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Interpreting and Communicating about Phenomena with Negative Rates of Change

Prof. Helen M Doerr, Syracuse University

Professor

Dr. Jonas Bergman Arleback, Syracuse University
Mrs. AnnMarie H O'Neil, C.S. Driver Middle School

Interpreting and Communicating about Phenomena with Negative Rates of Change

Much research over the last twenty years has documented the difficulties that students encounter when reasoning about and interpreting rates of change[1, 2, 3]. The complexity of such reasoning has proven difficult for high achieving undergraduate mathematics students[4] and students studying physics[5, 6]. To reason about rates of change, students must be able to simultaneously attend to both the changing values of the outputs of a function and changing values of the inputs to the function[7, 8]. In addition, students must be able to distinguish between the values of the outputs of a function and the values of the function's average rate of change over subintervals of the domain. When reasoning about changing phenomena, students often confuse these two quantities[1, 2, 9]. Furthermore, students have difficulty in distinguishing between the amount of change in a function's output value over a subinterval and the average rate of change of the function over that subinterval. To meaningfully interpret the graph of a function that represents two quantities that co-vary, students need to be able to simultaneously attend to and distinguish among three quantities: the value of the output of a function, the change in the values of the function's output over a subinterval, and the change in values of the input to the function. Reasoning about the latter two quantities is a foundational understanding for average rates of change in pre-calculus and instantaneous rates of change in calculus.

An equally important educational objective for engineering students is the ability to interpret and communicate their mathematical reasoning about rates of change in terms of the context of physical phenomena[10]. Researchers have found that students' understandings of rate in one context (such as kinematics or work) do not necessarily translate to another context[11, 12]. Engineering students will encounter many contexts where they need to create, interpret and communicate their reasoning about models of changing physical phenomena, including phenomena with changing rates of change and with negative rates of change. However, little research has attended to the particular challenges students may encounter when reasoning about models of physical phenomena with negative rates of change. Hence, in this qualitative research study, we investigated the question: how did students interpret and communicate their reasoning about negative rates of change across differing contexts of physical phenomena?

We begin this paper by describing the theoretical perspective that informed the design of the sequence of modeling activities. This sequence became the basis for a modeling-based mathematics course that was implemented in a summer bridge program for pre-engineering students. We then describe this research setting and the participants, the data sources and our methods of analysis. Then we report our findings on how students interpreted and communicated their reasoning about phenomena with negative rates of change in three contexts: motion along a straight path, light intensity at a distance from a point source, and the discharge of a capacitor in a simple circuit. We conclude with a discussion of the results and their implications.

Theoretical Perspective

Using a modeling perspective on teaching and learning mathematics[13-17], we designed and implemented a sequence of modeling activities[18, 19, 20] to support the development of students' abilities to create and interpret models of changing phenomena and the development of their understandings of average rates of change. Model eliciting activities are the beginning point of a sequence of model development tasks[14, 18]. A model eliciting activity (MEA) is designed to elicit the students' ideas about a meaningful and realistic problem situation, in such a way that students engage in an iterative process whereby they express, test and revise their ideas. Solutions to MEAs generally involve creating a generalizable model that reveals the underlying mathematical structure of the problem situation that can be further explored and subsequently applied in other contexts. Students' difficulties in reasoning about motion along a straight path are well known in the physics education research[24, 25]. Thus, we began this particular model development sequence with an MEA using motion detectors to initially elicit the notion of negative velocity with bodily motion along a straight path. We followed the MEA with a model exploration activity using a computer simulation[22] to explore the relationship between and interpretation of positive constant velocity and its associated position graph, followed by a focus on non-constant and negative velocities and the use of structured exercises[23] to explore the representations of negative rates. Model exploration activities focus on the mathematical structure of the elicited model, on the strengths of various representations, and on productive ways of using representations.

Following the model exploration activities related to motion along a straight path, the students engaged in two model application activities, applying their model of negative average rate of change to other changing phenomena. Often when a model is applied in a new context, this can result in further adaptations to the model, extending representations, deepening understandings or refining language for describing phenomena. In the first model application activity, the students created and interpreted a model of the intensity of light with respect to the distance from the light source; in the second model application activity, students created and interpreted a model of the voltage drop across a discharging capacitor in a simple resistor-capacitor circuit. Both of these model application activities involved non-linear decreasing functions over the domain of the positive real numbers. These functions have negative average rates of change over any subinterval of the positive domain. Over a sequence of subintervals, however, these negative average rates of change are increasing. Hence, in discriminating between the values of the function and the values of the average rate of change of the function over particular subintervals, and how each of these quantities is changing, students needed to attend to both the magnitude of the quantity and its sign. In addition, meaningful interpretations of these values needed to be linked to the phenomena being modeled.

The Model Exploration Activity Design: Motion along a Straight Path

The model exploration activity used a computer simulation environment, *SimCalc Mathworlds*[22]. Using the simulation, the students created velocity vs. time graphs to generate the motion of simulated characters. From this motion, the students created position graphs, thus developing an understanding of how the position graph could be constructed by calculating the area between the velocity graph and the x-axis. In exploring the linked relationship between the velocity vs. time and position vs. time graphs, students began to reason about the position of characters solely from information about their velocities. The exploration tasks included constructing velocity graphs from piecewise linear position graphs, interpreting descriptions of constant and non-constant motions in the positive and negative direction and creating appropriate position and velocity graphs, and investigating negative and linearly decreasing velocities and their associated position graphs. This model exploration activity was designed to develop the students' abilities to interpret velocity information from a position graph, to interpret position information from a velocity graph, and to interpret linearly increasing and decreasing velocities. Throughout the activity, the students were engaged with interpreting graphical representations of positive and negative average rates of change and changing velocity.

The Model Application Activity Design: Light Intensity

Prior to the light intensity model application activity, the students' work on the model eliciting and exploration activities almost exclusively involved different representations of one-dimensional motion along a straight path. The light intensity task was designed to provide the students with an opportunity to apply their understandings of average rate of change, initially elicited in the context of linear motion, in a new context and one in which the independent variable was distance rather than time. The first part of the activity aimed at making explicit the students' intuitive and initial models about the relationship between the light intensity and the distance from a point source. The students were asked to consider the scenario of an approaching car and to sketch a graph of how the intensity of the car's headlights varied depending on the distance you are from the car.

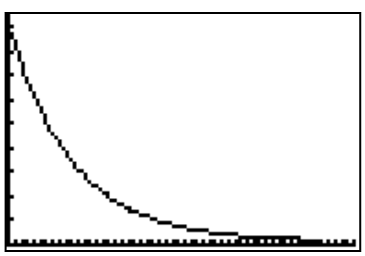
In the second part of the activity, students collected light intensity data using a flashlight with the focusing cap removed, a meter stick, and a light sensor connected to their graphing calculator. They collected 15 measurements of the intensity at one cm intervals from the light source and transferred the data to their computers. The light sensor measured light intensity in a relative and arbitrary unit called "light intensity units" (LIU). The students were asked to make a scatter plot of their data and to write descriptions of how the intensity of the light changed with respect to distance from the light source. They were encouraged to compare this relationship to their predictions from the first part of the activity. The students were also asked to calculate the average rates of change of the light intensity data in one cm intervals; to describe the average rates of change of the data over subsequent subintervals as the distance from the light source increases; and to create a separate rate graph of the calculated average rates of change over the one cm intervals.

In the third part of the activity, the students were asked to determine a function fitting their collected data and to analyze the average rates of change of their function. Due to the physical limitations of the light sensor, students needed to construct a piecewise function, consisting of a horizontal linear piece for the first two or three data points and an inverse squared piece for the remainder of the data set. The students worked with a partner to write a report that described the values of the light intensity in terms of a piecewise linear and inverse squared function; computed and interpreted average rates of change over one cm intervals of distance; graphically represented these average rates of change; and described how the values of the average rates of change of the function, over one cm intervals, changed as the distance from the light source increased.

The Model Application Activity Design: Discharging Capacitor

The second model application activity was an investigation of the rate at which a fully charged capacitor in a simple resistor-capacitor circuit discharged with respect to time, a phenomena which is governed by an exponential decay function. The students built the RC circuits, charged a capacitor, and measured the voltage drop across the capacitor as it discharged. Students were given a set of resistors and capacitors and were asked to develop a model they could use to answer these three questions: (1) How does increasing the resistance affect the rate at which a capacitor discharges? (2) Compare the rates at which the capacitor is discharging at the beginning, middle and end of the total time interval. How does the average rate of change of the function change as time increases? (3) How does increasing the capacitance affect the rate at which a capacitor discharges? Similar to the multiple tasks within the light intensity application activity, the students engaged in several iterations of interpreting and communicating their reasoning about three quantities: (1) the values of the exponential decay function that represented the voltage drop across the discharging capacitor; (2) the values of the average rates of change of the voltage drop computed over five second intervals; and (3) how the function values and the sequence of average rates of change values were changing as the capacitor discharged. As in the design of the model application activity on light intensity, at the completion of the activity, the students worked in pairs to complete a written report communicating their analysis of the discharging capacitors.

Following this model application activity, the students were given the following question on a written test: Given a data set of the voltage drop across a discharging capacitor for 50 seconds (see Figure 1A), (a) find an equation of the form $y = a \cdot b^x$ that could be used to describe the data; (b) give an interpretation of the constants a and b in (a); (c) find the point in time when the voltage across the capacitor was 0.05 V; (d) compute the average rate of change over three subintervals, from t = 5 to t = 10 seconds, t = 20 to t = 25 seconds, and t = 40 to t = 45 seconds respectively; and (e) write two or three sentences interpreting the negative average rate of change data in (d).



A

$$t = 5 \text{ to } t = 10 : \frac{2.0529 - 4.2245}{10 - 5} = -0.43 \text{ v/s}$$

$$t = 20 \text{ to } t = 25 : \frac{.27252 - .51666}{25 - 20} = -0.05 \text{ v/s}$$

$$t = 40 \text{ to } t = 45 : \frac{.04363 - .06406}{45 - 40} = -0.004 \text{ v/s}$$

B

Figure 1: A plot of the voltage data and calculations of three average rates of change.

In this paper, we restricted ourselves to an analysis of the students' responses to the last part of this question. Figure 1B shows the calculations of the average rate of change over the three intervals asked for in part (d) of the test item. Since this is an exponential decay function, the average rate of change over any subinterval of the domain is negative. However, the sequence of average rates of change over the 50 second domain is getting less negative and closer to zero and, hence, the sequence of average rates of change is increasing. In terms of the phenomena, a negative average rate of change means that the voltage across the capacitor is decreasing for that subinterval; but, as we will show in the results below, expressing the meaning of the sequence of increasing average rates of change in terms of the voltage was considerably more difficult for students.

Setting, Participants, Data Sources and Analysis

The sequence of model development activities formed the basis for a six-week modeling-based mathematics course, offered as part of a summer bridge program, for students who were preparing to study engineering[21]. Over three years, a total of 86 students (26 female and 60 male) participated in the study; 50 students had studied calculus in high school and 36 had not studied any calculus. All but one student had taken a high school course in physics. The students worked in small groups to complete the sequence of model development activities. The students' written work was collected and analyzed. Class discussion following the tasks focused on the relationships among different representations of negative rates of change and students' interpretations of change in different contexts. In the second and third year of implementation, all of the lessons were video taped. Each year, the students completed a pre- and post-test designed to measure their understanding of the concept of average rate of change. The results of two of these items, which focus on negative rates of change, are reported below. The students' written work was analyzed qualitatively, following the principles of grounded theory[26]. Codes were developed to categorize the students' reasoning about negative average rates of change and their interpretations of these rates of change in terms of physical phenomena.

For the model exploration activity, we report on the analysis of the student data collected in the third year of the study, from n=35 students. We analyzed three data sources: (1) the video taped discussion from two classes where graphical representations of negative velocities were the focus of discussion; (2) the students' responses to two items from a pre- and post-test that was completed at the beginning and end of the course; and (3) the students' written work on test questions given at the end of the model exploration activity.

For the light intensity model application activity, we report on the analysis of the student data collected in the third year of the study from n=35 students. The analysis was done in two phases. First, codes were developed to categorize the students' reasoning and answers on each of the questions in the first activity involving the intensity of a car's headlight with respect to the distance to the car. This initial analysis focused on capturing the students' initial models of how the light intensity varies with distance from the light source and how light disperses from a point source. In the second phase of the analysis, the students' written reports (n=18) were read and coded, focusing on interpretations and descriptions of how the intensity varied with the distance from the light source and its average rates of change, the graphs the students produced, and how they attended to the context throughout their writing.

For the discharging circuit model application activity, we report on the analysis of the student data collected in the first two years of the study from n=49 students, using three phases of coding to examine the extent to which the students made references to and distinctions among the values of the function, the values of the negative average rates of change over sub-intervals of the domain of the function, and, if and how these quantities were related to the physical phenomena.

Results

The results show that interpreting and communicating about negative rates of change was particularly difficult for the students. We found that many students were able to clearly distinguish between the function values and the values of the rates of change when the rates were positive, but then confounded these quantities when the rates were negative. Nearly all students were able to interpret and communicate about negative rates of change within the context of motion along a straight path. However, additional obstacles for students arose when describing decreasing non-linear functions and sequences of negative average rates of change in other contexts, such as with the light intensity and the discharging capacitor. We report our findings on students' abilities to interpret and communicate descriptions of changing phenomena in the three contexts that were addressed in the course: motion along a straight line path, light intensity with respect to distance from a light source, and voltage across a discharging capacitor.

Motion along a Straight Path

As we had anticipated, the students initially had difficulty in reasoning about velocity as the rate of change of position. While working in the computer simulation environment, the students gained proficiency in reasoning about and interpreting the linked graphical representations of velocity and position when both quantities were positive. However, student difficulties in carefully distinguishing between velocity and position surfaced when the rate (velocity) was negative and when the rate of change (velocity) was linear. The results of our study showed that understanding negative rates when the rate was changing was particularly difficult for the students. Partly this was due to the students conflating position and velocity graphs and their respective properties in their arguments. This was illustrated by the students' responses to the question about the velocity for the position graph shown in Figure 2.

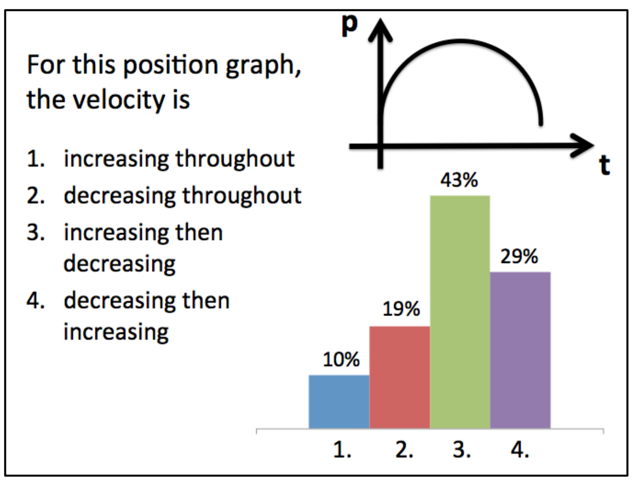


Figure 2. A question posed to the students and their responses.

As shown in Figure 2, 43% of the students described the velocity of the position graph as "increasing then decreasing." This is a correct description of the position graph, but an incorrect description of the velocity, which is decreasing throughout. The 29% of the students who responded "decreasing then increasing" appeared to be reasoning about the magnitude of the velocity (or speed), as evidenced in the class discussion that followed this question. These students argued that the velocity was decreasing initially; this is also true about the speed since the velocity is positive. However, when the velocity became negative, these students continued to reason about the speed which is increasing, rather than the velocity which is decreasing. The ensuing class discussion led to a clarification of the distinction between the velocity graph and the position graph and between velocity as a signed quantity and speed as the magnitude of velocity.

Students' difficulties in maintaining these distinctions continued to surface throughout the model exploration activity. When given the position graph shown in Figure 3A, nearly all of the students correctly constructed the corresponding velocity graph shown in Figure 3B. (We note that the simulation world allowed discontinuous velocity graphs; prior class discussion had addressed the difference between what is possible in the real world versus what is possible in a simulation environment.) However, when asked to interpret the velocity graph with its negative rates, many

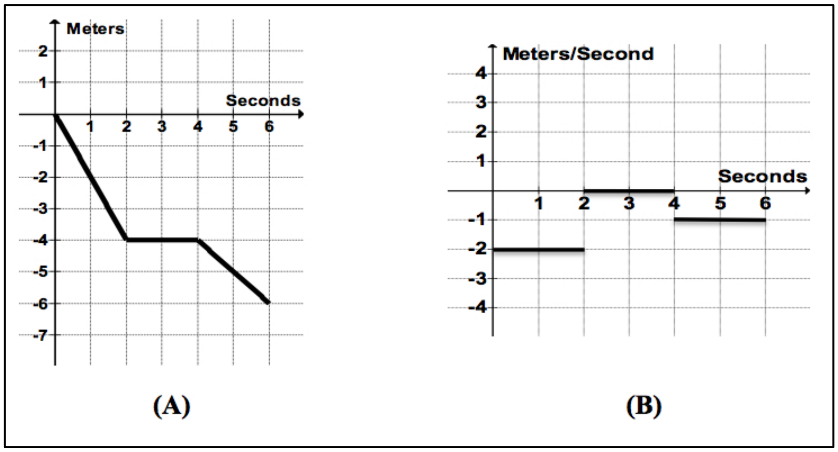


Figure 3. Creating and interpreting a velocity graph from a position graph.

students were not able to identify in which of the three intervals the velocity was the greatest. While 45% of the students correctly identified the greatest velocity occurring in the interval between t=2 and t=4 seconds, 45% of the students identified the greatest velocity as occurring between t=0 and t=2 seconds. These students were attending to the magnitude of the velocity since this is the interval where the magnitude of the velocity (speed) is the greatest. One of the students argued that the first interval provided the greatest velocity because the sign of the number only provided direction of the movement not how fast the object was moving. This student was reasoning about the velocity as if it were two separate quantities: a sign that indicated direction and a numerical value that indicated how fast the object was moving. It would appear that the student did not see the value of the velocity as a single negative quantity that could be compared to other negative quantities. The 10% of the students who identified the interval between t=4 and t=6 seconds as having the greatest velocity were correctly comparing the relative velocities of the first and third intervals, but not regarding a value of zero as greater than either of the negative quantities. Again, the ensuing class discussion resolved these distinctions for the students.

At the conclusion of the model exploration activity, as evidenced by responses to a set of examination questions, most students were able to successfully interpret velocity information from a position graph and position information from a velocity graph, for positive and negative velocities and for constant and non-constant velocities. This success was also evidenced in the substantial improvement on two of the pre- and post-items, given at the beginning and the end of the course. These two items address the interpretation of a linearly decreasing velocity graph that takes on negative values and the interpretation of a position graph where the velocity changes over sub-intervals of the domain and takes on negative values.

The first item asked students to identify which of a set of position versus time graphs would best represent an object's motion during the same time interval for the motion shown in the velocity graph in Figure 4. To successfully answer this item requires an understanding of how to reason about position when given a velocity graph. On the pre-test, we found that *n*=10 (29%) of the students were able to correctly identify an appropriate position graph. On the post-test, we found that *n*=25 (71%) of the students correctly identified an appropriate position graph.

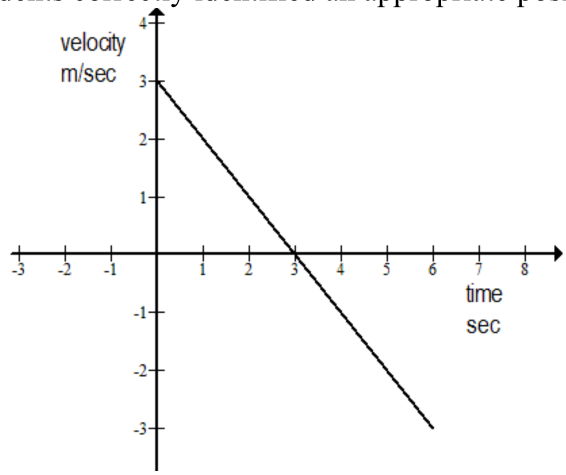


Figure 4: Reasoning about position from a velocity graph.

On an analogous item reported by Beichner (1994) only 29% of students were able to correctly use areas to reason about velocity from an acceleration graph, after instruction in kinematics in a college physics course. The substantial 42% gain on this question suggests that the model exploration activity helped the students understand how to reason graphically about non-constant and negative velocities (or rates of change) such as shown in Figure 4.

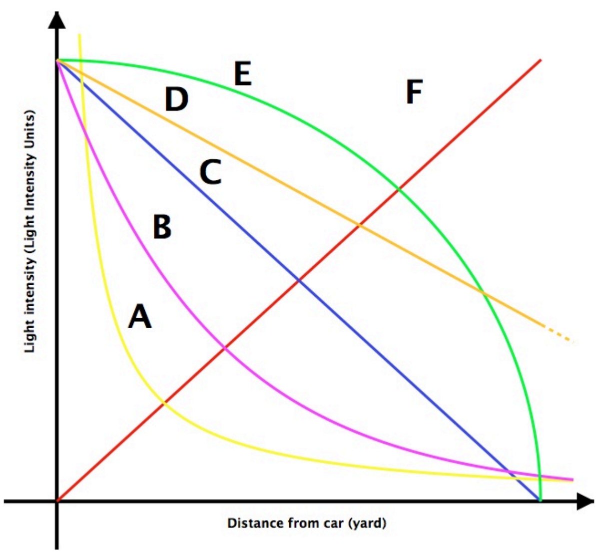
The second item asked the students to choose a written description of the motion of an object whose position is shown in Figure 5.

Figure 5: Describing an object's motion when given a position graph.

On the pre-test, *n*=12 (34%) of the students selected a correct description; the post-test results showed that *n*=18 (80%) of the students were able to select a correct description. Results reported by Beichner (1994) on this item showed that 37% of the students were able to describe the motion correctly after kinematics instruction. We attribute the substantial gain in achievement on this item to the emphasis on descriptions of motion in the model exploration activities.

Light Intensity

We found that the students' difficulties in distinguishing between the values of the function and the values of the average rates of change of the function subintervals of the domain, that were overcome in the context of motion, with its associated position and velocity graphs, resurfaced as they engaged in the new context of the light intensity model application activity. Although all of the students had had a course in high school physics, nearly all of the students (n=28, 83%) drew a linear relationship between the intensity of light and the distance from the source. The students' initial models of the relationship between light intensity and the distance from the light source are summarized in Figure 6. All but one student correctly described how the intensity at 1000 yards compared to the intensity at 2000 yards based on the graph they drew.

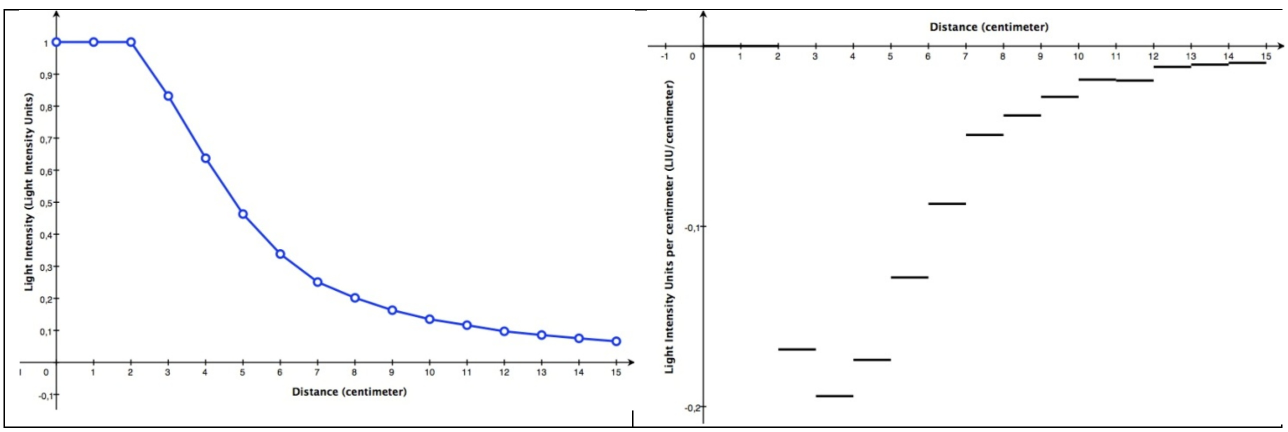


Graph	Number of students
A	4 (12%)
B	1 (3%)
C	22 (65%)
D	5 (15%)
E	1 (3%)
F	1 (3%)

Figure 6: Students' initial models of intensity vs. distance from light source.

However, when asked to "Compare the rate at which the intensity is changing at 1000 yards and 2000 yards", only half of the students (n=17 of the 34 students) correctly described or calculated the rate of change based on their graphical representation. For example, one student correctly concluded from his incorrect C graph that: "The rate at which the intensity is changing at 1000 yards and 2000 yards is the same." But nearly half of the students (n=16 out of the 34 students) did not compare the rate at which the intensity was changing, but rather compared the values of the function at the two distances. The remaining one of the 34 students simply gave an equation of a linear function that did not correspond to his graph. In other words, our analysis of the students' answers on this task showed that half of the students conflated the values of the function with the rate at which the function is changing. This means that for about half of the students the earlier distinction, in the context of motion along a straight path, between the value of the output of the function value (e.g., position) and the value of its rate of change (e.g., velocity) was not readily applied in this new context of light intensity.

In our analysis of the students' written reports completed at the end of the model application activity on light intensity, we found that nearly all of the students' reports (89%) included correct descriptions of correct graphs of the values of the light intensity (such as that shown in Figure 7A). Nearly all reports (89%) also included a correct graph of the average rate of change over the 1 cm intervals (similar to that in Figure 7B). Almost half of the students (44%) made it explicitly clear how to see the average rates of change (shown in Figure 7B) in the scatter plot (shown in Figure 7A) by referring to the slope of the line segment connecting two consecutive points. This is illustrated by the description written by one pair of students: "As the distance from the source increases the intensity of the light decreases. For example, at 3 cm the light intensity reads .832 LIU [Light Intensity Units]; at 4 cm the light intensity reads .637 LIU; and at 5 cm the light intensity reads .463 LIU. It is also noted that the rate at which the data changes as the distance from the light source increases changes." This description was accompanied by a table listing the light intensity values, the calculated average rate of change values and a scatter plot of the data with the average rates of change drawn in connecting consecutive points. In their description, the students clearly distinguished among the values of the function (the light intensity at given distances), how those values are changing ("the intensity of the light decreases"), and that the rate at which the function values change also changes.



A B

Figure 7: A typical scatter plot of the light intensity data (A) and its corresponding average rate of change graph (B).

Over half (56%) of the students correctly interpreted a correct graph of the average rates of change of the light intensity over one cm subintervals. We found that some students were able to correctly describe the values of the sequence of average rates of change and to refer to the magnitude of the values when describing the phenomena. For example, one pair of students wrote: "Although the majority of the average rates of change are increasing, the absolute value of average rates of change are decreasing which tells us that as the distance from the light source increases the intensity of the light decreases quickly at first and then decreases more slowly." In this statement, the students reasoned from the graph of the average rates of change values (shown in Figure 7B), and correctly articulated that the "majority" of these values are increasing, seemingly focusing on the average rate of change values when the distance from the light source is greater than 3 cm. The students considered the absolute values of the negative average rates of change to provide the qualitative statement about the function values ("intensity of the light") that is "decreasing quickly at first and then decreases more slowly." This pair of students is able to maintain and communicate the distinction between the function values (the light intensity) and the average rates of change values. In addition, by shifting to the language of the magnitude of the average rates of change, this pair of students correctly interprets and communicates how the rate itself changes: "quickly" and then "slowly."

Many students encountered difficulty in maintaining these distinctions and correctly expressing them in terms of the phenomena. In some cases, this led to contradictions within the students' description. For example, one pair of students wrote: "The intensity of the light decreases at a decreasing rate with respect to the distance from the light source, as predicted earlier. The average rates of change increase at a decreasing rate as distance from the light source increases." Within this description, the students first refer to the rate as a "decreasing rate" and then in the second sentence claim that the "average rates of change increase." Many students who displayed this difficulty used phrases such as "decreasing at a decreasing rate" rather than writing two separate statements: one about the function and one about its rate of change. Examples such as this point to the complexity of expressing the ideas involved with negative rates and their different representations, especially when used in an applied context.

Discharging Capacitor

As in the light intensity application activity, a significant number of students exhibited conflated ideas with respect to expressing their understanding of the behavior of the average rate of change and the behavior of the function in the context of the discharging capacitor. The most common occurrence was for students to make an incorrect assertion about the behavior of the average rate of change that would have been a correct assertion about the behavior of the function. This is illustrated by the following typical example, where the student refers to the function and calculated average rates of change shown in Figure 1: "From the data that I have just found I can recalculate that the average rate of change is decreasing. It is decreasing at an increasing rate." This claim about the average rate of change is incorrect, but if it were instead stated about the behavior of the function, then it would be a correct statement.

Some students described the magnitude at which the rate of change changed rather than describing this change as a signed quantity. This may have blurred the interpretation of the average rate of change since the magnitude (the absolute value) of the average rate of change is decreasing whereas the signed average rate of change is increasing. The following example shows how easy it is to shift between conflicting formulations: "In (d) the average rate of change is negative and thus we know that voltage is decreasing. However, the rate at which voltage is decreasing is less and less each second. This statement can be concluded because comparing the first, middle, and ending intervals shows that the average rate of change is becoming less negative. Thus we see that the average rate of change is decreasing at an increasing rate." In the first and third sentence of this student's answer, the average rate of change is seen and used as a signed quantity, enabling the student to correctly describe both the behavior of the function ("voltage is decreasing") and the average rate of change ("is becoming less negative"). In the second sentence, the average rate of change is seen and used in a magnitude sense, leading the student to incorrectly conclude that the rate of change of the voltage is decreasing ("less and less"). In the fourth sentence, the student's statement ("the average rate of change is decreasing") is true about the magnitude of the average rate of change, but not about the signed quantity.

The difficulty in seeing the average rate of change as a signed quantity rather than as a magnitude seems to be one of the factors underlying the students' conflated ideas about the behaviors of the function and the average rate of change. This can be seen in the following example: "The average rate of change is decreasing at an increasing rate. This is because the numbers are getting closer to zero." It is not the average rate of change that is decreasing, but rather the function values are decreasing. The values that the student correctly calculated for the average rate of change were negative and increasing since those numbers were "getting closer to zero." This student is not using the signed values of the average rates of change to infer the behavior of the function but rather the student incorrectly concludes that it is the average rate of change itself that is decreasing at an increasing rate. This error points to the difficulty of simultaneously attending to changes in the function values and changes in the average rate of change values.

Discussion and Implications

An underlying assumption of this study has been the importance for engineering students to create, interpret and communicate their thinking about models of changing phenomena. The sequence of model development activities that was implemented in this study was designed to support the development of engineering students' abilities to interpret and communicate their reasoning about changing phenomena, particularly when the rates of change are negative. We began the sequence of tasks with the everyday and familiar context of motion along a straight line path. While working through a set of model exploration activities in a computer simulation environment, the students developed their abilities to distinguish between the values of the output of the function and the values of the average rates of change of a function and their abilities to distinguish between and interpret velocity graphs and position graphs. However, when we shifted from the phenomena of motion to the new phenomena of light intensity and voltage drop across a capacitor, student difficulties in distinguishing between the function and its average rates of change re-surfaced for many of the students. But, by the completion of the model application activities, nearly all students were proficient at constructing function graphs and corresponding average rates of change graphs in all three of the contexts of motion, light intensity and a discharging capacitor.

Almost all of the students were able to extend their reasoning about average rates of change to include negative rates and linearly changing rates in the context of motion. This reasoning was supported by their work in the computer simulation environment and class discussions that required them to create and interpret graphs, and communicate and justify their reasoning. However, additional obstacles for students arose when encountering decreasing functions and their negative rates of change in the context of light intensity and a discharging capacitor. The conceptual challenges reside, in part, in attending simultaneously to the global features of the behavior of the function and the average rate of change, and in coordinating one's understanding of the change in function values with the rate of that change over various subintervals. The difficulty of such comparisons becomes more complex when the rates are negative but increasing. In the case of the light intensity and the discharging capacitor models, the magnitude of the average rates of change over a sequence of subintervals decreased, while the sequence of signed average rates of change increased as they became less negative. For many students, language for describing this change in terms of the light intensity and the capacitance appeared in conflict with formal mathematical language for describing that change.

The language of magnitude (or absolute value) appeared to be helpful to students in using careful and precise language to describe changing phenomena. Unlike in the case of motion, where there are different words for describing the signed quantity (velocity) and its magnitude (speed), for other phenomena, such as light intensity and the voltage drop across a capacitor, there are no such distinct and familiar words. This suggests that greater attention needs to be paid to the use of the language of magnitude (or absolute value) to help students in interpreting and communicating about phenomena with negative rates of change. Expressions like "increasing at a decreasing rate" and "decreasing at an increasing rate" appeared to be especially problematic for many students. It would appear that separating such a statement about a function into two statements (one about the function increasing and the other about the rate of change decreasing) would help students in articulating precise mathematical formulations about changing phenomena and in avoiding the easy slip into conflating the function and its rate of change.

This study also suggests that explicit attention needs to be paid to helping students learn to communicate about the context of the changing phenomena, not simply to construct the graphical representation of the phenomena. Since motion is a familiar, everyday experience for students, this would appear to be a useful and productive context for introducing negative rates and distinguishing between the magnitude of the average rate of change and the average rate of change as a signed quantity. However, in shifting to other contexts, students will likely need support in order to communicate their ideas about the changing phenomena. This study would suggest that in order for engineering students to become more proficient at communicating their mathematical reasoning about changing phenomena they will need more experiences in doing so.

Conclusions

The model development sequence supported many students in attending simultaneously to the behavior of the function and its average rate of change, but many students experienced difficulties in distinguishing between these concepts when the average rates of change were negative. For some students, these difficulties persisted through to the end of the model development sequence. Nearly all students became proficient at constructing graphical representations of changing phenomena and their associated rate of change graphs, but some students continued to have difficulties in interpreting such graphs in terms of the phenomena using precise, careful mathematical language. This suggests the need for closer attention to interpreting and communicating the concept of a negative rate of change in the context of changing phenomena.

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