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Aliasing Effect Near Sampling Frequency

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Abstract

In the *Signals and Systems* course, the aliasing effect is often discussed in the frequency domain, which leads to the Nyquist-Shannon sampling theorem. However, students often do not have an intuitive understanding of the theorem. Therefore, the *wagon wheel illusion* can be used to demonstrate the aliasing effect when the condition of this theorem is not met. This phenomenon can be investigated with a simple Simulink model, and various scenarios can be demonstrated. In addition, the analysis of this model can also help students understand the new technology of direct sampling receiver.

Introduction

In modern signal processing systems, the input analog signal is usually digitized first and then fed into a digital signal processor. The digitizing process can be divided into two steps: sampling and quantizing. The sampling process can be described mathematically as a multiplication with a comb function, which is a periodic pulse train with very narrow pulse width. In the frequency domain, the multiplication becomes a convolution. The comb function is very interesting in Fourier transform, since the result is still a comb function. Therefore, the effect of the convolution is generating many copies of the transformed input signal with the spacing of the sampling frequency, which is shown in Fig. 1 [1]. If the sampling rate is not high enough, there would be an overlap between two neighbors, and this is the cause of the aliasing effect.

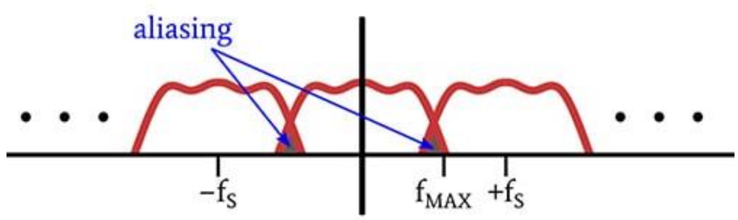


Fig. 1. Alising phenomenon in frequency domain.

In practical situations, the input signal is always band limited, so we can assume that the upper limit of the frequency is f_{MAX}. The Nyquist-Shannon sampling theorem [2] indicates that the sampling frequency should be at least twice of f_{MAX}, i.e. $f_S \geq 2 \cdot f_{MAX}$, which can be understood easily from Fig. 1. On the other hand, if the sampling frequency is fixed, the bandwidth of the input signal is required to be no greater than $f_s/2$, which is called the Nyquist frequency and it is an attribute of the electronic device.

Students can understand the Nyquist-Shannon sampling theorem very well in frequency domain, but they often have trouble to relate it to the situations in the time domain. In nature, the aliasing effect does not cause us much trouble, so it is very hard to develop intuitive understanding of it. Fortunately, movie and TV come for a rescue, and the aliasing effect can be illustrated vividly in the phenomenon of *wagon wheel illusion* [3, 4].

In many movies and TV shows, people see that the wheels of a car seemingly rotate backward while the car is moving forward. As we know, movies are shot as discrete still pictures at the rate of 24 frames per second, and they are shown on the screen with the same rate. Therefore, the motions in movies are illusions of our eyes, which need at least 50 ms to perceive a still picture. From the point of view of signal processing, the process of shooting a movie is just like digitizing a signal at the sampling rate of 24 Hz. With our eyes serving as a lowpass filter, the original motion is perceived in the same way as the reconstructed signal [5, 6].

When we watch a rotating wheel, our attention is usually focused on the spokes. For example, with a six-spoke wheel rotating at the speed of 4 revolutions per second, the pictures of the wheel will be identical if a movie is shot at the rate of 24 frames per second. Therefore, the wheels rotating at this speed will look like stationary in the movie. If the rotating speed is a little lower than that, then it appears that the wheel is rotating backward.

A lab assignment was designed in the course of *Signals and Systems* to demonstrate this phenomenon, and Simulink was used for simulation. The original signal is a sine wave, and it is sampled by multiplying it with a square wave with very low duty cycle, which is just like a pulse train or comb function. Figuratively speaking, the effect of this multiplication is just like taking snap shots with a fixed rate. The sampled signal goes through a low pass filter, and the original signal can be recovered if it meets the Nyquist criterion. However, if the frequency of the signal is close to the sampling frequency, a low frequency sine wave will be reconstructed, and the frequency is equal to the difference of the two.

System Diagram

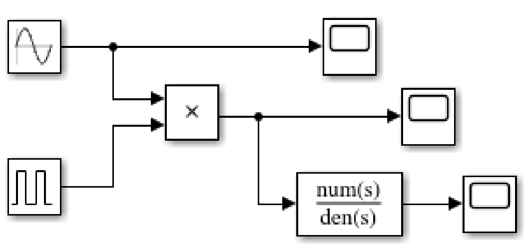


Fig. 2. System diagram of a sampling and reconstruction system

The system diagram is shown in Fig. 2, which is implemented with Simulink. On the top left is the function generator, which can create a sine wave with different frequencies. At the bottom left is another function generator, which creates a square wave at 20 Hz. Its duty cycle is set at

5%, so the square wave becomes a pulse train. In order to retain the power of the sampled signal, the height of the pulses is 20, which is the reciprocal of the duty cycle. The outputs of these two function generators go into a multiplier, and the output becomes the sampled signal, which then goes through a low pass filter (LPF) at the bottom. On the right there are three display modules, which can show the original waveform, the sampled waveform, and the reconstructed waveform, respectively.

Low Pass Filter

One of the challenges in this system is the low pass filter. Since the slope of the lower order filters is not steep enough, the high frequency components created by the sampling process cannot be rejected effectively. On the other hand, if a higher order filter is used, too many parameters need to be set up. Therefore, the lowest order filter with an acceptable result is preferred. The transfer functions of Butterworth filters at different orders are shown in the Fig. 3.

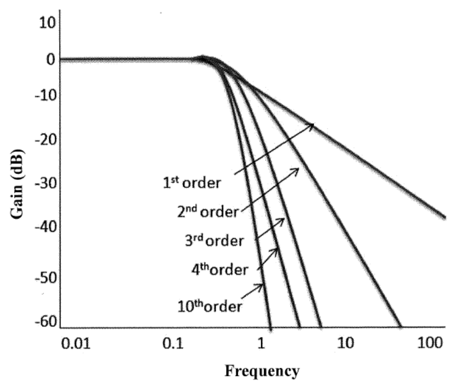


Fig. 3. Transfer functions of Butterworth lowpass filters.

Fortunately, the original signal is a sine wave in this simulation, so the cutoff frequency of this filter can be moved to a lower frequency at f_0 = 5 Hz. If an ideal LPF is used, the cutoff frequency should be at the Nyquist frequency (10 Hz).

First, a second order Butterworth LPF is engaged, and its transfer function is expressed as below, where $\omega_0 = 2\pi f_0 = 10\pi$.

$$H(s) = \frac{\omega_0^2}{s^2 + \sqrt{2}\omega_0 s + \omega_0^2} \qquad (1)$$

In order to test the validity of this LPF, a sine wave with f = 2 Hz is used as the input signal, which is shown on the left of Fig. 4. On the right is the reconstructed waveform, where some residue of the higher frequency components are superposed onto the original sine wave. This result indicates that the second order LPF is not good enough, so we need to use a higher order LPF. Besides the distortions in the reconstructed waveform, there is also a small shift or delay, which is caused by the system. However, this delay is harmless from the point of view of signal reconstruction.

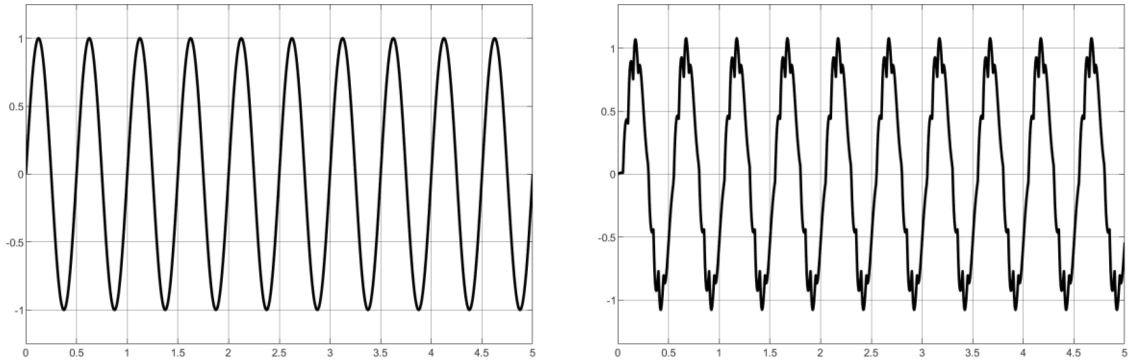


Fig. 4. Original and reconstructed sine wave with second order LPF.

In order to suppress the higher frequency components, a third order Butterworth LPF can be used, and its transfer function is expressed as below:

$$H(s) = \frac{\omega_0^3}{s^3 + 2\omega_0 s^2 + 2\omega_0^2 s + \omega_0^3} \qquad (2)$$

Fig. 5 shows the simulation result, and the reconstructed signal looks reasonably good, though the higher frequency components still cause some minor distortions.

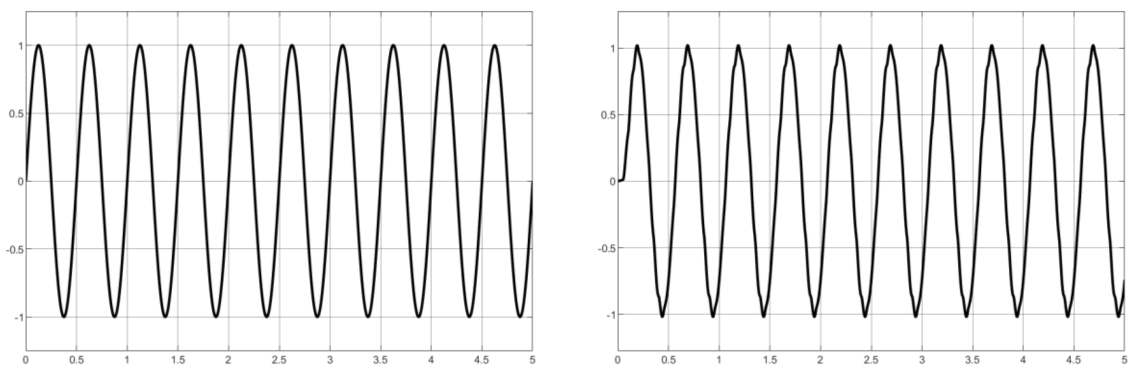


Fig. 5. Original and reconstructed sine wave with third order LPF.

Input Signal at Sampling Frequency

If the input signal is at the sampling frequency, the reconstructed signal will have a constant value. As we know, sine and cosine functions can be derived as x- and y-components from a unit vector rotating around the origin with a constant angular velocity. In mathematics it is equivalent to the Euler's equation:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \qquad (3)$$

A familiar object that rotates with a constant speed is a traditional clock with hands. If our attention is fixed at the minute hand and we check it every hour, and then we will find that this hand is not moving at all, provided that the clock runs accurately. This is the case when the signal and sampling function have the same frequency.

On the left of Fig. 6 is the simulation result of this situation, and the frequency of the input signal is 20 Hz, which is the same as the sampling frequency. On the right of Fig. 6 is the setting of the pulse train, and there is no phase delay in the sampling process. Due to the limited rejection capability of the LPF, the 20 Hz signal still leaks into the output signal. If this small oscillation is ignored, the output becomes a constant.

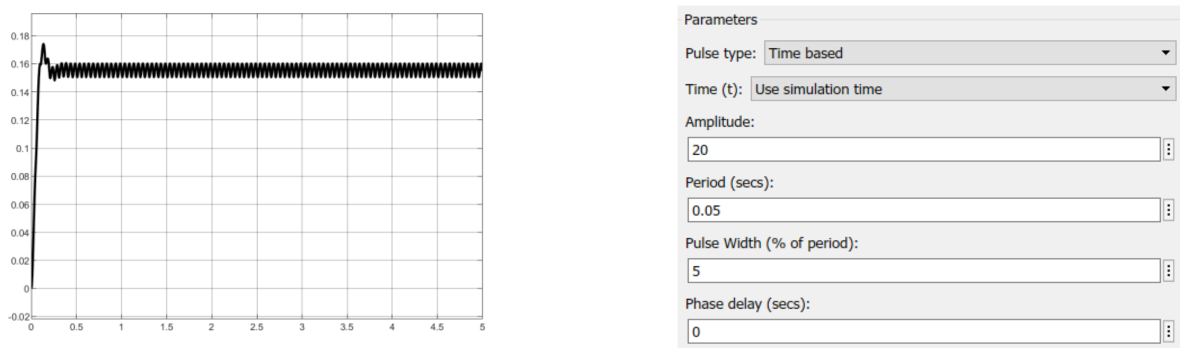


Fig. 6. Input signal at sampling frequency and no delay in sampling.

In the example of the clock, the minute hand can be at any initial position, depending on the time we check it. For example, if we check the clock on the hours, the position of the minute hand will point upwards, and it seems remaining there if we check it every hour. If interpreted from the point of view of Euler's equation, the position of the minute hand corresponds to the initial phase angle. Therefore, if we change the phase delay of the sampling signal, the resulted constant will change.

If we check the clock at 30 minutes after the hours, we will find the minute hand pointing downwards. Fig. 7 shows the simulation result with the phase delay of a half period (0.025 s), and the result indicates that the DC level is shifted from a positive value to a negative value, but the absolute value remains the same. This can be understood easily with the property of the trigonometric functions: $\sin(\theta + \pi) = -\sin(\theta)$.

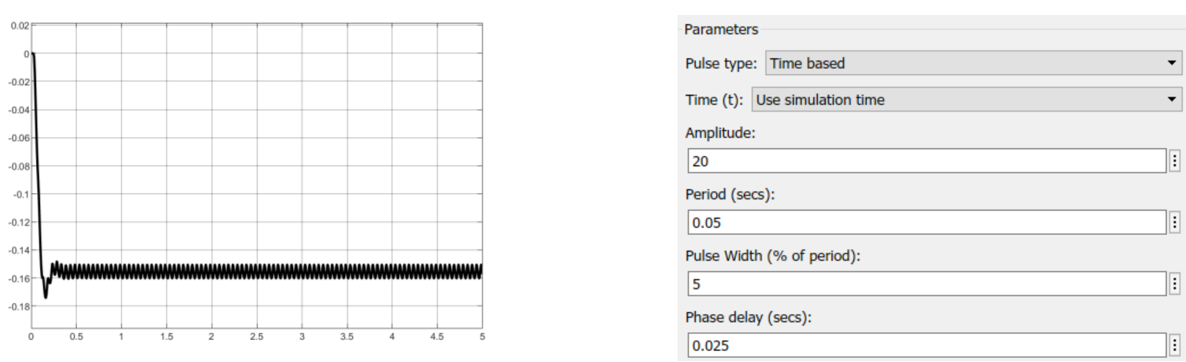


Fig. 7. Input signal at sampling frequency and with delay in sampling.

This frequency locking effect can be used in measuring rotating speed of machines with a strobe light. If the machine and strobe light have the same period or frequency, the machine will look like stationary. Since the frequency of the strobe light can be changed continuously in a wide range and its value can be measured very accurately, this approach is very easy to implement [7, 8].

Input Signal off Sampling Frequency

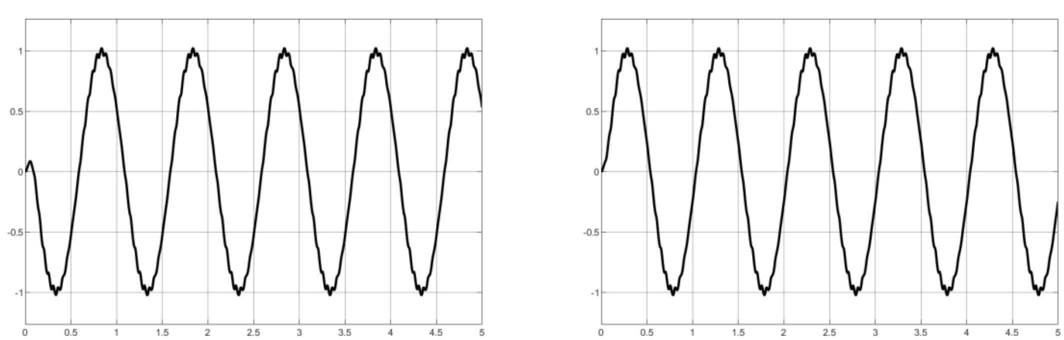
If the input signal is at a frequency slightly different from the sampling frequency, the reconstructed signal will be a sine wave with the frequency equal to the difference of the two. This can be understood intuitively with the analogy of the clock hand. Assuming we still check the clock every hour, but the clock is running faster at the rate of 105%. In other words, it advances 3 minutes every hour. If we start with the minute hand at 0, after one hour it will point to the position at 3 minutes, and then 6, 9, 12 minutes, etc. in the following hours. Similarly, if the clock is running slower at the rate 95%, the minute hand will point to the positions at -3, -6, -9, -12 minutes, etc. If we take a picture every hour and then show them one after another with a short interval, just like playing a movie, it will look like that the clock is running forward or backward. Imagine that the clock hand is replaced by the spokes of a wheel, in the same way it will look like rotating forward or backwards at a much slower rate.

With the input signal as a sine wave with the angular of frequency ω_{sg} close to the sampling frequency ω_{sp}, the output is also a sine wave with much lower frequency, $\omega_d = \omega_{sg} - \omega_{sp}$.

$$v_o(t) = A\sin(\omega_d t) \tag{4}$$

The situation of a faster clock corresponds to the situation $\omega_{sg} > \omega_{sp}$, while a slower clock corresponds to the opposite case. Since $\sin(-\omega t) = -\sin(\omega t)$, the output signals of these two different situations will be inverted or with 180° phase shift.

Fig. 8 shows the reconstructed signals with the input signals at 19 Hz and 21 Hz, respectively. With the sampling frequency at 20 Hz, these two waveforms have the same frequency of 1 Hz. However, there is a 180° phase shift between these two reconstructed signals: $v_{19}(t) = -v_{21}(t)$.



(a) (b)

Fig. 8. Reconstructed signals with the input signals at (a) *f* = 19 Hz and (b) *f* = 21 Hz.

Discussion

In recent years, the sampling process was used for down conversion in direct sampling receivers [9, 10], which can replace the mixer in the traditional heterodyne demodulation scheme [11]. In the system diagram shown in Fig. 2, the pulse train generated for sampling can be expanded into the following Fourier series, provided the phase delay is optimally adjusted.

$$v_{pt}(t) = A_0 + A_1 \cos(\omega_{sp} t) + \sum_{k=2}^{\infty} A_k \cos(k\omega_{sp} t) \tag{5}$$

The multiplication block in the system diagram plays the role of a mixer, which shifts the frequency of the input signal to different frequencies: $\omega_{mix} = \omega_{sg} \pm k\omega_{sp}$.

In the case of input signal with frequency close to the sampling frequency, the low pass filter in the systems will keep only one component that resulted from mixing with the fundamental frequency term: $A_1 \cos(\omega_{sp} t)$. Therefore, in the analysis all the other terms in Eq. (5) can be ignored.

$$A_{sg} \sin(\omega_{sg} t) \cdot A_1 \cos(\omega_{sp} t) = A\{\sin[(\omega_{sg} - \omega_{sp})t] + \sin[(\omega_{sg} + \omega_{sp})t]\} \tag{6}$$

The second term in Eq. (6) has a frequency above the cutoff frequency of the LPF, so it will be rejected. Therefore, the result is a sine wave with $\omega_d = \omega_{sg} - \omega_{sp}$, which can be expressed in Eq. (4).

In the case of an input signal with a very low frequency, such as the situation discussed in the section of “Low Pass Filter”, $|\omega_d| = |\omega_{sg} - \omega_{sp}|$ is higher than the cutoff frequency of the LPF, and thus only the DC term in Eq. (5) has a contribution to the output signal.

$$LPF\{v_{sg}(t) \cdot v_{pt}(t)\} = A_0 A_{sg} \sin(\omega_{sg} t) \tag{7}$$

In order to keep the amplitudes of the input and output signals the same, $A_0 = 1$ is required. Therefore, the height of the pulse train is set as the reciprocal of the duty cycle, in this way the integral of the pulse train signal over one period is equal to one.

Conclusion

The aliasing effect of the *wagon wheel illusion* is investigated with a simple Simulink model, where the sampling process is simulated as a multiplication with a pulse train, which is a square wave with very low duty cycle. In addition, a third order Butterworth LPF is good enough to replace an ideal LPF, provided that the cutoff frequency is shifted lower. When the input signal is a sine wave with the same frequency of the sampling signal, the reconstructed signal becomes a constant, and its value depends on the phase delay of the sampling pulse train. If the frequency of the input signal is slightly higher or lower than that of the sampling process, the resulting signal will be a sine wave with a frequency equal to the difference between the two. With the analogy between the clock hand and the spokes of a wheel, as well as the relationship between the phase angle and the position of the clock hand, the simulation results can explain the *wagon wheel illusion* very well.

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