120th ASEE Annual Conference & Exposition
FRANKLY, WE DO GIVE A D*MN
June 23-26, 2013

Paper ID #7430

Cross-disciplinary coherence: Mathematics and Physics for Engineering

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Cross-Disciplinary Coherence;
Mathematics and Physics for Engineering

Abstract

There is a pressing need for students to learn and understanding how to use mathematics in physics and engineering. In particular, engineering students must be able to express ideas that arise in physics and engineering in mathematical terms and then use their problem-solving skills to understand the consequences. Based on my experiences in teaching ordinary differential equations to engineering students, students see their mathematical education as simply a vast collection of specific procedures. The question raised here is whether better coordination of the content in first-year math and physics courses could improve student ability to use math in subsequent engineering courses. If this is so, then the mathematical content used in the physics course must be documented before changes in the content in the math course can be planned. At the same time, the physics course might benefit from a better illustration of important mathematics concepts, helping students to appreciate what they need to know mathematically. This paper presents just such a documentation of the mathematical content in a typical first-year physics course.

1. Introduction

The Department of Mathematics of a typical large mid-western university teaches a large number of students each year, of whom about 70% are engineering students. The Department of Physics teaches also teaches many student each year, of which about 75% are engineering students. Many of these engineering students are enrolled in the basic first-year courses in physics and mathematics, and to accommodate such large numbers, course enrollments are split into multiple large lectures and supplemented with smaller recitation sections.

To ensure uniform teaching, the curriculum in physics and mathematics (as well as other subjects) is determined by the choice of textbooks. All sections must follow a standard syllabus that is set by course coordinators. The syllabus is created to parallel the topics covered in the textbooks and often in the order they are presented. Not surprisingly, these textbooks have a generic content with even similar presentation styles, such as colorful graphical illustrations, boxed text to highlight definitions or specific statements. The standard design in textbooks is to divide material into parts or chapters covering broad topics, and each part or chapter is subdivided into sections that cover material appropriate for one lecture. These sections present the material sequentially, piecing together the quantitative skills and concepts needed for an overall description of the topic.

Results from education research now appear in modern textbooks with improved methods of visual presentation and new arrangements of material that introduce concepts in digestible pieces. For example, the popular Stewart text[1] for calculus, now in its sixth edition, has its origins in the calculus reform movement that swept collegiate mathematics in the 1990s[2,3]. The intent is to provide the instructor with a textbook flexible enough to support a wide range of syllabi and to suit a wide range of teaching styles. Since faculty members in different institutions choose to emphasize different combinations of topics, the textbooks are written so that chapters are as independent of prior topics as much as possible. As a consequence, the connections between the topics are downplayed. One such example is Boyce & DiPrima's *Elementary Differential Equations and Boundary Value Problems* [4]. In this text, each topic has its own set of independently developed objectives and its own set of canonical examples and exercises that, once covered, may have little explicit relevance to subsequent topics. Through such presentations of the material, the student is encouraged to learn the many topics as separate parts, through pattern recognition, imitation, and memorization. The organization of the material, and the authors' emphasis on disconnected, self-contained units of content leads to the students' compartmentalization of knowledge[5,6] and inhibits transfer[7]. Those who develop courses and syllabi, as well as those who create curricula, seem to believe that a wide knowledge base of physics and mathematics topics will enable students to adapt their knowledge to the demands of subsequent engineering courses.

Current experiences demonstrate that this belief is unfounded as engineering faculty members question the meaning of *mathematical competence* [8]. Educators note, both through personal experience and in the research literature[9,10] that engineering students are not adapting the skills they learn in their math and physics classes to their engineering studies under traditional pedagogies. Often, the students just *don't see* how problem-solving strategies and skills learned in one setting apply to exercises and support thinking in another setting. Moreover, the separateness of knowing about math, knowing about physics, and knowing about engineering concepts is both emphasized and encouraged by the curricula in place in each of these disciplines. Indeed, transfer to new contexts of higher-level cognitive abilities from one domain to another is not automatic and may require special attention[11].

If one accepts that the connection between math and physics understanding is an important part of engineering students' preparation, then it is then worthwhile to ask just how those connections are currently presented in the core physics and math courses during the first year of undergraduate engineering education. That is the purpose of the study reported here. As a start, the broad outlines of the first-quarter physics and math course as adopted at my University are listed with a subsequent more detailed examination of the curriculum that attempts to assess how math concepts are used in physics education. The hope is that it serves as a basis for interaction between physics and math educators in the future. As I have argued above, the use of math in physics, and subsequently in engineering, is what should drive the interaction among the disciplines. The analysis that follows the curricula overviews is motivated by this view.

2. Overview of curriculum

To be as specific as possible, my study is restricted to the textbooks used at my University. For physics, the textbook is "Fundamentals of Physics (8^{th} ed)" by Halliday/Resnick/Walker, while for math, it is "Calculus: Early Transcendentals, Volume 1. (6^{th} ed)" by Stewart. The outline of the curriculum for the first-quarter physics course that covers Chapters 1 – 11 is:

Chapters 1 – 2 (4 lectures), motion along a straight line;

- Motion, position and displacement, average velocity
- Instantaneous velocity
- Acceleration and free fall
- Graphical integration

Chapters 3 – 4 (2 lectures), vectors and two-dimensional motion;

- Vectors and scalars, addition of vectors
- Components of vectors, unit vectors
- Scalar and vector multiplication
- Instantaneous velocity and acceleration, projectile motion
- Uniform circular motion
- Relative motion

Chapter 5 (4 lectures) forces;

- Newton's first law, force, mass
- Newton's second law, free-body diagrams
- Gravitational force, normal force, friction, tension
- Newton's third law and applications

Chapter 6 (4 lectures) friction and circular motion;

- Properties of friction, static and kinetic, drag forces
- Uniform circular motion

Chapters 7 – 8 (4 lectures), energy;

- Kinetic energy
- Work and kinetic energy
- Work done by gravitational force, spring force and applied forces
- Work done by variable forces, power
- Work and potential energy
- Conservative forces, gravitational and elastic energies
- Conservation of mechanical energy
- Work done on a system by external forces
- Conservation of energy

Chapter 9 (3 lectures), systems of particles;

- Center of mass and linear momentum
- Collisions and impulse
- Momentum and kinetic energy in collisions
- Elastic and inelastic collisions

Chapters 10 – 11 (6 lectures) rotation;

- Angular position, displacement, velocity
- Constant angular acceleration
- Relating linear and angular variables
- Kinetic energy of rotation, rotational inertia
- Newton's second law for rotation, work and rotational kinetic energy
- Rolling as translation and rotation
- Forces and kinetic energy of rolling, torque and angular momentum
- Newton's law in angular form
- Angular momentum of a system of particles, rigid bodies

- Conservation of angular momentum

The outline for the first-quarter sequence in math is:
Chapter 1 (2 lectures), functions and inverses;
- Exponentials and logarithms
- Trigonometric functions and inverses

Chapter 2.1-6 (5 lectures), limits and continuity;
- Tangents
- Limit of a function
- Limits laws
- Continuity
- Limits at infinity, horizontal asymptotes

Chapter 2.6-8 (2 lectures), derivatives
- Tangents and rates of change
- Derivatives as functions

Chapter 3.1-6 (6 lectures) properties of derivatives;
- Polynomials and exponentials
- Product and quotient rules
- Trigonometric functions
- Chain rule
- Implicit differentiation
- Logarithm functions

Chapter 4.1-9 (9 lectures), miscellaneous;
- Maximum and minimum
- Mean value theorem
- Derivatives and the shape of curves
- Curve sketching
- Optimization
- Anti-derivatives
- Indeterminate forms
- Newton's method

There are 27 lectures in the Physics course and 24 lectures in the math course, the difference is the addition of several review lectures in the math course that have not been listed.

3. Mathematical content

The two curricula differ significantly in scope and purpose. In physics, the heart of the subject is the use of vectors and differentiation and integration of vector-valued functions of time to describe motion. In math, the topic is essentially the mathematical nature of scalar functions of x, restricted to derivatives. While there is a clear motivation for motion to be the starting topic in physics, the motivation to restrict attention to scalar functions and their derivatives in math is less clear. In the Preface of the physics textbook, the stated intent is to connect physics to the real world. At the beginning of each chapter, there is an introduction explaining the purpose of the chapter and how it connects to previous material. There is no motivation for the study of calculus in the preface in the math textbook; the introduction to each chapter describes mathematical motivation such as the need to study limits because that is how derivatives are defined. There are many other important applications to limits, for example, limiting behavior of expressions containing physical parameters.

A concept central to both physics and math is measurement. In physics, the starting point is the representation of an object as a particle whose position is recorded as a single number. In the physics textbook, an object moves like a particle if every portion of the object moves in the same direction and at the same rate. Thus any fixed location of the object can serve as a reference point. This idealization is very similar to that of geometry, where a point is considered to have location but no size. Indeed, this idealization is vital to the use of vectors and the measurement of distances, intervals of time, etc., and is the first key connection between math and physics. Curiously, the math textbook doesn't discuss the continuity of real numbers nor the fact that we deal with significant digits in much of science and engineering. By choosing integers or simple numbers most of the time, the examples in the math textbook can be expressed exactly without the need to indicate approximation. Also, aside from a few scattered examples, quantities very rarely have units.

The next important concept is fundamental to the notion of continuity. The location of a particle may be recorded at various time intervals, but we can imagine that it has a specific location at any moment in time. Since the particle cannot suddenly jump, its location is a continuous function of time. Here then, is a wonderful opportunity to emphasize that a small enough change in time must imply a small change in location. The physics textbook assumes the continuity of the location of a particle in time with very little discussion. Even when an example shows a discontinuous velocity or acceleration, the graphical illustration draws a continuous curve. There is little discussion of discontinuity and where it might be appropriate. On the other hand, the math textbook gives a brief discussion of continuity as continuous changes in time or drawing a graph without lifting the pen from the paper, but it misses the many cases where outputs depend continuous with inputs or changes in physical parameters.

In the physics textbook, the location of the particle is not usually expressed explicitly as a function of time $x(t)$. Students' difficulties with the nature of variable and parameter have been described[12,13] and their difficulties with interpreting and manipulating functions are well documented and theorized[14,15,16]. It is disappointing, then, that the physics textbook avoids the use of notation for functions throughout the material, instead favoring statements and expressions that appear algebraic. As an example, the location of a particle is expressed as $x = vt + 0.5at^2$ rather than $x(t) = vt + 0.5at^2$. The first form appears algebraic, while the second emphasizes the functional dependency of the location on time. Variables in equations are referred to as quantities without regard to whether they are dependent or independent or whether they are parameters. For example, the parameters in $x(t)$ are constants as suggested by the notation (they are not written as functions of time), while the first form lends itself to multiple interpretations.

Even the definition of instantaneous velocity in the physics textbook is expressed as the limit of the change in location divided by the change in time but without clearly identifying that the limit takes place at a specified moment. In other words, the velocity is not generally written as $v(t)$. If it were, jump discontinuities in the velocity that may occur for example when a ball bounces or when particles collide could be expressed as one-sided limits, another important math concept that is ignored in the physics textbook.

Derivatives in the physics textbook are discussed as rates of change, whereas derivatives in the calculus textbook are presented first as slopes of tangent lines. This is a key difference in how mathematicians and users of mathematics, like engineers and physicists, interpret mathematical concepts and these differences are implicitly passed on to students[17]. Students draw on different experiences when they imagine slopes or rates of change, and these experiences determine the students' concept images[18]. A concept image is the total cognitive structure that is associated with the concept and the concept's definition specifies that concept In the context of derivatives, the students are learning to associate one concept image, slope-of-the-tangent, with the concept definition in their math classes. On the other hand, the students are being expected to use the other concept image, rate-of-change, in their physics classes. Neither of the two curricula work to coordinate these two concept images and neither of the two curricula work to match the concept images with the formal concept definition. My experience with students in a course on differential equations confirms this interpretation since many still don't grasp that velocity is a rate of change. The reason may be that the calculus textbook continues with exploring the properties of derivatives as techniques to understand the behavior of functions and their graphs, while the physics textbook leaves kinetics behind and concentrates on forces, energy and momentum where velocity is a quantity in its own right without regard to it being a derivative. Overall, the calculus textbook emphasizes the mathematical properties of derivatives, such as the chain rule, by many examples without any physical context, while the physics textbooks, uses a few examples, which are nearly all simple polynomials without much reference to the properties of differentiation. The two extremes may serve only to inhibit the students' conceptual growth.

Similarly, integration is introduced in the physics textbook as the method to find the distance traveled if the velocity is known. Since integration appears early in the physics curriculum, well before its treatment in calculus, graphical techniques are used to calculate the integral as areas under the curve. Provided acceleration and velocity are simple enough, these areas are easily calculated by geometrical means. Geometric methods are simple and intuitive, but the downside is that the students are not explicitly shown the use of integration techniques and the power of integration as a solution strategy in their physics classes. Furthermore, the integrands are not even expressed as functions. The student rarely sees differentiation or integration as operators on functions, a very important conceptual development if the student is to progress satisfactorily in advanced courses.

Another important mathematical topic that is central to the study of physics is vectors. Vectors are foundational to the development of the ideas in the physics curriculum and provide also a symbolic language through which physical laws and principles are expressed. Mathematics educators have noticed that students are efficient at resolving horizontal and vertical components of vectors, but show difficulty in manipulating vectors that represent forces[19].

In the physics textbook, vectors are introduced as arrows, geometrical entities with magnitude and direction. They may be freely moved about as in the process of addition. But that means for the student some uncertainty about the nature of the origin of the arrow and its placement. Certainly, the zero vector is not clearly defined and notation such as $\vec{a} + \vec{c} = 0$ adds to the confusion about the origin of the vector space, for example, where is the origin of the velocity vector? Unfortunately, the standard calculus class does not cover vector spaces. Indeed, the opportunity to relate vectors directly with measurements and the need to establish clearly a reference point or origin is missed. For a vector in the plane, two measurements are required irrespective of the nature of the vector whether it is a location, a velocity, etc. The zero vector (two measurements of zero) is very important in a mathematical understanding of vectors, and its nature would have to be adequately addressed.

The view that vectors are lists of numbers (components in physics) may help students to appreciate the algebraic nature of vectors and so contribute to their facility in manipulating vectors through right-angle trigonometry or through vector arithmetic. Including scalar functions of time as the components (vector-valued functions of the independent variable time) might re-enforce the understanding that it is the independent time variable that connects the vectors of location, velocity and acceleration. Specific measurements are made at the same time so that we can say the particle is here with the given velocity and acceleration. Expressing them as arrows then serves to give a geometric interpretation of the measurements. Perhaps the greatest confusion arises when forces are expressed as vectors. Where is their origin and where do they apply? A clearer understanding of vectors used in physics may depend on a sound mathematical description of vectors that harnesses the different interpretations of vectors, such as describing vectors as actions that have the same effect[19].

4. Discussion

Physics and math educators are very focused on improvements in teaching their discipline. Their purpose is to ensure that students understand the concepts relevant to the field. They also appreciate that most students enrolled in the first year in physics and math classes will be continuing in engineering, and so acknowledge that the topics covered should be appropriate for engineering students. However, in math, the emphasis is on improving skills and simplifying conceptual development without explicit attention to its connections to physics and engineering. The selected physics curriculum reduces the complexity of the underlying mathematical content by keeping it at the level of algebra or trigonometry. In comparing the math content between the calculus and the physics textbooks, there is evidence that each holds differing epistemological and paradigmatic commitments, which are not incompatible but whose connections are not made explicit to the students[20,21,22].

In order to address these concerns and to move forward in improving the foundational education of our engineers, we must jump the barriers between disciplines and administration and work together to design a new curriculum that draws together key concepts and problem-solving skills within the context of understanding science and its connections to engineering. Some results[23] suggests that reorganizing the foundational engineering education sequences so that the instructors cooperate and coordinate content, rather than just presenting disparate views of the subject matter in parallel. Others have suggested that providing engineering students with experiences in modeling may help bridge the gaps among disciplines[24,25]. Any new core curriculum in engineering education must be sustained by an ongoing investment by mathematicians, physicists, engineers and other scientists. All must respond by bringing their ideas to the table and being flexible in how they might be integrated into a curriculum that benefits student learning in engineering (and perhaps science too).

How might such a curriculum be? One possibility that retains much of the current material but in a different order and with a different emphasis might be:

- Start by discussing the location of a particle and the *uncertainties* in its measurement (physics). Describe the mathematical idealization that the location is given *precisely* (mathematics). This illustrates a *fundamental* difference in the treatment of quantities in science and mathematics and it would help students to realize and appreciate the difference. At his stage, the motion should be restricted to one dimension.

- Introduce the idea that the position of a particle changes continuously, although its motion may not: there could be jumps in the velocity, for example in a collision (physics). Connect this intuitive notion of continuity with the more formal mathematical perspective (mathematics). Here is the opportunity to emphasize the nature of functions and the notation we use to describe them. These ideas can be introduced by considering particle motion that is given by polynomials since the operations are mostly algebraic.

- Now introduce the possible ways to measure the average velocity and its idealization as an instantaneous measurement. Now is a good time to re-enforce the difficulties in measuring instantaneous velocity and its uncertainties (physics) compared to the precision in mathematics that is contained in the derivative as a limit (mathematics). On the other hand, this is also a good time to acknowledge that there are many ways to define the derivative as a limit, and that these ways arise naturally in science and engineering, for example, the limit of forward differences (standard definition), the limit of backward differences (more natural in the computation of average speed), and the limit of centered differences (a natural way to measure slopes).

The outline above is just a start and is only intended to promote some of the considerations in a changed curriculum. It does raise the possibility of coordinating physical concepts with mathematical concepts by considering importance phenomena such as kinematics of particle motion as the driving force to develop the necessary concepts – physics places emphasis on the principles that guide particle motion and mathematics provides idealizations that allow precise description of the motion. It seems quite possible for such a curriculum to be developed, but it will require a coordinated effort by physicists, mathematicians and engineers.

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