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Enriching Engineering Education with Relations

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Abstract

We describe how the algebra of relations provides a suitable framework for the study of interconnected dynamic systems and enriches students' understanding of systems, circuits, machines, processes, and feedback control. Compared to the traditional approach based on transfer functions, the theory is shown to be simpler yet more general and rigorous. Previously introduced in the technical literature within the setting of abstract algebra, the theory is presented here in a less general but more accessible manner. We also introduce some new concepts and constructs that increase its utility and pedagogical value. These include relation diagrams (the counterpart of traditional block diagrams) and impedance relations. Examples illustrate applications of the theory and its potential benefits for engineering education.

1 Introduction

Engineers use problem solving to invent, design, build, and improve structures, machines, devices, systems materials, and processes. Thus, a central goal of engineering education is to develop the problem solving abilities of students. Since mathematics is the basis for modeling, reasoning, and communicating solutions of technical problems, the quality of this foundation is crucial to engineering education.

In prior work[1], mathematics education has been evaluated from the viewpoint of relevance and rigor, where rigor is defined as difficulty or cognitive load. Here, we instead use rigor to mean the validity of the justification provided for a theory. This justification is the understanding (the "why?") underlying our beliefs, which is arguably the most important aspect of knowledge from an educational standpoint. We distinguish rigor from difficulty since, for example, a justification (proof) that 0 + 1 = 1 may be simple yet rigorous.

Simplicity is another important quality of a theory. For example, a 10 page proof that 0 + 1 = 1 which involves polynomials, calculus, and trigonometry could be rigorous (valid), but would be unnecessarily complex. While such a proof might fair well in a journal article, it has low educational value. In contrast, the clear thinking and penetrating insight of a simple proof has higher educational value. When evaluating the simplicity of a proof or explanation, the required prerequisite knowledge should be also be taken into account.

A third important quality is generality–the mathematical scope or power of a theory. We will use ‘depth’ as a comprehensive term for generality, simplicity, and rigor, although how one weights these attributes to judge the depth of a theory is a subjective matter of preference and purpose, and trade-offs typically occur between these attributes. For example, a rigorous proof might not be as simple as a non-rigorous one (as evidenced by the fact that some authors use rigor to mean difficulty[1]). However, one theory may outperform another in all three depth attributes.

In the context of engineering education, a theory can also be evaluated in terms of its relevance to engineering problems. Increasing depth can also improve relevance, since a more general theory may apply to a larger class of engineering problems, a simpler theory may be more practical to learn and implement, and a more rigorous theory gives more reliable results.

Many difficult junior and senior undergraduate courses in electrical, mechanical, aerospace, and chemical engineering involve the study of interconnected dynamic systems modeled by differential or difference equations, such as feedback control systems. The traditional framework for the analysis and design of such systems is based on the transfer function, which models single-input single-output (SISO) linear time-invariant (LTI) systems. It can be defined by taking the Laplace transform of a differential equation (in continuous time) or the z-transform of a difference equation (in discrete time).

In the continuous LTI case, the differential equation may be written the form $ua(D) = yb(D)$, where $u \in C_\infty$ is the input signal, $y \in C_\infty$ is the output signal, $a, b \in \mathbb{R}[x]$ are real polynomials with $b \neq 0$, and D is the differential operator, applied in postfix notation. To obtain the transfer function of this system, one assumes that the initial conditions of the input and output signals are zero and applies the Laplace transform to both sides of this differential equation to give $U(s)a(s) = Y(s)b(s)$, where $U(s)$ and $Y(s)$ are the Laplace transforms of $u(t)$ and $y(t)$, respectively, and s is a complex variable. This yields the transfer function $Y(s)/U(s) = a(s)/b(s)$, which may be multiplied by a particular transformed input $U(s)$ to find the corresponding transformed output $Y(s)$.

Transfer functions are appealing in that they model dynamic systems as rational functions that can be added, multiplied, and inverted to reduce networks of interconnected subsystems. However, the educational value of the transfer function is deficient with respect to both relevance and depth. It lacks generality since it applies only to LTI systems, and even for LTI systems, it does not model the free response. This is a problem because the free response of a system determines the essential property of stability. Although textbooks, instructors, and students often draw conclusions about a system’s free response and stability from a system’s transfer function, we will see that such conclusions can be wrong, and even when they are right, the reasoning is not. This illustrates another common trade-off in the depth attributes: the (perceived) gain in generality achieved by applying a theory beyond its domain of validity comes at the expense of rigor.

A third way that the transfer function lacks generality is that it operates only on Laplace transformable signals (or z-transformable signals in the discrete case). Even the constant function $y(t) = 1$ on $t \in (-\infty, \infty)$ has no (two-sided) Laplace transform, as its transform does not converge. Consequently, transforms are essentially limited to causal signals, i.e. signals that are zero for

$t < 0$, and even these are not transformable if not exponentially bounded. This causality constraint leads to the use of the one-sided Laplace transform or the left-truncation of the input using the unit step function, resulting in loss of information about system behavior at negative times. This truncation introduces another problem if there are system zeros, namely the differentiation of a discontinuous input, which requires Laplace transforms of distributions. Since a rigorous theory of distributions is largely inaccessible to undergraduate engineers, there is much confusion around this issue[2], and suggested remedies vary from rigorous but impractical[3] to non-rigorous but practical[2].

Beyond these issues, the Laplace transform itself is a difficult concept for engineering students to grasp[4]. It involves an improper contour integral (i.e. with at least one limit approaching infinity) in the complex plane and models signals in the mysterious s-domain. Laplace transforms can be avoided by using an alternative operational calculus[5 6 7], but the resulting transfer functions have the same basic limitation: they do not model the free response of a dynamic system.

In this paper, we present an alternative theory for the analysis of dynamic systems and compare its educational value with the traditional theory. The theory was originally introduced in[8] based on the algebra of relations. Here, we present it from a more pedagogical perspective and introduce some new elements, such as relation diagrams and generalized impedances, which have counterparts in the traditional theory.

We present no empirical data, as this work addresses the question of what material to teach and why, rather than how to teach. While this question does not necessarily have empirical answers, it is nonetheless an important question in educational research. Improved techniques for teaching and learning, while also important, lose their value if the quality (relevance and depth) of the content is poor. Hence, our goal is to present a new theory, discuss its educational value, and solicit the perspectives of fellow educators.

2 Relation to Prior Work

The use of relations to model and analyze dynamic systems[8] is related to the behavioral approach[9] of mathematical systems theory, but adopts the input-output viewpoint of classical control by emphasizing binary relations, which can be composed, added, and inverted (via the relational converse). The theory[8] is presented here in a less formal and more accessible style suitable for undergraduate engineering students, with an emphasis on pedagogy and engineering applications, and with some new elements (relation diagrams and impedance relations).

We focus mainly on SISO LTI dynamic systems, which are represented as a *rational relations*. These are written as ratios of differential operators (or shift operators in the discrete case) and are distinct from rational symbols or rational matrices [10 11 12]. Rational relations look like transfer functions, but model the system directly in the time domain and include its free response. The algebraic rules for addition, multiplication, and inversion of rational relations are similar to the corresponding rules for transfer functions but prescribe when to cancel common pole-zero pairs that arise in parallel, series, and feedback interconnections. Uncancelled pairs represent uncontrollable modes in the system output.

Rational relations complement the operational calculus of[5 6 7]. Whereas relations are used to reduce networks of interconnected systems, operational calculus allows one to solve for the output of such systems in terms of the input and the initial conditions. Rational relations can be extended to discontinuous functions by using the operational calculus to define a generalized derivative. Together, these complementary theories cover (and extend) all aspects of the classical treatment of LTI dynamic systems (i.e. based on Laplace transforms and transfer functions).

3 In a Nutshell

Although relations apply to very general systems, we will briefly introduce some key ideas in the specific context of SISO LTI dynamic systems. The differential operator D (on the space of smooth signals C_∞) has no inverse because it is not injective: there are multiple signals that have the same derivative, all differing pairwise by a constant signal. The integration operator (i.e. with limits from zero to $t \in \mathbb{R}$) also does not have an inverse because it is not surjective: there are some signals that lie outside of its range (i.e. output set), namely, any signal that does not pass through the origin. Since neither of these operators has an inverse, they are certainly not inverses of each other. Transfer functions neglect this basic fact by incorrectly treating differentiation and integration as inverses.

More generally, a (non-scalar) polynomial operator $b(D)$ has no inverse for the same reason. However, if $b(D)$ is viewed as a relation between its input and output, then it has a converse relation $b(D)^{-1}$, whose input is the output of $b(D)$ and conversely. This converse relation is not an operator because it is nondeterministic: it relates each input value to many output values, all differing pairwise by the kernel of $b(D)$ (i.e. the set of signals annihilated by $b(D)$). Hence, $b(D)^{-1}$ could be called a multi-valued operator (although this is technically a contradiction in terms).

Consider a differential equation of the form

$$ua(D) = yb(D), \tag{1}$$

where $u \in C_\infty$ is the input signal, $y \in C_\infty$ is the output signal, $a, b \in \mathbb{R}[x]$ are real polynomials with $b \neq 0$, and D is the differential operator, applied in postfix notation. This defines a binary (input-output) relation R between u and y, and so we write it as uRy. This relation is obtained from (1) as $R = a(D)b(D)^{-1}$, which is the (postfix) composition of two relations: the operator $a(D)$ and the converse relation $b(D)^{-1}$. This *rational relation*[8], also written as $R = a(D)/b(D)$, represents the set of all $(u, y) \in C_\infty^2$ satisfying (1).

Like transfer functions, rational relations may be added, composed, and inverted (via the converse) to model parallel, series, and feedback interconnections of subsystems, and algebraically reduced to find the relation between any input and any output of the network. The algebraic rules for doing so are simple but differ from those used for transfer functions. Differences occur whenever transfer functions hide certain behaviors, called hidden modes. For example, consider the unstable system $\dot{u}(t) - u(t) = \ddot{y}(t) - y(t)$, which allows the unstable *free response* $y(t) = e^t$ (i.e. when $u(t) = 0$). Its transfer function $Y(s)/U(s) = 1/(s+1)$ hides the unstable mode e^t via a pole-zero cancellation. In contrast, this system is represented exactly by the rational relation

$$\frac{(D-1)}{(D^2-1)} \neq \frac{1}{(D+1)}. \tag{2}$$

This inequality, which simply says that $\dot{u}(t)-u(t)=\ddot{y}(t)-y(t)$ and $u(t)=\dot{y}(t)+y(t)$ are different systems, illustrates one way in which rational relations differ from transfer functions. The system on the left has a hidden mode (and is thus uncontrollable[9]), while the system on the right has no hidden modes (and is thus controllable[9]). Unfortunately, even systems composed entirely from controllable subsystems can be uncontrollable. Unlike transfer functions, rational relations keep track of any hidden modes, including those introduced by series, parallel, and feedback connections.

The transfer function $T(s)=Y(s)/U(s)$ does not model hidden modes because it does not model the free response at all: $U(s)=0$ implies that $Y(s)=0$. This problem has nothing to do with the Laplace transform. The problem is that the initial conditions in (1) are set to zero (before taking Laplace transforms) in order to get the transfer function. This produces a unique $y(t)$ for each $u(t)$ and turns the relation between u and y into an operator R_0. But the relation $R=a(D)/b(D)$ defined by (1) is not an operator but rather a nondeterministic relation that relates each value of the input signal u to (infinitely) many values of the output y, which all differ pairwise by a free response $0R \neq 0$ of the system (1). Hence, the (deterministic) operator R_0 only approximates the actual relation R.

We will see that there is no need to approximate relations as operators, as relations can be composed, added, and inverted as easily as operators and in a more general (inclusive) manner. Rational relations, such as those in (2), provide more precise models of LTI systems and represent them in the familiar time domain.

4 Relations

Here, we will formalize some of the above concepts in the general context of relations. The relation is a primitive of mathematical logic. For example, set theory is founded on the binary relation of membership $\in$. It is called binary because it relates two objects; for example $u \in y$. The relational framework in[8] deals specifically with relations on sets (of signals) and focuses on binary relations that relate the inputs and outputs of a dynamic system.

A binary relation (or graph) R is simply a subset of $U \times Y$, where the set U is called the source, and the set Y is called the target. In this paper, the source and target are usually the same set M, and so $R \subseteq M^2$. If $(u,y) \in R$, then we say that u and y are related by R, and express this in infix notation as uRy. Examples include $u<y$, $u=y$, and u is-married-to y. The left object (u) in these expressions is called the input of the binary relation and the right object (y) is the output.

By definition, R is the set of all $(u,y) \in M^2$ such that uRy. For example, the binary relation $<$ on $\mathbb{R}$ (the reals) is the set of all points $(u,y) \in \mathbb{R}^2$ that lie above the line given by $u=y$ (with input u plotted on the horizontal axis). Similarly, the identity relation $=$ on $\mathbb{R}$ is simply the line given by $u=y$. This identity relation is also written as 1, so $u1y$ means $u=y$. These examples illustrate why a relation is called a graph.

The domain of a binary relation $R \subseteq M^2$ is the set of all $u \in M$ such that uRy for some $y \in M$. In other words, it is the set of values that the relation allows the input to have. If the domain of R is all of M, we say that the relation is total. Most dynamic systems encountered in engineering do not constrain the input and are therefore total relations. In fact, having a freely specified value is normally considered a basic characteristic an input (versus an output).

A relation is called *deterministic* if each input corresponds to a unique output. A relation that is both total and deterministic is called a function. When the domain of a function is a vector space (e.g. a signal space), it is called an operator. However, a dynamic system is not an operator because it is nondeterministic: even if its input is zero, its output can be any transient response generated by the system. This observation is the key motivation for the relational approach to dynamic systems[8]. The classical approach is to approximate such systems as operators (usually as transfer functions in a transformed domain), instead of as the nondeterministic relations that they actually are. We will show that modeling these systems correctly (as nondeterministic relations) is easy and avoids several problems.

Binary relations can be composed just like functions (or operators). The composition of $R_1 \subseteq M^2$ and $R_2 \subseteq M^2$ is formally defined as[8]

$$R_1 R_2 = \{(u,y) \in M^2 : \exists x \in M, uR_1x, xR_2y\}. \tag{3}$$

This definition just says that the output (x) of R_1 is the input of R_2. Here the order of composition is written in postfix and denoted by juxtaposition. Composition is associative and has identity $1 = \{(u,u) \in M^2\}$.

The converse R^{-1} of a binary relation $R \subseteq M^2$ is the relation

$$R^{-1} = \{(y,u) \in M^2 : (u,y) \in R\}. \tag{4}$$

Thus, the converse is found simply by swapping the input and output, i.e. flipping the graph (relation). For example, as a binary relation on $\mathbb{R}$, the converse of $<$ is $>$. This generalizes the inverse of a function (or operator): if the converse of a function is a function, then it is the inverse of the function. Otherwise the inverse (function) does not exist (i.e. the converse is not total or not deterministic or both). This is another advantage of relations: every relation (including every operator) is invertible in the sense of converse. (Even the zero operator has a converse relation!) From (3) and (4), we obtain

$$(R_1 R_2)^{-1} = R_2^{-1} R_1^{-1}, \tag{5}$$

which generalizes the familiar identity for invertible functions.

In the context of dynamic systems, M is a vector space of real or complex-valued signals in continuous or discrete time. Addition of such signals naturally defines addition and negation, respectively, of binary relations $R_1, R_2 \subseteq M^2$ as follows[8]:

$$R_1 + R_2 = \{(u, y_1 + y_2) \in M^2 : uR_1y_1, uR_2y_2\}, \tag{6}$$
$$-R_1 = \{(u, -y) \in M^2 : uR_1y\}. \tag{7}$$

Stated simply, we add relations by adding their outputs, and negate them by negating their outputs, just as we would for functions (or operators). Addition of relations is commutative and associative. The additive identity is $0 = \{(u,0) : u,0 \in M\}$. A useful equivalent definition of addition is

$$R_1 + R_2 = \{(u,y) \in M^2 : \exists y_1, uR_1y_1, uR_2(y-y_1)\}. \tag{8}$$

Addition, composition, converse, and scaling are closed operations on the set S of all binary relations on a vector space M and thus form an algebraic system. When M is a signal space, S includes every dynamic system having inputs and outputs in M, including nonlinear time-varying systems and differential-algebraic (or descriptor) systems. Input-output relations of systems composed of series, parallel, and feedback interconnections of subsystems can be expressed algebraically in terms the subsystems (binary relations) using these operations. We will give some examples of this after introducing relation diagrams.

5 Image Notation and Operators

Nondeterministic total relations can be viewed as multi-valued functions (or operators), as formalized here using image notation. The image of a point $u \in M$ under a relation R is defined as $uR = \{y \in M : uRy\}$. Hence, we may write uRy as $y \in uR$. If R is an operator, we can simply write $uR = y$. Similarly, the *image* of a subset $M_1 \subseteq M$ is defined as $M_1R = \{y \in M : \exists u \in M_1, uRy\}$. In this notation, we may write the *range* of R as MR, and the *domain* of R as MR^{-1}. The zero image of R is $0R$, and the *kernel* of R is $0R^{-1}$. A relation $R \subseteq M^2$ is called *surjective* (or onto) if $MR = M$ and is called *injective* (or one-to-one) if u_1Ry and u_2Ry imply that $u_1 = u_2$. It is *total* if its converse is surjective (i.e. $MR^{-1} = M$), and it is *deterministic* if its converse is injective.

These above definitions can be expressed algebraically as follows. A binary relation $R \subseteq M^2$ is surjective if $1 \subseteq R^{-1}R$, injective if $RR^{-1} \subseteq 1$, total if $1 \subseteq RR^{-1}$, and deterministic if $R^{-1}R \subseteq 1$. It thus follows that R is total and injective iff $RR^{-1} = 1$, and R is deterministic and surjective iff $R^{-1}R = 1$.

A deterministic relation is called a partial function, and, as mentioned, a total deterministic relation R is called a function (or operator). We will henceforth use lowercase letters for deterministic relations (i.e. partial operators and operators). From the preceding identities, an injective operator g on M satisfies $gg^{-1} = 1$, while a surjective operator g satisfies

$$g^{-1}g = 1 \tag{9}$$

The set of all total relations on M is closed under composition, addition, and scaling, but not closed under converse. The same is true of the set of all deterministic relations.

6 Linear Relations

The algebra of relations gains additional structure when restricted to relations that are total and linear. To define a linear relation on M, first define a space $(M^2, +, \mathrm{R})$, where addition and scalar multiplication are defined as $(u_1,y_1) + (u_2,y_2) = (u_1+u_2, y_1+y_2)$ for $u_1,y_1,u_2,y_2 \in M$ and

$\alpha(u_1, y_1) = (\alpha u_1, \alpha y_1)$ for $\alpha \in \mathrm{R}$. Then, a linear relation on M is defined simply a subspace of $(M^2, +, \mathrm{R})$.

The set of all linear functions (operators) that map M into M is called $\mathrm{End}(M)$, which forms a ring (called the endomorphism ring) under addition and composition. Let $\mathrm{L}(M)$ be the set of all total linear relations on M. Whereas $\mathrm{End}(M) \subset \mathrm{L}(M)$ is a ring, $\mathrm{L}(M)$ is only a seminearring[13], as it lacks additive inverses, and only the right distributive law holds[8]: for all $R, S, T \in \mathrm{L}(M)$, $(R+S)T = RT + ST$. The reason the left distributive law fails is because these relations may be nondeterministic[8].

7 Relation diagrams

Relation diagrams generalize the block diagrams of classical control theory and are useful for representing and reducing interconnections of subsystems. They are introduced here in a general setting that includes nonlinear time-varying systems.

Relation diagrams consist of variables, binary relations, and summing junctions. The notation is illustrated here for a series connection:

$$\begin{aligned} &\xrightarrow{u} R_1 \xrightarrow{x} R_2 \overset{y}{\text{———}} \\ \Leftrightarrow\ &\xrightarrow{u} R_1 R_2 \overset{y}{\text{———}} \end{aligned} \tag{10}$$

and for a parallel connection:

$$\begin{aligned} &\begin{tikzcd} R_1 \arrow[r, "y_1"] & \bigcirc \arrow[r, dash, "y"] & {} \\ R_2 \arrow[u, leftrightarrow, "u"] \arrow[ur, "y_2"'] \end{tikzcd} \\ \Leftrightarrow\ &\xrightarrow{u} R_1 + R_2 \overset{y}{\text{———}} \end{aligned} \tag{11}$$

Edges such as $u, x, y \in M$ are variables, and vertices such as $R_1, R_2, \bigcirc$ are relations on these variables. Arrowheads at the binary relations R_1 and R_2 indicate the inputs to these relations. The tertiary relation $\bigcirc$, called a summing junction, sums its edges to zero, negating edges that have no arrowheads at $\bigcirc$. For example, the summing junction in (11) represents the relation $y_1 + y_2 - y = 0$.

The equivalences diagrammed in (10) and (11) are the definitions of relational composition and addition given by (3) and (6), respectively. The diagrams in (10) justify the use of postfix notation $R_1 R_2$ (versus prefix $R_2 R_1$) for composition and infix $u(R_1 R_2)y$ for membership.

Linear relation diagrams may be reduced using algebraic identities, such as the following. If $T \in \mathrm{L}(M)$ (i.e. T is a total and linear), then for all $x, y, z \in M$,

$$(x+y)Tz \Leftrightarrow \exists x_1, xTx_1, yT(z - x_1). \tag{12}$$

Proof: the existence of x_1 in xTx_1 follows from totality of T, while $yT(z - x_1)$ follows from linearity of T, by subtracting xTx_1 from $(x+y)Tz$. Conversely, adding xTx_1 to $yT(z - x_1)$ gives $(x+y)Tz$.

This identity is diagrammed below:

$$\begin{tikzcd}[row sep=small] {} \arrow[d, "x"'] & & & & {} \arrow[r, "x"] & T \arrow[d, "x_1"'] \\ \bigcirc \arrow[r] & T \arrow[r, dash, "z"] & {} & \Longleftrightarrow & & \bigcirc \arrow[r, dash, "z"] & {} \\ {} \arrow[u, "y"] & & & & {} \arrow[r, "y"] & T \arrow[u] \end{tikzcd} \tag{13}$$

The directions of arrowheads entering T are critical to the identity, but the directions of arrows at the summing junction are not (as these merely denote sign). A corollary is that if T is surjective (and not necessarily total), then T^{-1} is total, and so the identity in (13) holds for T^{-1} in place of T, or equivalently, with all arrowheads at T reversed in (13). A familiar special case of (13) is when T is a linear operator, such as a transfer function.

This identity is used below to the right distributive law of total linear relations, which is implied by the following more general result: Given any relations $R, S \subseteq M^2$ and any total linear relation $T \in \mathrm{L}(M)$,

$$(R+S)T = RT + ST. \tag{14}$$

Proof: Diagrams (15) to (18) are all equivalent with respect to (u, y), where the equivalence of (17) and (18) is obtained by moving T across the summing junction via (13).

$$\xrightarrow{u} (R+S)T \xrightarrow{y} \tag{15}$$

$$\begin{tikzcd} R \arrow[r] \arrow[d, leftrightarrow, "u"'] & \bigcirc \arrow[r] & T \arrow[r, dash, "y"] & {} \\ S \arrow[ur] \end{tikzcd} \tag{16}$$

$$\begin{tikzcd} RT \arrow[r] \arrow[d, leftrightarrow, "u"'] & \bigcirc \arrow[r, dash, "y"] & {} \\ ST \arrow[ur] \end{tikzcd} \tag{17}$$

$$\xrightarrow{u} RT + ST \xrightarrow{y} \tag{18}$$

Application of (13) also gives immediately the identity

$$\begin{tikzcd} R \arrow[r, dash] \arrow[d, leftrightarrow, "u"'] & T \arrow[d, leftrightarrow] & & \\ S \arrow[r] & \bigcirc \arrow[r, dash, "y"] & {} & \Longleftrightarrow \\ R \arrow[r] \arrow[d, leftrightarrow, "u"'] & \bigcirc \arrow[r, dash] & T & {} \arrow[l, "y"'] \\ S \arrow[r] & T \arrow[u] & & \end{tikzcd} \tag{19}$$

which says that for any relations $R, S \subseteq M^2$ and for any total linear relation $T \in \mathrm{L}(M)$,

$$RT^{-1} + S = (R + ST)T^{-1}. \tag{20}$$

8 Feedback Loops

Consider the feedback system

$$\begin{tikzcd} {} \arrow[r, "r"] & \bigcirc \arrow[r, "e"] & C \arrow[d, "u"] \\ & & P \arrow[ul, dash, "y"] \end{tikzcd} \tag{21}$$

which is equivalent to a system of three relational equations:

$$r - e - y = 0 \tag{22}$$

$$eCu \tag{23}$$

$$uPy \tag{24}$$

Note that it is incorrect to write $uP = y$ for (24) unless P is deterministic (i.e. an operator).

In the context of feedback control systems, P in (21) represents a (nonlinear) plant, C a controller, y the output to be controlled, r a reference to track, and $e = r - y$ the tracking error. The relation P, defined by differential equations, is generally nondeterministic: each value of the input u corresponds to (infinitely) many values of the output y. The feedback system (21) yields the binary relations R_{re}, R_{ru}, and R_{ry} between the reference r and the outputs e, u, and y, respectively. For example, $R_{re} \subseteq M^n \times M^n$ is the set of all (r, e) that satisfy (22), (23), and (24), or equivalently, Diagram (21).

Multiplying (22) through by -1 shows that reversing the sign of all variables at the summing junction has no effect. Therefore, the summing junction of (21) is equivalent to the one shown in (25) below. Also, eCu in (21) has been replaced in (25) by the equivalent proposition $uC^{-1}e$:

$$\begin{tikzcd} {} \arrow[r, dash, "r"] & \bigcirc & C^{-1} \arrow[l, "e"'] \arrow[d, leftrightarrow, "u"] \\ & & P \arrow[ul, "y"] \end{tikzcd} \tag{25}$$

Comparing (25) with (11), we see that the feedback connection (25) is just a parallel connection from u to r given by $u(C^{-1} + P)r$, which is equivalent to $r(C^{-1} + P)^{-1}u$. Hence, we conclude that

$$R_{ru} = (C^{-1} + P)^{-1}. \tag{26}$$

This insight that a feedback connection is just a parallel connection in disguise does not generally hold in an operator-theoretic framework, since C may not be invertible. For example, if $C = 0$ and P is total, then $R_{ru} = (0^{-1} + P)^{-1} = 0$. This expression is undefined in an operator-theoretic framework since the relation 0^{-1} is not an operator.

Other closed-loop relations are found similarly. For example, since C and P are in series, we may redraw (21) as

$$\begin{tikzcd} {} \arrow[r, dash, "r"] & \bigcirc \arrow[r, shift left, "e"] & CP \arrow[l, shift left, "y"] \end{tikzcd} \tag{27}$$

where we have again inverted the signs at the summing junction. This may be written as

$$\begin{tikzcd} {} \arrow[r, dash, "r"] & \bigcirc & 1 \arrow[l, "e"'] \arrow[d, leftrightarrow, "e"] \\ & & CP \arrow[ul, "y"] \end{tikzcd} \tag{28}$$

since $e1e$ (i.e. $e = e$). This graph is a parallel connection from e to r given by $e(1+CP)r$. Equivalently, $r(1+CP)^{-1}e$, and so

$$R_{re} = (1+CP)^{-1}. \tag{29}$$

To find R_{ry}, we replace $eCPy$ in (27) with $y(CP)^{-1}e$ (reversing the arrows at $(CP)^{-1}$ to swap input and output):

$$\begin{tikzcd} {} \arrow[r, dash, "r"] & \bigcirc \arrow[r, shift right, "y"'] & (CP)^{-1} \arrow[l, shift right, "e"'] \end{tikzcd} \tag{30}$$

This may be written as

$$\begin{tikzcd} {} \arrow[r, dash, "r"] & \bigcirc \arrow[dr, leftrightarrow, "y"'] & (CP)^{-1} \arrow[l, "e"'] \\ & & 1 \arrow[u, "y"'] \end{tikzcd} \tag{31}$$

which is a parallel connection from y to r given by $y(1+(CP)^{-1})r$. Equivalently, $r(1+(CP)^{-1})^{-1}y$, and so

$$R_{ry} = (1+(CP)^{-1})^{-1}. \tag{32}$$

If $C,P \in \mathrm{L}(M)$ (i.e. total and linear), then other forms of R_{ru} and R_{ry} are also valid. In particular, applying (20) to (26) gives

$$\begin{aligned} R_{ru} &= (C^{-1}+P)^{-1} && (33)\\ &= ((1+PC)C^{-1})^{-1} && (34)\\ &= C(1+PC)^{-1}. && (35) \end{aligned}$$

Similarly, applying (20) to (32) gives

$$\begin{aligned} R_{ry} &= (1+(CP)^{-1})^{-1} && (36)\\ &= ((1+CP)(CP)^{-1})^{-1} && (37)\\ &= CP(1+CP)^{-1}. && (38) \end{aligned}$$

Table 1 summarizes closed-loop relations under various restrictions. For example, if $C,P \in \mathrm{End}(M)$ (i.e. C and P are linear operators), then the expressions in the last column (which are also valid for transfer functions) are obtained. Since the restrictions decrease from right to left, any expression to the left of a valid expression is also valid. These expressions are also valid for MIMO feedback loops.

Table 1: Expressions for Closed-loop Relations

Restrictions:	None (i.e. nonlinear and partial)	$C,P \in \mathrm{L}(M)$ (i.e. linear and total)	$C,P \in \mathrm{End}(M)$ (i.e. linear, total, and deterministic)
$R_{re} =$	$(1+CP)^{-1}$	$(1+CP)^{-1}$	$(1+CP)^{-1}$
$R_{ru} =$	$(C^{-1}+P)^{-1}$	$C(1+PC)^{-1}$	$(1+CP)^{-1}C$
$R_{ry} =$	$(1+(CP)^{-1})^{-1}$	$CP(1+CP)^{-1}$	$(1+CP)^{-1}CP$

9 Rational Relations

Consider the continuous SISO LTI system

$$ua(D) = yb(D), \tag{39}$$

where $u \in C_\infty$ is an input signal, $y \in C_\infty$ is an output signal, $a, b \in \mathbb{R}[x]$ are real polynomials with $b \neq 0$, and D is the differential operator, applied in postfix notation. This defines a relation on $M = C_\infty$, so we write (39) as uRy, where the relation $R = a(D)b(D)^{-1}$ between u and y is called a *rational relation*. In what follows, we will omit the D and write $R = a/b \equiv ab^{-1}$.

Let $B = \mathbb{R}[D]$ be the set of all real polynomials in D. Then for every nonzero $b \in B$ and $u \in C_\infty$, the differential equation $u = yb(D)$ has a solution y. Thus, every nonzero $b \in B$ is surjective. This can be written in image notation as

$$C_\infty b = C_\infty. \tag{40}$$

By (9), this equivalent to $b^{-1}b = 1$.

Also, every $b \in B$ is determined (up to a scalar factor) by its kernel. That is, for all nonzero $b_1, b_2 \in B$,

$$\ker(b_1) = \ker(b_2) \Leftrightarrow b_1 \sim b_2, \tag{41}$$

where $b_1 \sim b_2$ means that there is a real scalar α such that

$$b_2 = \alpha b_1. \tag{42}$$

10 Algebra of Rational Relations

Two polynomials are called coprime if they have no common factors other than scalars. A simple proof of the following lemma follows from the Bezout property of polynomials[8]:

Lemma 10.1 *If $a, b \in B$ are coprime, then $b^{-1}a = ab^{-1}$.*

Algebraic rules for the addition, composition, and inversion of rational relations will now be stated and proved. The following theorem extends that given in[8] by adding a rule for the equality of two rational relations. Also, the proof presented here uses relational identities.

Theorem 10.2 *The set of rational relations* $(Q,+,\cdot)$ *is a subseminearring of* $\mathrm{L}(C_\infty)$. *For all* $a_1,a_2 \in B$, *and all* $b_1,b_2,g \in B\backslash\{0\}$,

$$\operatorname{coprime}(b_1,a_2) \Rightarrow \frac{a_1}{b_1 g}\frac{g a_2}{b_2} = \frac{a_1 a_2}{b_1 b_2}, \tag{43}$$

$$\operatorname{coprime}(b_1,b_2) \Rightarrow \frac{a_1}{g b_1} + \frac{a_2}{g b_2} = \frac{a_1 b_2 + a_2 b_1}{g b_1 b_2}, \tag{44}$$

$$\left(\frac{a_1}{b_1}\right)^{-1} = \frac{b_1}{a_1}, \tag{45}$$

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \Leftrightarrow (b_1 \sim b_2 \wedge a_1 b_2 = a_2 b_1). \tag{46}$$

Proof If $\operatorname{coprime}(b_1,a_2)$, Lemma 10.1 gives $b_1^{-1}a_2 = a_2 b_1^{-1}$. This and (9) give $a_1 b_1^{-1} g^{-1} g a_2 b_2^{-1} = a_1 a_2 b_1^{-1} b_2^{-1}$, which proves (43). Now, suppose $\operatorname{coprime}(b_1,b_2)$. Then

$$\begin{aligned}
\frac{a_1}{g b_1} + \frac{a_2}{g b_2} &= a_1 b_1^{-1} g^{-1} + a_2 b_2^{-1} g^{-1} && (47)\\
&= (a_1 b_1^{-1} + a_2 b_2^{-1}) g^{-1} && (48)\\
&= (a_1 + a_2 b_2^{-1} b_1) b_1^{-1} g^{-1} && (49)\\
&= (a_1 + a_2 b_1 b_2^{-1}) b_1^{-1} g^{-1} && (50)\\
&= (a_1 b_2 + a_2 b_1) b_2^{-1} b_1^{-1} g^{-1} && (51)\\
&= (a_1 b_2 + a_2 b_1)(g b_1 b_2)^{-1} && (52)\\
&= \frac{a_1 b_2 + a_2 b_1}{g b_1 b_2}, && (53)
\end{aligned}$$

where (47) and (52) follow from (5), (48) follows from (14), (49) and (51) follow from (20), and (50) follows from Lemma 10.1. The rule for inversion (45) follows from (4). (N.B. This converse relation is in Q iff $a_1 \neq 0$.) Finally, suppose $a_1/b_1 = a_2/b_2$, or equivalently

$$a_1 b_1^{-1} = a_2 b_2^{-1}. \tag{54}$$

Postmultiplying (54) by $b_1 b_2$ gives $a_1 b_2 = a_2 b_1$, as required by (46). Also, taking the image of $0 \in M$ under each side of (54) gives $0 b_1^{-1} = 0 b_2^{-1}$, which is equivalent to $\ker(b_1) = \ker(b_2)$. By (41), this implies that $b_1 \sim b_2$, as required. Conversely, suppose that $b_1 \sim b_2$ and $a_1 b_2 = a_2 b_1$. Substituting (42) into this last expression gives $a_1 \alpha b_1 = a_2 b_1$, which implies that $a_1 \alpha = a_2$. This and (42) give

$$\begin{aligned}
a_1 b_1^{-1} &= a_1(\alpha\alpha^{-1}) b_1^{-1} && (55)\\
&= a_1 \alpha (b_1 \alpha)^{-1} && (56)\\
&= a_2 b_2^{-1}. \quad \blacksquare && (57)
\end{aligned}$$

Although the signals in Theorem 10.2 are assumed to be smooth (C_∞), this theorem generalizes straightforwardly to discontinuous signals by applying the notion of generalized derivative introduced in our companion paper.

11 Examples in Rational Relation Algebra

The example in Section 3 demonstrates the identity (46) for equality of two rational relations. The two relations in (2) are not equal because their denominators differ by the nonscalar factor $D-1$ (so $b_1 \not\sim b_2$).

The following examples further demonstrate how the algebra of rational relations $(Q,+,\cdot)$ differs from that of transfer functions. Applying (43) shows that composition of rational relations is not commutative in general. For example, the following (postfix) compositions are different:

$$\frac{1}{D-1}\frac{D-1}{D+1} = \frac{1}{D+1}, \tag{58}$$

whereas

$$\frac{D-1}{D+1}\frac{1}{D-1} = \frac{D-1}{D^2-1} \supset \frac{1}{D+1}. \tag{59}$$

A pole-zero cancellation occurs in (58), but a zero-pole cancellation does not occur in (59).

The rational relation $1/(D+1)$ has no additive inverse since (44) and (46) give

$$\frac{1}{D+1} + \frac{-1}{D+1} = \frac{0}{D+1} \neq 0. \tag{60}$$

The reciprocal of a rational relation is its converse, but is in general not its left or right inverse. For example,

$$\left(\frac{D+1}{D+2}\right)^{-1} = \frac{D+2}{D+1}, \tag{61}$$

but

$$\frac{D+1}{D+2}\left(\frac{D+1}{D+2}\right)^{-1} = \frac{D+1}{D+1} \supset 1. \tag{62}$$

Composition is not left-distributive over addition, but it is right-distributive. An example of the failure of the left-distributive law is

$$0 = \frac{0}{D}0 = \frac{0}{D}(1-1) \neq \frac{0}{D}(1) + \frac{0}{D}(-1) = \frac{0}{D}. \tag{63}$$

In contrast, right distribution always holds; for example,

$$\frac{0}{D} = 0\frac{0}{D} = (1-1)\frac{0}{D} = (1)\frac{0}{D} + (-1)\frac{0}{D} = \frac{0}{D}. \tag{64}$$

The algebra in these examples differs from that of transfer functions, which do not model the free response or the hidden modes.

12 Discrete-time Case

Another example of a class of rational relations is the set of discrete-time LTI systems described by difference equations. Let M be the set of all causal (i.e. one-sided) discrete-time signals mapping $\mathbb{N}$ to $\mathbb{R}$. Let L denote the left-shift operator, defined as $xL(k) = x(k+1)$, where $x \in M$. Then, given polynomial operators $a(L)$ and $b(L) \neq 0$, the discrete-time rational relation $a(L)/b(L) \equiv a(L)b(L)^{-1}$ is the set of all $(u,y) \in M^2$ such that $ua(L) = yb(L)$. If Q is the set of all such relations, then an analogous version of Theorem 10.2 applies, and the resulting seminearring Q is isomorphic to that obtained in the (noncausal) continuous-time case. Hence, all the examples of Section 11 apply with D replaced by the left-shift operator L.

These relations have the same advantages over transfer functions as those in the first example. For example the rational relation

$$\frac{L-2}{L-2} \neq 1, \tag{65}$$

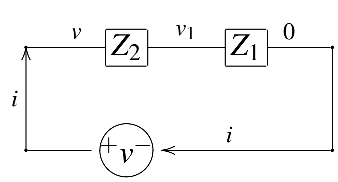
has an unstable (and noncausal) output 2^k when its input is zero. In contrast, the corresponding transfer function reduces to the identity operator 1, which fails to model the uncontrollable mode 2^k.

In the noncausal case, discrete time input/output signals are defined on the full time interval $\mathbb{Z}$ instead of on the non-negative interval $\mathbb{N}$. In this case, the left shift operator L has a trivial kernel and has an inverse L^{-1} (the right-shift operator). The set of operators generated by L is not the ring of polynomials, but rather the ring of Laurent polynomials, which contain negative powers of the operator L. A slightly modified version of Theorem 10.2 still applies in this case, but the obtained seminearring Q of rational relations are is not isomorphic to that obtained for the causal discrete-time case (and for the noncausal continuous-time case). In particular, common factors of L can be cancelled from the numerator and denominator of a noncausal discrete-time rational relation. For example, $L/L = LL^{-1} = 1$, whereas $(L-2)/(L-2) \neq 1$, as this last relation has an unforced response of 2^k, where now $k \in \mathbb{Z}$.

13 Relational Impedance

An electrical or mechanical impedance has a relational model. The normal rules for reducing an impedance network, such as an electrical circuit, can be applied to relational impedances, as shown below using relation diagrams and demonstrated on circuits.

a) Series Circuit In the circuit below, i is the current through the voltage source v, v_1 is the voltage across impedance Z_1, and $v_2 = v - v_1$ is the voltage across Z_2. These impedances could represent single electrical components or could be the equivalent impedances of sub-circuits containing several elements.



(66)

The relations describing the circuit are iZ_1v_1, iZ_2v_2, and $v = v_1 + v_2$, which are diagrammed as:

$$
\begin{tikzcd}
Z_1 \arrow[r, "v_1"] \arrow[d, leftrightarrow, "i"'] & \bigcirc \arrow[r, dash, "v"] & {} \\
Z_2 \arrow[ur, "v_2"'] & &
\end{tikzcd}
\tag{67}
$$

$$
\Leftrightarrow \xrightarrow{\ i\ } Z_1 + Z_2 \overset{v}{\text{———}}
$$

Thus, two impedances in series give a parallel connection (11) between i and v. Hence, (67) reduces to iZv, where $Z = Z_1 + Z_2$ is the equivalent impedance. This result is readily obtained without a relation diagram: by (6), simply add iZ_1v_1 and iZ_2v_2 to obtain $i(Z_1 + Z_2)v$.

The same expression for Z is obtained using transfer function impedance models. The difference is that adding relations Z_1 and Z_2 yield the full system Z, including any uncontrollable modes (as demonstrated in the examples below). This circuit is analogous to a feedback loop (21), with v as reference, v_2 as output, v_1 as error, i as control signal, Z_2 as plant, and the admittance Z_1^{-1} as the controller. The equivalent impedance Z is the closed-loop relation R_{vi} from reference v to control signal i.

b) Voltage Division To find the relation between v and v_1 in circuit (66), we eliminate i and v_2 from (67):

$$
\begin{tikzcd}
{} \arrow[r, dash, "v"] & \bigcirc & Z_1 \arrow[l, "v_1"'] \arrow[d, leftrightarrow, "i"] \\
& & Z_2 \arrow[ul, "v_2"]
\end{tikzcd}
\tag{68}
$$

$$
\begin{tikzcd}
{} \arrow[r, dash, "v"] & \bigcirc \arrow[r, shift left, "v_1"] & Z_1^{-1}Z_2 \arrow[l, shift left, "v_2"]
\end{tikzcd}
\tag{69}
$$

$$
\begin{tikzcd}
{} \arrow[r, dash, "v"] & \bigcirc & 1 \arrow[l, "v_1"'] \arrow[d, leftrightarrow, "v_1"] \\
& & Z_1^{-1}Z_2 \arrow[ul, "v_2"]
\end{tikzcd}
\tag{70}
$$

$$
\overset{v}{\text{———}} (1 + Z_1^{-1}Z_2) \xleftarrow{\ v_1\ } \tag{71}
$$

$$
\xrightarrow{\ v\ } (1 + Z_1^{-1}Z_2)^{-1} \overset{v_1}{\text{———}} \tag{72}
$$

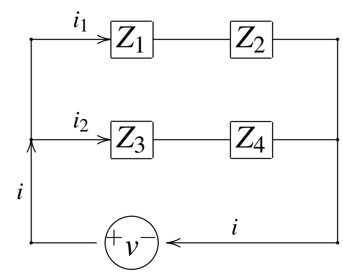
and thus $R_{vv_1} = (1 + Z_1^{-1}Z_2)^{-1}$. In the feedback analogy of Section 8, this is the closed-loop relation between reference v and error v_1 (compare (70) with (28)).

As a concrete example, suppose $Z_1 = Z_2 = D^{-1}$ are unit capacitors. Then, $R_{vv_1} = (1 + DD^{-1})^{-1}$, which reduces to $R_{vv_1} = D/2D \equiv 0.5DD^{-1}$ by the rules of Theorem 10.2. This gives the (uncontrollable) differential equation $\dot{v} = 2\dot{v}_1$. In particular, v or v_1 can be a nonzero constant when the other is zero, whereas a transfer function model gives $v = 2v_1$. The uncontrollability in this example is structural, as it occurs for any capacitance values. An analogous mechanical system is two parallel springs, which can have equal and opposite tensions without an external force applied.

Since relations are more precise than transfer functions, care is needed in their computation. For example, (68) gives $R_{vi} = (Z_1 + Z_2)^{-1}$ and iZ_1v_1, but it is incorrect to conclude that $R_{vv_1} = R_{vi}Z_1 = (Z_1 + Z_2)^{-1}Z_1$. This equals the correct expression $R_{vv_1} = (1 + Z_1^{-1}Z_2)^{-1}$ if Z_1 is surjective and deterministic, but may not if Z_1 is nondeterministic. This is because the value of v_1 in iZ_1v_1 may disagree with the value of v_1 implicit in $R_{vi} = (Z_1 + Z_2)^{-1}$. Although every true solution (v, v_1) lies in $R_{vi}Z_1$, this relation also contains false solutions $(v, v_1) \notin R_{vv_1}$ since $R_{vv_1} \subset R_{vi}Z_1$.

To demonstrate this, suppose $Z_1 = D^{-1}$ is a unit capacitor and $Z_2 = 1$ is a unit resistor. Then $R_{vi}Z_1 = (D^{-1} + 1)^{-1}D^{-1} = D/(D(D+1))$, whereas the exact relation is $R_{vv_1} = (1 + Z_1^{-1}Z_2)^{-1} = (1+D)^{-1} \subset D/(D(D+1))$. The correct relation $R_{vv_1} = (1+D)^{-1}$ is controllable, but $R_{vi}Z_1$ is not.

c) Series and Parallel Circuit Suppose we wish to find the equivalent impedance $Z = R_{iv}$ of the following circuit:



(73)

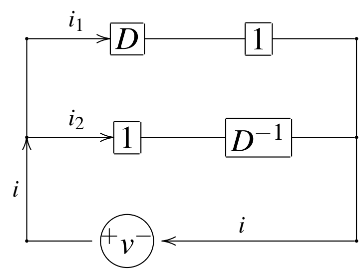
If v_i is the voltage across each Z_i, then the system equations are $i_1Z_1v_1$, $i_1Z_2v_2$, $i_2Z_3v_3$, $i_2Z_4v_4$, $i = i_1 + i_2$, and $v = v_1 + v_2 = v_3 + v_4$.

Adding the first two of these equations gives $i_1(Z_1 + Z_2)(v_1 + v_2)$, which gives (from the sixth equation) $i_1(Z_1 + Z_2)v$, or equivalently, $v(Z_1 + Z_2)^{-1}i_1$. Similarly, $v(Z_3 + Z_4)^{-1}i_2$. Adding these last two equations gives $v\left((Z_1 + Z_2)^{-1} + (Z_3 + Z_4)^{-1}\right)i$ since $i = i_1 + i_2$. Inverting this gives iZv, where

$$Z = \left((Z_1 + Z_2)^{-1} + (Z_3 + Z_4)^{-1}\right)^{-1} \tag{74}$$

is the equivalent impedance between i and v.

Equation (74) is identical to that obtained by treating the impedances as transfer functions or phasors but gives an exact model of the system behavior, including the free response. For example, suppose in circuit (66) that $Z_1 = D$ is a unit inductance, $Z_2 = 1$ and $Z_3 = 1$ are unit resistances, and $Z_4 = D^{-1}$ is a unit capacitance. Replacing each impedance element in (73) with its impedance relation gives



(75)

Substituting these values into (74) and simplifying via the algebraic rules of Theorem 10.2

gives

$$Z = \left((D+1)^{-1} + (1+D^{-1})^{-1}\right)^{-1} \tag{76}$$

$$= \left(\frac{1}{D+1} + \frac{D}{D+1}\right)^{-1} \tag{77}$$

$$= \frac{D+1}{D+1} \neq 1. \tag{78}$$

From (78), the differential equation relating v and i is

$$v(D+1) = i(D+1). \tag{79}$$

For a given $v \in C_\infty$, the response of (79) is $i \in v + \mathbb{R}e^{-t} \equiv \{v + ce^{-t} : c \in \mathbb{R}\}$. In contrast, the transfer function of (79) is $T_{vi} = 1$, which yields the response $i = v$. The system (79) is uncontrollable (as defined in[9]). Whether or not a system is controllable, there is no mathematical basis for recovering the differential equation of a reduced system from its transfer function, since the assumption of zero initial conditions (used to reduce the system) cannot be reversed after the reduction.

14 Analysis of LTI Systems

Stability and system type are two key concepts in the classification of dynamic systems. The relational framework provides a simple and rigorous basis for their analysis.

For a given signal space $\mathscr{S}$, we define the set of stable signals $\mathscr{S}_s = \{y \in \mathscr{S} : \lim_{t\to\infty} y(t) = 0\}$. A rational relation R is stable if $0R \subseteq \mathscr{S}_s$, i.e. if its unforced (free) response is stable. Since $R = a(D)b(D)^{-1}$, we have

$$0R = 0b(D)^{-1} = \ker b(D) = \sum_{j=1}^{n} \sum_{k=0}^{m_j-1} \mathrm{Re}(\mathbb{C}t^k e^{\lambda_j t}). \tag{80}$$

where each $\lambda_j \in \mathbb{C}$ is a root of b having multiplicity $m_j \geq 1$, $\mathbb{C}t^k e^{\lambda_j t} = \{ct^k e^{\lambda_j t} \in \mathscr{S} : c \in \mathbb{C}\}$, and where the Re operator and addition are applied pointwise to the sets of functions in the summand. It follows that a necessary and sufficient condition for stability is that the roots of b are in the open left-half plane.

As shown in the examples of the previous section, the denominator of the transfer function does not necessarily match the denominator of the actual relation. If an unstable mode in the relation is missing from the transfer function, then the system may incorrectly be determined to be stable. Even if (by luck) this does not occur, a stability analysis based on transfer functions is not rigorous.

The internal model principle (IMP) of control theory[14] characterizes the set of reference trajectories that can be tracked by a given plant and controller, which is related to the system type. Rational relations provide a simple derivation of this principle. The relation between tracking error e and reference r was obtained in (29) as $e(1+CP)r$. Exact tracking ($e=0$) gives $0(1+CP)r$, so the set of references that can be tracked is given by

$$
\begin{aligned}
r &\in 0(1+CP) && (81)\\
&= 0CP && (82)\\
&= 0\frac{a_C a_P}{b_C b_P} && (83)\\
&= 0(b_C b_P)^{-1} && (84)\\
&= \ker b_C b_P, && (85)
\end{aligned}
$$

where b_C and b_P are the denominator polynomials of the controller and plant, respectively.

For examples, the feedback system can track a ramp iff $b_C b_P$ contains a factor of D^2 and can track a sinusoid at frequency ω iff $b_C b_P$ contains a factor of $D^2+\omega^2$. Of course, for arbitrary initial conditions, *asymptotic* tracking additionally requires that the free response $0(1+CP)^{-1}$ is stable, which requires that the numerator of the relation $1+CP$ has all of its roots in the open left half-plane.

We define the system type n of the relation CP as the number of factors of D in the polynomial $b_C b_P$. Then (85) shows that the closed-loop system can track a constant reference iff $n \geq 1$, a ramp if $n \geq 2$, etc. Moreover, the system type of P tells us how do design C to track a given reference.

15 Discussion and Conclusions

This presentation of the relational approach to dynamic systems is aimed at readers interested in mathematics for engineering education and as such may not represent the ideal presentation for undergraduates. While our somewhat *top-down* presentation, from abstract concepts to particular examples, emphasizes the generality of relations, a more *bottom-up* presentation may be more suitable in engineering courses.

When the material is integrated with a specific course topic (such as feedback control systems), it might occupy forty lectures, so extrapolating this paper to that setting requires some thought. One approach is to mirror the pedagogical order typically followed within the traditional framework, wherein students first become acquainted with the behavior of specific systems (e.g. using operational calculus to solve initial value problems) and then investigate the stability and performance of interacting subsystems (such as feedback loops). The latter activity would be facilitated by using (nondeterministic) relations to model, reduce, and analyze interconnected dynamic systems.

We conclude with some brief observations regarding the educational value of relations compared to transfer functions, beginning with the attributes of depth (generality, rigor, and simplicity). Relations are more general because they include operators (such as transfer functions) as a special case. The algebra of relations (addition, composition, and inversion) generalizes that of operators, and the relation diagram generalizes the block diagram of the operator-theoretic framework.

As applied to dynamic engineering systems, relations are more rigorous because they model such systems exactly as nondeterministic relations instead of approximating them as operators. In particular, relations model the free response of dynamic systems, including any hidden modes. Since transfer functions exclude the free response, conclusions drawn about the response and stability of the systems they model are invalid and sometimes incorrect. This was demonstrated by circuit examples, but pertains to any type of dynamic system.

The relational approach is simpler than the transfer function approach since there is no need to approximate relations as operators before manipulating them: one simply adds, composes, and inverts them as they are. Moreover, relations avoid transforms (and operational calculus) and model systems directly in the time-domain. While the algebraic rules for rational relations are more subtle than those of transfer functions, the differences arise only when hidden modes occur. Hence, they are only as complicated as they need to be.

Finally, relations are more relevant to engineering systems because they model a larger class of them and model them more precisely. But beyond its utility in the context of dynamic systems analysis, the relation is a fundamental concept that applies to a larger world of knowledge than the standard engineering problems that may come to mind when we think of relevance. A deep understanding of such fundamentals may improve knowledge retention, innovation, and original problem solving in new or unfamiliar contexts.

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