Verified Solutions using Rotation Operators for Combining Rotations

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ABSTRACT

Combining two rotations into one is described in detail in the book *Advanced Dynamics* by Shuh Jing Ying. The solutions are available, but not explicitly verified. This is a very important subject. The repeated use of the combination of two rotations into one allows the combination of many rotations into one. This is especially useful when time is of the essence, such as during calculations for a guided missile in mid-air. In this paper, the solutions are verified by substituting results back into the original equation. The process is tedious and details will be presented in the conference. In addition, an implementation strategy using a ball joint is proposed so the theoretical equations may be related to a practical application, a specific example will be presented in the conference.

INTRODUCTION

If this is the first time that you have seen how two rotations are combined into one rotation, it is not surprising since Ying's *Advanced Dynamics* has a limited distribution. So, the ideas expounded below may seem innovative and new, yet they are not. The purpose of this paper is to demonstrate the use of rotation operators to solve this problem and to inspire readers to create other innovative solutions. This is the educational purpose of this paper. Since this solution is based on rotation operators, and rotation operators are often overlooked in dynamics, let us begin with a brief historical overview of rotation operator. Then, I will start from the definition of rotation operator, provide examples of operations, and then verify the solutions.

Rotation operator was first introduced by J. W. Gibbs in 1901 as mentioned in Ying's *Advanced Dynamics* [1]. A search of the online scientific literature revealed no papers directly related to this study. However, there are some papers parallel to the rotation operator in this paper, for example, "Beyond Euler angles: Exploiting the angle-axis parametrization in a multipole expansion of the rotation operator" by Mark Siemens et el [2] uses angle and rotating axis as the arguments for the operator, similar to the rotation operator in this paper but the application is in a totally different field, quantum mechanics. Other useful references related to the vector and tensors analyses include the following: *Advanced Dynamics* by D. Greenwood [3]; *Vector analysis and Cartesian Tensors* by D. Bourne [4] and *From Vectors to Tensors* by J. R. Ruiz-Tolosa. [5].

REVIEW OF ROTATION OPERATORS

Definition of Rotation Operator

The notations used in the equations are as follows: bold letters represent vectors, a double arrow on the top of a letter indicates a dyad or dyadic. A pair of vectors written in a definite order, such as **ij**, is called dyad and a linear combination of dyads is known as a dyadic. Now, consider that a position vector **r** is rotated with respect to vector **n** by angle β to **r'.** The angle β is measured in the plane perpendicular to **n,** containing the ends of vectors **r** and **r'** in that plane as shown in Fig. 1. Let **a** be a vector with the direction of **n** and the magnitude of the component of **r** along **n**, so that

$$\mathbf{a} = \mathbf{n}\,(\,\mathbf{r}\cdot\,\mathbf{n}\,)$$

Let **b** and **c** be vectors in the circular plane, which is the top view of Fig. 1a looking down directly along – **n**. Hence

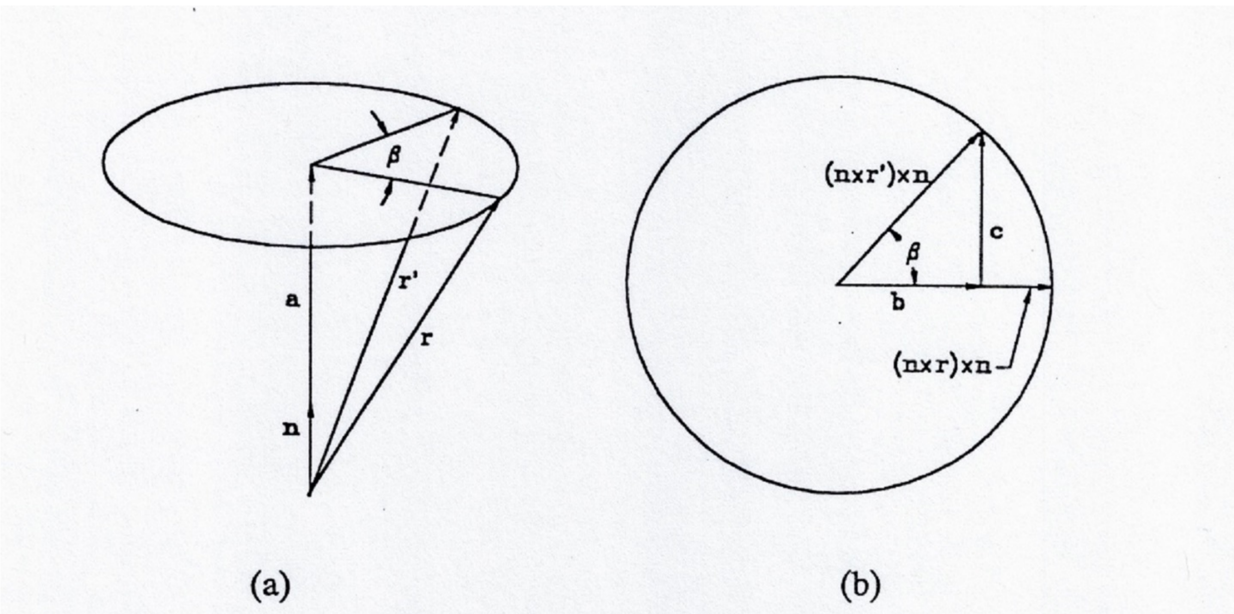

Fig. 1 Rotation of **r** about **n.**

$$\mathbf{r}' = \mathbf{a} + \mathbf{b} + \mathbf{c}$$

The radius of the circle is r $\sin\theta$ where θ is the angle between **r** and **n,** or

$$|\boldsymbol{n} \times \mathbf{r}| = \;|(\,\mathbf{n}\;\times \mathbf{r}\,) \times \mathbf{n}| = \;|(\,\mathbf{n}\;\times \mathbf{r}'\,) \times \boldsymbol{n}|$$

The vectors **b** and **c** are

$$\mathbf{b} = \;[(\mathbf{n} \times \mathbf{r}\,) \times \mathbf{n}]\;\cos\theta$$

$$\mathbf{c} = (\mathbf{n} \times \mathbf{r}) \sin\theta$$

Finally we have

$$\mathbf{r}' = \mathbf{n}(\mathbf{n} \cdot \mathbf{r}) + \cos\beta\ (\mathbf{n} \times \mathbf{r}) \times \mathbf{n} + \sin\beta\ (\mathbf{n} \times \mathbf{r})$$

$$= \mathbf{n}(\mathbf{n} \cdot \mathbf{r}) + [-\mathbf{n}(\mathbf{n} \cdot \mathbf{r}) + \mathbf{r}(\mathbf{n} \cdot \mathbf{n})] \cos\beta + (\mathbf{n} \times \mathbf{r}) \sin\beta$$

$$= (1 - \cos\beta)\,\mathbf{n}(\mathbf{n} \cdot \mathbf{r}) + \cos\beta\ \mathbf{r} + \sin\beta\ (\mathbf{n} \times \mathbf{r})$$

By defining a rotation operator as

$$\overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) = (1 - \cos\beta)\,\mathbf{nn} + \cos\beta\ \overleftrightarrow{\mathbf{1}} + \sin\beta\ (\mathbf{n} \times \overleftrightarrow{\mathbf{1}}) \qquad (1)$$

we obtain

$$\mathbf{r}' = \overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) \cdot \mathbf{r} \qquad (2)$$

So the rotation operator is a dyadic and is defined as a function of rotating axis **n** and angle β **.**

Properties of the Operator

1) As $\beta = 0$,

$$\overleftrightarrow{\mathbf{R}}(\mathbf{n}, 0) = \overleftrightarrow{\mathbf{1}} \qquad (3)$$

When **n** is rotated about **n** itself, **n'** is **n** or

$$\overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) \cdot \mathbf{n} = \mathbf{n} \qquad (4)$$

2) The consecutive rotations about the same axis **n** by angles of α and β will expect a result of

$$\overleftrightarrow{\mathbf{R}}(\mathbf{n}, \alpha) \cdot \overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) = \overleftrightarrow{\mathbf{R}}(\mathbf{n}, \alpha + \beta) \qquad (5)$$

3) $\overleftrightarrow{\mathbf{R}}^{T}(\mathbf{n}, \beta) = (1 - \cos\beta)\,\mathbf{nn} + \cos\beta\ \overleftrightarrow{\mathbf{1}} - \sin\beta\left(\mathbf{n} \times \overleftrightarrow{\mathbf{1}}\right) = \overleftrightarrow{\mathbf{R}}(\mathbf{n}, -\boldsymbol{\beta}) \qquad (6)$

4) $\overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) \cdot \mathbf{V} = \mathbf{V} \cdot \overleftrightarrow{\mathbf{R}}^{\mathrm{T}}(\mathbf{n}, \beta) \qquad (7)$

5) $\overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) \cdot \overleftrightarrow{\mathbf{T}} \cdot \overleftrightarrow{\mathbf{R}}^{\mathrm{T}}(\mathbf{n}, \beta) = \overleftrightarrow{\mathbf{T}'} \qquad (8)$

   Proof:

   Consider $\overleftrightarrow{\mathbf{T}} = \mathbf{AB}$

$$\overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) \cdot \mathbf{AB} \cdot \overleftrightarrow{\mathbf{R}}^{\mathrm{T}}(\mathbf{n}, \beta) = \left(\overleftrightarrow{\mathbf{R}} \cdot \mathbf{A}\right)\left(\mathbf{B} \cdot \overleftrightarrow{\mathbf{R}}^{\mathrm{T}}\right)$$

$$= \mathbf{A}'(\overleftrightarrow{\mathbf{R}} \cdot \mathbf{B})$$

$$= \mathbf{A}'\mathbf{B}' = \overleftrightarrow{\mathbf{T}'}$$

6) $\left[\overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) \cdot \mathbf{V}\right] \times \overleftrightarrow{\mathbf{1}} = \overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) \cdot \left(\mathbf{V} \times \overleftrightarrow{\mathbf{1}}\right) \cdot \overleftrightarrow{\mathbf{R}}^{\mathrm{T}}(\mathbf{n}, \beta) \qquad (9)$

   Proof of the formula is omitted here.

7) If a unit vector **n** is rotated to **n'** by $\overleftrightarrow{\mathbf{R}}(\mathbf{m}, \alpha)$

$$\mathbf{n}' = \overleftrightarrow{\mathbf{R}}(\mathbf{m}, \alpha) \cdot \mathbf{n} \qquad \textbf{then}$$

$$\overleftrightarrow{\mathbf{R}}(\mathbf{n}', \beta) = \overleftrightarrow{\mathbf{R}}(\mathbf{m}, \alpha) \cdot \overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) \cdot \overleftrightarrow{\mathbf{R}}^{\mathrm{T}}(\mathbf{m}, \alpha) \qquad (10)$$

Proof of the formula is omitted here.

8) Some commonly used formulas in dyadic operation

$$\mathbf{A} \cdot \left(\mathbf{n} \times \overleftrightarrow{\mathbf{1}} \right) = \mathbf{A} \times \mathbf{n} \tag{11}$$

$$\left(\mathbf{n} \times \overleftrightarrow{\mathbf{1}} \right) \cdot \mathbf{A} = \mathbf{n} \times \mathbf{A} \tag{12}$$

$$\left(\mathbf{n} \times \overleftrightarrow{\mathbf{1}} \right) \cdot \left(\mathbf{n} \times \overleftrightarrow{\mathbf{1}} \right) = \mathbf{nn} - \overleftrightarrow{\mathbf{1}} \tag{13}$$

$$\mathbf{nn} \cdot \left(\mathbf{n} \times \overleftrightarrow{\mathbf{1}} \right) = \mathbf{nn} \times \mathbf{n} = 0 \tag{14}$$

COMBINATION OF TWO SUCCESSIVE ROTATIONS ABOUT DIFFERENT AXES BY ONE ROTATION

Suppose a rigid body to be rotated by two steps. First it is rotated about the **k** axis by an angle of ϕ and then it is rotated about **k'** axis by an angle of ψ. The directions of **k** and **k'** are known, and the plane containing them is determined. Choose the x axis perpendicular to the plane. Suppose the true angle between **k** and **k'** is θ, as shown in Fig. 2, then

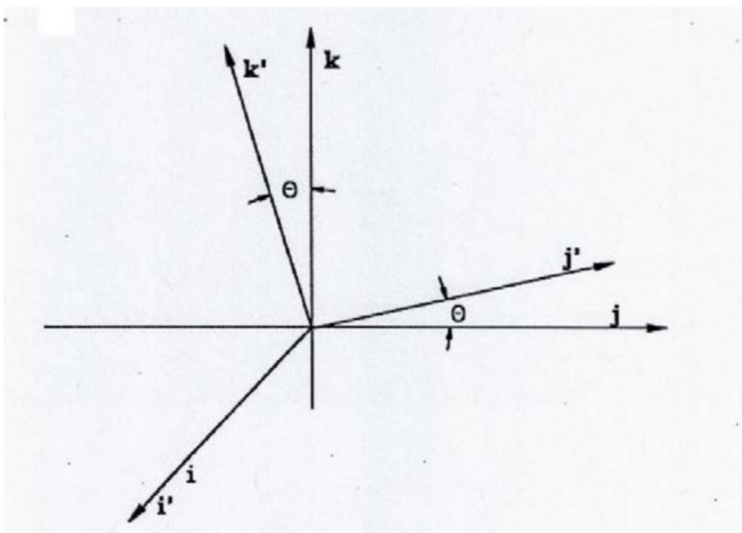

Fig. 2 True angle θ between axes **k** and **k'**

$$\mathbf{k}' = \overleftrightarrow{\mathbf{R}} \cdot \mathbf{k} = -\sin\theta\ \mathbf{j} + \cos\theta\ \mathbf{k} \tag{15}$$

And the two consecutive rotations may be expressed by

$$\overleftrightarrow{\mathbf{R}}_1 = (1 - \cos\emptyset)\,\mathbf{kk} + \cos\emptyset\ \overleftrightarrow{\mathbf{1}} + \sin\emptyset\,(\mathbf{k} \times \overleftrightarrow{\mathbf{1}}) \tag{16}$$

and

$$\overleftrightarrow{\mathbf{R}}_2 = (1 - \cos\psi)\,\mathbf{k}'\mathbf{k}' + \cos\psi\ \overleftrightarrow{\mathbf{1}} + \sin\psi\,(\mathbf{k}' \times \overleftrightarrow{\mathbf{1}}) \tag{17}$$

According to Euler's theorem that the most general displacement of a rigid body with one point fixed is equivalent to a single rotation about some axis through the point, these two rotations can be combined into one, i.e.,

$$\overleftrightarrow{\mathbf{R}}(\mathbf{n}, \beta) = \overleftrightarrow{\mathbf{R}}_2 \cdot \overleftrightarrow{\mathbf{R}}_1 \qquad (18)$$

The theorem is established if **n** and β are determined uniquely. Through some manipulations as shown in Ying's *Advanced Dynamics* [1] it is found that

$$\cos\frac{\beta}{2} = \cos\frac{\psi}{2}\cos\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\phi}{2}\cos\theta \qquad (19)$$

$$\mathbf{n} = \frac{1}{\sin(\beta/2)}\left[\cos\frac{\psi}{2}\sin\frac{\phi}{2}\,\mathbf{k} + \sin\frac{\psi}{2}\cos\frac{\phi}{2}\,\mathbf{k}' + \sin\frac{\psi}{2}\sin\frac{\phi}{2}\,(\mathbf{k}' \times \mathbf{k})\right] \qquad (20)$$

Normally one algebraic equation just can determine one unknown. But Eq. (18) is a dyadic equation, from magnitude and direction two unknowns are determined.

To verify the solutions we need to substitute Eqs. (19) and (20) back into Eq. (18) or the equivalent. From Eq. (18) we have

$$(\mathbf{1} - \mathbf{\cos\beta})\,\mathbf{nn} + \mathbf{\cos\beta}\,\overleftrightarrow{\mathbf{1}} + \mathbf{\sin\beta}\left(\mathbf{n} \times \overleftrightarrow{\mathbf{1}}\right) = \overleftrightarrow{\mathbf{R}}_{\mathbf{2}} \cdot \overleftrightarrow{\mathbf{R}}_{\mathbf{1}}$$

and taking the transpose of both sides

$$(1 - \cos\beta)\,\mathbf{nn} + \cos\beta\,\overleftrightarrow{\mathbf{1}} - \sin\beta\left(\mathbf{n} \times \overleftrightarrow{\mathbf{1}}\right) = \left(\overleftrightarrow{\mathbf{R}}_2 \cdot \overleftrightarrow{\mathbf{R}}_1\right)^{\mathrm{T}} = \overleftrightarrow{\mathbf{R}}_1^{\mathrm{T}} \cdot \overleftrightarrow{\mathbf{R}}_2^{\mathrm{T}}$$

The subtraction of the preceding two equations gives

$$\sin\beta\left(\mathbf{n} \times \overleftrightarrow{\mathbf{1}}\right) = \frac{1}{2}\left[\overleftrightarrow{\mathbf{R}}_2 \cdot \overleftrightarrow{\mathbf{R}}_1 - \overleftrightarrow{\mathbf{R}}_1^{\mathrm{T}} \cdot \overleftrightarrow{\mathbf{R}}_2^{\mathrm{T}}\right] \qquad (21)$$

Note that from Eq. (18) to Eq. (21), no approximation is made in the process so the solutions **n** and β satisfy Eq. (21) is equivalent to Eq. (18). The left hand side of Eq. (21) is

$$\text{LHS} = \sin\beta\,(\mathbf{n} \times \overleftrightarrow{\mathbf{1}}) \qquad (22)$$

and the right hand side is

$$\text{RHS} = \frac{1}{2}\left[\overleftrightarrow{\mathbf{R}}_2 \cdot \overleftrightarrow{\mathbf{R}}_1 - \overleftrightarrow{\mathbf{R}}_1^{\mathrm{T}} \cdot \overleftrightarrow{\mathbf{R}}_2^{\mathrm{T}}\right]$$

$$\begin{aligned}
= \frac{1}{2}\{ & [(1-\cos\psi)(1-\cos\phi)\,\mathrm{k'k}\,\cos\theta + (1-\cos\psi)\sin\phi\,\mathrm{k'k'}\cdot\left(\mathrm{k} \times \overleftrightarrow{1}\right) \\
& + \sin\psi\,(1-\cos\phi)\left(\mathbf{k}' \times \overleftrightarrow{\mathbf{1}}\right)\cdot\mathbf{k}\mathbf{k} + \mathbf{\cos}\psi\,\sin\phi\,(\mathbf{k} \times \overleftrightarrow{\mathbf{1}}) \\
& + \sin\psi\cos\phi\left(\mathbf{k}' \times \overleftrightarrow{\mathbf{1}}\right) + \sin\psi\sin\phi\,(\mathbf{k}' \times \overleftrightarrow{\mathbf{1}})\cdot(\mathbf{k} \times \overleftrightarrow{\mathbf{1}})] \\
- & [(1-\cos\phi)(1-\cos\psi)\,\mathbf{k}\mathbf{k}'\,\mathbf{\cos}\theta - (1-\cos\phi)\sin\psi\,\mathbf{k}\mathbf{k}\cdot(\mathbf{k}' \times \overleftrightarrow{\mathbf{1}}) \\
& - \sin\phi\,(1-\cos\psi)(\mathbf{k} \times \overleftrightarrow{\mathbf{1}})\cdot\mathbf{k}'\mathbf{k}' - \sin\phi\cos\psi\,(\mathbf{k} \times \overleftrightarrow{\mathbf{1}}) \\
& + \sin\phi\sin\psi\,(\mathbf{k} \times \overleftrightarrow{\mathbf{1}})\cdot\left(\mathbf{k}' \times \overleftrightarrow{\mathbf{1}}\right) - \cos\phi\sin\psi\left(\mathbf{k}' \times \overleftrightarrow{\mathbf{1}}\right)]\}
\end{aligned} \qquad (23)$$

Through some tedious manipulations and with the use of the following identities

$$\mathbf{i} \times \overleftrightarrow{\mathbf{1}} = \mathbf{k}\,\mathbf{j} - \mathbf{j}\,\mathbf{k}, \qquad \mathbf{j} \times \overleftrightarrow{\mathbf{1}} = \mathbf{i}\,\mathbf{k} - \mathbf{k}\,\mathbf{i}, \qquad \mathbf{k} \times \overleftrightarrow{\mathbf{1}} = \mathbf{j}\,\mathbf{i} - \mathbf{i}\,\mathbf{j},$$

$$\left(\mathbf{k}' \times \overleftrightarrow{\mathbf{1}}\right) \cdot \mathbf{k} = \mathbf{k}' \times \mathbf{k} = -\sin\theta\ \mathbf{i}$$

$$\left(\mathbf{k} \times \overleftrightarrow{\mathbf{1}}\right) \cdot \left(\mathbf{k}' \times \overleftrightarrow{\mathbf{1}}\right) = \left(\mathbf{k} \times \overleftrightarrow{\mathbf{1}}\right) \times \mathbf{k}' = -\mathbf{j}\,\mathbf{j}' - \mathbf{i}\,\mathbf{i}' \cos\theta$$

and

$$\mathbf{j}' = \overleftrightarrow{\mathbf{R}}(\mathbf{i},\theta) \cdot \mathbf{j} = \cos\theta\,\mathbf{j} + \sin\theta\ \mathbf{k}$$

Eq. (23) finally becomes

$$\text{RHS} = 2\left[\cos\frac{\psi}{2}\cos\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\phi}{2}\cos\theta\right] \cdot$$

$$\left[\cos\frac{\psi}{2}\sin\frac{\phi}{2}\left(\mathbf{k} \times \overleftrightarrow{\mathbf{1}}\right) + \sin\frac{\psi}{2}\cos\frac{\phi}{2}\left(\mathbf{k}' \times \overleftrightarrow{\mathbf{1}}\right) + \sin\frac{\psi}{2}\sin\frac{\phi}{2}\left(\mathbf{k}' \times \mathbf{k}\right) \times \overleftrightarrow{\mathbf{1}}\right] \tag{24}$$

Substituting the solutions found in Eqs. (19) and (20) into Eq. (21) gives

$$\text{LHS} = 2\left[\cos\frac{\psi}{2}\cos\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\phi}{2}\cos\theta\right] \cdot$$

$$\left[\cos\frac{\psi}{2}\sin\frac{\phi}{2}\,\mathbf{k} + \sin\frac{\psi}{2}\cos\frac{\phi}{2}\,\mathbf{k}' + \sin\frac{\psi}{2}\sin\frac{\phi}{2}\left(\mathbf{k}' \times \mathbf{k}\right)\right] \times \overleftrightarrow{\mathbf{1}} \tag{25}$$

Therefore

LHS = RHS

The solutions of **n** and β are verified. To prove Eq. (20) truly with magnitude of one for the unit vector **n** is very difficult but from many numerical examples given in the *Advanced Dynamics* by Ying [1], we do find that the magnitude of **n** is one.

PRATICAL MECHANISM FOR THE SUPPORT PROPOSED

It is expected that the initial position of the missile is approximately at the right point. That is, the adjustment to the precise launch position is very small. The proposed support mechanism is a type of ball joint with a thrust bearing between the inner ball and the support to avoid high friction on the contact surface due the weight of the missile. The contact surface may be 30 % of the bottom surface of the ball. The missile is attached directly above the ball. So the concept for general construction is established. Detail design of the rotating mechanism is beyond the scope of this paper. Also because many different designs may be chosen for given external construction limitations, no specific design for the rotating mechanism is proposed. Take the center of the ball joint as the origin of the coordinates. With the ball joint, the rotating mechanism can rotate the missile to a proper position along the axis **n** and angle β. A specific example will be presented in the conference.

CONCLUSION

Because the rotation operator is not a common subject in dynamics, it was presented at the beginning of this paper. This additional background material increased the paper length, but was necessary to reach a broader audience. The author hopes that from here readers may be able to develop other innovative uses of rotation operator. The solutions for the combined rotations are verified so it is safe to use. A ball joint for the support is proposed so the rotation can be done according to any axis through the center of the ball in the space.

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