented accurately by means of circular functions. However, for large amplitudes, the period is represented by Jacobi's complete elliptic integral of the first kind and varies with the initial amplitude, while the corresponding angle of swing is represented by elliptic functions of Jacobi. It is shown that the period of the linearized motion is always smaller than, or equal to, that from the nonlinear motion. The errors induced by the linearization process are determined analytically and represented graphically. It is demonstrated that those in the magnitude and phase of swing vary with time and the initial amplitude of the pendulum. Consequently, as a general rule, it is inaccurate to use the error in the angle as an estimate of the accuracy of how well the linearized solution approximates the actual motion.


## 1. Introduction

The motion of a pendulum is studied in the first college physics course; and its governing differential equation is amongst the first ones that are solved in an introductory course on ordinary differential equations. This equation is encountered again and again in courses such as dynamics, controls, vibrations, and acoustics. In all these cases, however, it is linearized by assuming that the amplitude of oscillation is small. As a consequence, students do not see what happens to the oscillation of a pendulum when the amplitudes are large and the restoring force becomes nonlinear. More importantly, they do not know the limits of applicability of the linearized solution they have studied.

In this article, we present the solution to the oscillations of a pendulum that includes large amplitudes and compare the general solution to that which is valid only for small amplitudes. This allows one to determine the errors induced by and the limits of applicability of the linearized equation.

## 2. The Basic equation

Consider a rigid body that is suspended from a point O about which it oscillates in the vertical plane. Let the angular displacement about the vertical axis be denoted by θ, measured in radians. After applying either Newton's second law of motion, or the conservation of mechanical energy, it is found that undamped oscillations about point O can be obtained by solving the equation[1]

$$\ddot{\theta} + \omega_n^2 \sin(\theta) = 0, \qquad (1)$$

In general, the conditions at the starting time, t = $t_s$, are given by [2]

$$t = t_s, \theta(t_s) \equiv \theta_s, \dot{\theta}(t_s) \equiv \dot{\theta}_s. \qquad (1a)$$

In these equations, the dots represent differentiation with respect to time t and the quantity $\omega_n$, which has units of rad/s, is related to the natural frequency of the system.

As an example, for a compound pendulum swinging in the vertical plane about a horizontal axis that goes through point O,

$$\omega_n \equiv \sqrt{\frac{m_{total} g d}{J_0}}, \qquad (1b)$$

where, $m_{total}$ is the total mass of the pendulum; g is the acceleration of gravity; d is the distance between point O and the center of mass of the pendulum; and $J_0$ is the (polar) mass moment of inertia of the body about point O. It can be seen that $\omega_n$ is a physical parameter that does not depend on time.[1]

## 3. The solution for small angles: circular functions

For small amplitudes, it is conventional to linearize Eq.(1) by expanding the sin θ into a power series as shown below

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots + \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} + \cdots \qquad (2)$$

and replacing the sin θ with θ, the first term in that series. Doing so gives

$$\ddot{\theta} + \omega_n^2 \theta = 0 \qquad (3)$$

This is the equation that is used in all the courses mentioned above. Its solution is

$$\theta(t) = A\sin(\omega_n t) + B\cos(\omega_n t) \qquad (4)$$

In this case, $\omega_n$ is the circular frequency of the motion expressed in radians per second.

After the initial conditions given in Eq (1a) are used in Eq (4), the constants A and B are found to be given, respectively, by

$$\begin{aligned} A &= \theta_s \sin(\omega_n t_s) + \tfrac{\dot{\theta}_s}{\omega_n}\cos(\omega_n t_s) \\ B &= \theta_s \cos(\omega_n t_s) - \tfrac{\dot{\theta}_s}{\omega_n}\sin(\omega_n t_s) \end{aligned} \qquad (5)$$

In order to obtain a solution with a simple mathematical form, it is conventional to let α be the maximum amplitude of oscillation and set $t_s \equiv 0, \theta_s \equiv 0, \dot{\theta}_s \equiv \omega_n \alpha$.[2] Incorporating these assumptions into Eq. (5) leads to $A = \alpha$ and $B = 0$; and Eq (4) becomes

$$\theta(t) = \alpha \sin(\omega_n t) \qquad (6)$$

Here the period of oscillation, $\tau_n$, is related to the circular frequency, $\omega_n$, by

$$\tau_n = \frac{2\pi}{\omega_n} \qquad (6a)$$

It can readily be observed from Eq.(6) that the instantaneous position of the pendulum during oscillation is a circular function of time and is directly proportional to the amplitude of motion, α. From Eq.(6a), it is seen that the period of oscillation of the pendulum is a constant that is independent of the amplitude of motion. It follows that all amplitudes that are within the limits of applicability of the governing equation yield the same period of oscillation. Consequently, the period and frequency of oscillation are not affected by the initial conditions. We will compare these results to those obtained when the pendulum assumes large amplitudes of oscillation.

4. The solution for any angle: elliptic functions and integrals

When swinging angles may be large, Eq.(1) is transformed into Jacobi's elliptic integral of the first kind by two successive integrations and a change of variables [2]. The exact solution to Eq.(1) is [2]

$$\theta(t) = 2\,Arc\sin\left[\sin\left(\frac{\alpha}{2}\right)sn(\omega t)\right], \qquad (7)$$

where sn represents Jacobi's elliptic function with the elliptic modulus suppressed [3-7].

The elliptic functions of Jacobi are defined as inverses of Jacobi's elliptic integral of the first kind. Thus, if one writes

$$u = \int_0^{\phi}\frac{d\phi}{\sqrt{1-k^2\sin^2(\phi)}},$$

then, for example, $sn(u,k)=\sin(\phi)$, $cn(u,k)=\cos(\phi)$ and $dn(u,k)=\sqrt{1-k^2\sin^2(\phi)}$.

For the derivation of Eq. (7), it is conventional to transform the original differential equation into an integral as

$$\omega t = \frac{1}{2}\int_0^{\theta}\frac{d\phi}{\sqrt{\left[\sin^2\left(\frac{\alpha}{2}\right)-\sin^2\left(\frac{\phi}{2}\right)\right]}}. \qquad (7a)$$

Then, by setting

$$u \equiv \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}, \qquad (7b)$$

and using this change of variables in Eq. (7a), one gets

$$\omega t = \int_0^{\rho}\frac{du}{\sqrt{\left(1-u^2\right)\left(1-k^2u^2\right)}} \qquad (7c)$$

where

$$\rho \equiv \frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} , \qquad (7d)$$

and

$$k^2 \equiv \sin^2\left(\frac{\alpha}{2}\right) . \qquad (7e)$$

From Eq.(7c), the period of oscillation is given by Eq.(8) as

$$\tau = \frac{4}{\omega} K(k^2) , \qquad (8)$$

where K denotes Jacobi's complete elliptic integral of the first kind [3-7], which is defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} . \qquad (8a)$$

Expanding K into a power series [4], one gets

$$\tau = \tau_0 \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \left(\frac{1.3.5.7}{2.4.6.8}\right)^2 k^8 + \ldots \right\} , \qquad (9)$$

where

$$\tau_0 = \frac{2\pi}{\omega} .$$

Rearranging Eq.(9), the ratio of the two periods is found to be

$$\frac{\tau}{\tau_0} = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 + \left(\frac{1.3.5.7}{2.4.6.8}\right)^2 k^8 + \ldots , \qquad (9a)$$

It can readily be observed from Eq.(7) that the instantaneous position of the pendulum during oscillation is a nonlinear function of the amplitude of motion, α, and an elliptic function of time. From Eq.(8), it is seen that the period of oscillation of the pendulum depends upon the amplitude of motion. Although the period varies with of the amplitude in a nonlinear way, one can see from Eq.(9) that it increases monotonically with the amplitude. Consequently, the period and frequency of oscillation are affected by the initial conditions. We will compare results obtained assuming small amplitudes of oscillation to those obtained assuming large amplitudes.

5. Comparing the solutions

5.a. Comparing the two periods

It can be seen from Eq.(9) that, $\tau$, the period obtained from the nonlinear equation, increases with the amplitude; and from Eq.(9a) that $\tau$ is always larger than, or equal to $\tau_0$, the period obtained from the linearized equation. This relationship is illustrated graphically by the plot of $\frac{\tau}{\tau_0}$ vs. k that is shown in Fig. 1.

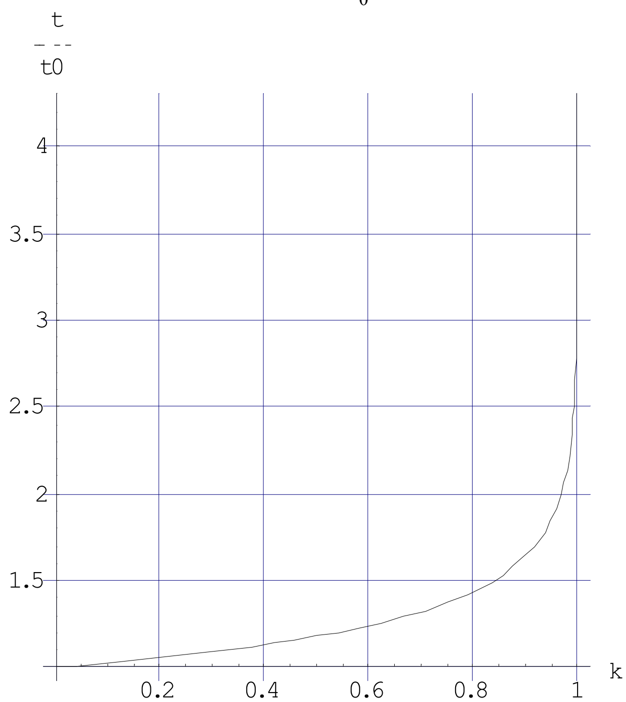


Fig.1. Plot of the ratio of periods: $\frac{\tau}{\tau_0}$ vs. k, Eq. (9a)

We define the error in the computation of the period as the difference between the exact and the approximate periods divided by the exact period, as shown in Eq (10).

$$Error_P = \frac{\tau - \tau_0}{\tau} \tag{10}$$

Similarly, we define the error in the computation of the initial swing angle as the difference between the initial angle of swing α and the sine of α divided by the sine of α, as shown in Eq.(11).

$$Error_\theta = \frac{\alpha - \sin\alpha}{\sin\alpha} \tag{11}$$

The ratio between the errors found in Eqs. (11) and (10) is shown in Eq.(12) and plotted in Fig. (2).

$$Ratio = \frac{Error_p}{Error_\theta} \tag{12}$$

It can be seen from that figure that the error in the angle is always larger than that in the period. Thus, since the error that is made in using the angle itself instead of its sine is much easier to compute, it can be obtained and used as an upper bound on the error to be expected in the period.

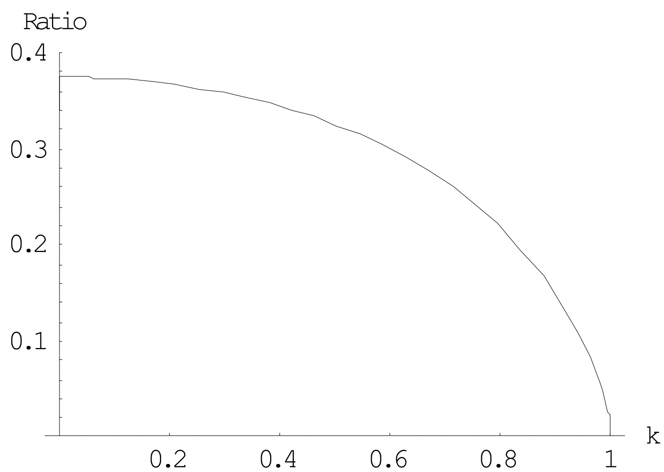


Fig. 2. A ratio of the error in the period to that in the angleα.

5.b. Comparing the swing angles

To illustrate the differences between the swing angles obtained from the nonlinear and linear equations, six starting angular amplitudes have been chosen; and, for each, a solution was obtained using the linearized equation and another with the nonlinear equation. The initial angles used are $\alpha \approx 10^o$, $30.32^o$, $63^o$, $88.420^o$, $121.3^o$, and $147^o$ and they have been identified in Fig.1b with dots. They correspond, respectively, to k = {0.08716, 0.26148, 0.52296, 0.69728, 0.8716, 0.95876}. Plots of the corresponding variations of the angular positions of the pendulum with time are shown in Fig. 3, where the solid lines represent the linear solution and the dashed lines the nonlinear (exact) solution.

From Fig. 3, it can be seen that, as the initial amplitude α, that is given to the pendulum to initiate its motion, increases (from $10^o$, to $30^o$, $63^o$, $88^o$, $121^o$, and $147^o$), so does, $\frac{\tau}{\tau_0}$, the ratio between the exact period τ and the approximate period $\tau_{0,}$ Eq.(9). The widening difference between the periods prevents the two curves from being in lock-step; this, in turn, increases the discrepancy between the corresponding angular positions of the pendulum.

For each initial angle used in Fig. 3, differences between the two solutions were computed using Eq.(13) and plotted in Fig. 4 in which the initial angle is a parameter.

$$\text{Differ} = 2\,Arc\sin\left[\sin\left(\frac{\alpha}{2}\right) sn(\omega t)\right] - \alpha\sin(\omega_n t) \qquad (13)$$

It can be seen from Fig. 4 that these differences vary considerably with both time and amplitude and can change algebraic signs during the motion.

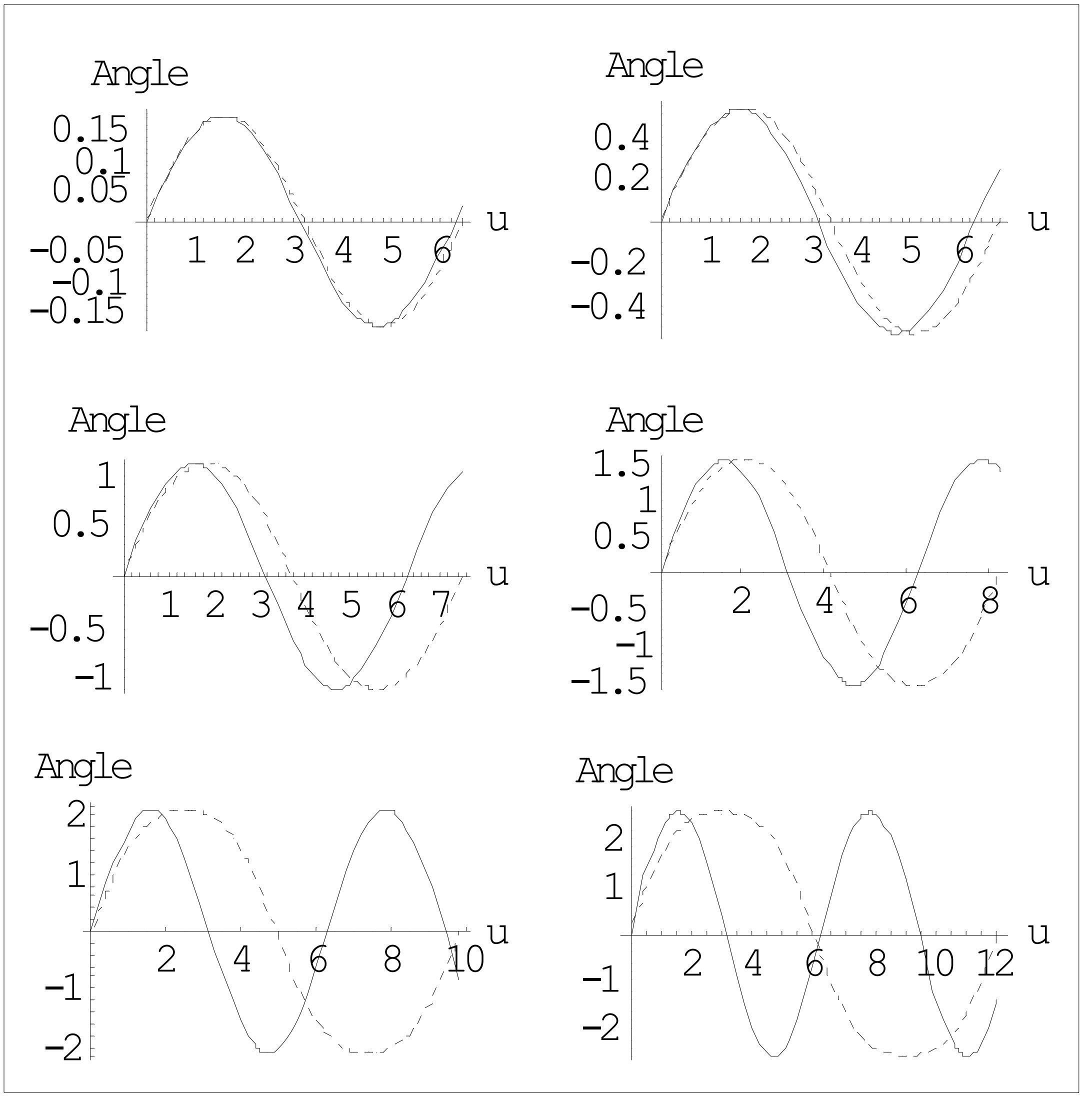


Fig.3. Swing angle vs. time, for α ≈ 10°, 30°, 60°, 90°, 120°, and 150°.
Exact values are for α ≈ 10°, 30.32°, 63°, 88.420°, 121.3°, and147°.
Solid lines (approximate solution); dashed lines (exact solution)

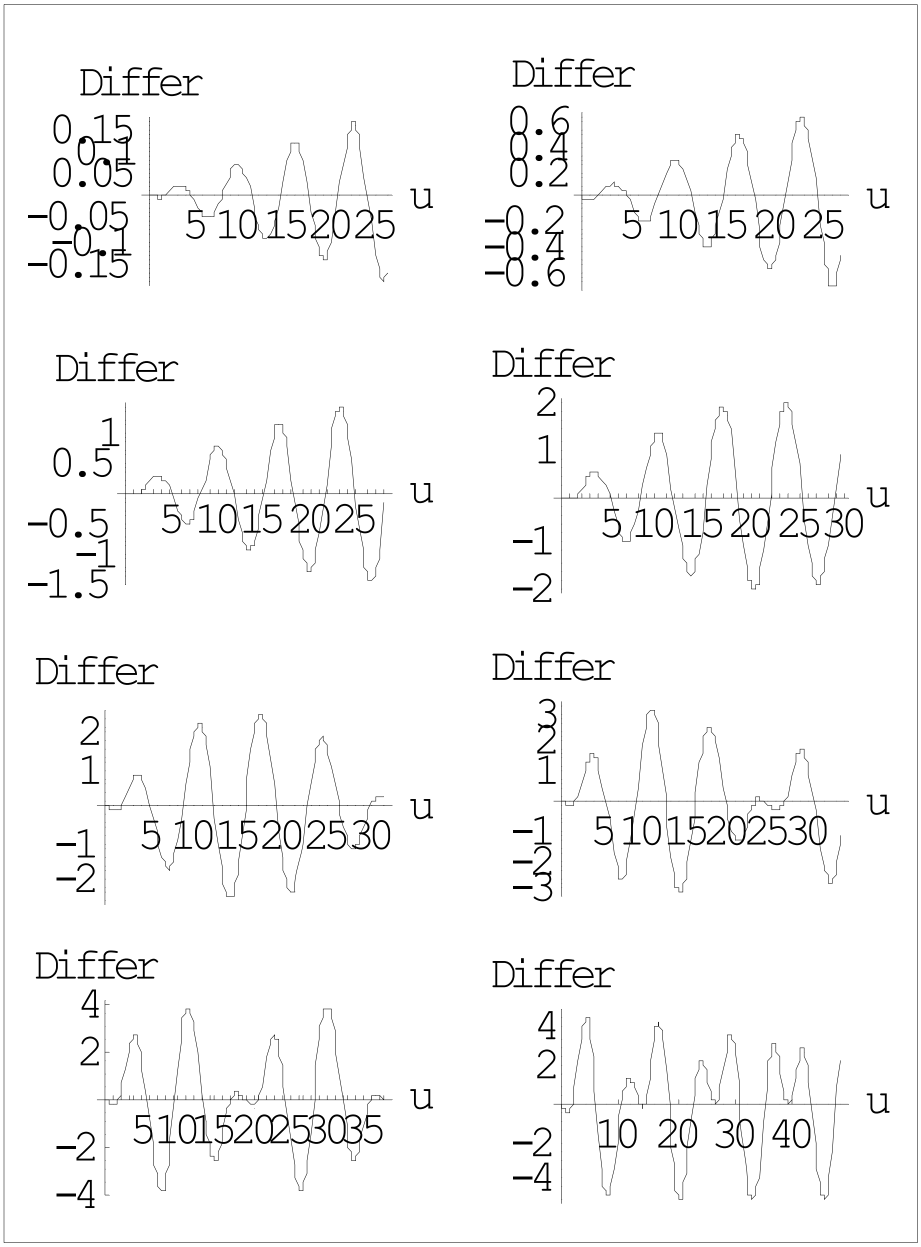


Fig. 4. Differences in Eq.(13) are plotted over four consecutive cycles, k is a parameter.
(k = 0.12, 0.24, 0.36, 0.48, 0.60, 0.72, 0.84, and 0.96, respectively).

6. Conclusions

We presented the solution to the oscillations of a pendulum that encompasses large amplitudes of swing and compared it to that which is valid only for small amplitudes. This allowed for the determination of the limits of applicability of the linearized equation.

It was shown that, in both cases, the angle of swing is a periodic function of time but the mathematical nature of the functions involved changes with the amplitude of motion. For small initial angular displacements, the period of oscillation is a constant that is independent of the initial displacement of the pendulum; and the ensuing angle of swing is represented accurately by circular functions. For large amplitudes, however, the period of oscillations is not a constant, for it varies with the initial amplitude given to the pendulum; it is represented mathematically by Jacobi's complete elliptic integral of the first kind; and the corresponding angle of swing is expressed by means of elliptic functions of Jacobi [2-8]. As can be seen from Eq.9, the period of the linearized motion is always smaller than, or equal to, that of the nonlinear motion.

The approximation $\sin\theta \approx \theta$ that is used to linearize the differential equation introduces three kinds of errors in the solution: one is in the magnitude of the period of oscillation; the second one is in the magnitude of the swing angle; and the third one is in the phase of motion. These errors were determined exactly and represented graphically.

When one uses the approximate equation, approximation errors that are introduced in the angle of swing affect the swing period of oscillation. The error induced in the period is fixed for a given motion of the pendulum (Eq. 10). So is the error in the angle itself (Eq. 11). These two errors were compared in Eq.(12). From the plot of their ratio that is shown in Fig. (2), it can be seen that the error in the period is always smaller than that made in approximating the $\sin(\theta)$ by the angle $\theta$. Therefore, the error in the angle can be used as shortcut to the determination of an upper bound on the error to be expected in the period. Numerical experimentation showed that the period of oscillation of the pendulum can be estimated reasonably well using the linearized equation up to angles of $30^{o}$.

Errors in the amplitude of swing and in the phase vary with time and the initial amplitude. If the time elapsed is large, even what might ordinarily be considered to be small angles of swing can lead to large errors in the predicted position of the pendulum. Indeed, numerical experimentation showed that the linearized equation represents the position of the pendulum reasonably well only up to about $10^{o}$ [8]. Consequently, care must be taken in determining the amplitude and the phase of swing of a pendulum from the linearized equation. As a general rule, therefore, it is inaccurate to use the magnitude of the difference between $\sin(\theta)$ and the angle $\theta$ as an indication of the accuracy of the solution obtained from the linearized equation.

7. References


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