Technology-Based Problems in Calculus From Science and Engineering

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ABSTRACT

We report on the development of a site[1] for complex, technology-based problems in calculus, from science and engineering. Five college faculty from mathematics and engineering and seven high school teachers from mathematics and science developed a collection of complex problems in mathematics which demand the use of technology. Specifically, the underlying technology is a computer algebra system (in our case, Mathematical). The intent of the project is to provide faculty with a source of such problems. Accordingly the problems, in the form of Mathematical notebooks, ASCII files, and .html files, are on the World Wide Web and can be accessed at

http: //www.rose-hulman. edu/Class/CalculusProbs

We would welcome further contributions of problems, critiques of in place problems, descriptions of use of the problems, and problem improvements and suggestions.

INTRODUCTION

We believe there are a number of weaknesses in current curricula.

1. There is an overemphasis on rote manipulation, hence lack of focus on the bigger picture and deeper problems. Students become lost in the vast amount of numeric and symbolic manipulations which they are asked to perform by hand.
2. Few opportunities exist to fully utilize current software (Mathematical, Maple, MathSoft, DERIVE, etc.)
3. Compartmentalization exists in which students see little substantive relationships between mathematics, science, and engineering.

Further, without technology students are reduced to memorizing collections of special techniques. Overemphasis on manipulative skills at the expense of concepts suppresses student curiosity and fails to develop required problem formulation skills. Moreover, error-fraught handwork stops students from going into depth or playing "What if?" games with problem formulation, solution, or parameters.

We believe one remedy for the these weaknesses is to develop complex, technology-based problems in calculus from science and engineering. We have done that! We present situations in which students formulate strategies to reach a solution, work towards a solution using technology (in our case Mathematical), and then evaluate the reasonableness of their solutions in the context of a science or engineering setting in most cases. We refer to these as technology-based problems.

These problems were developed during the summers of 1994, 1995, and 1996. A team of faculty and associate high school mathematics teachers developed problems, polished established problems, prepared standard formats for problem presentation, produced Mathematical resources to accompany problems, and wrote up guidelines for using problems in high school and college calculus settings. Dr. Klebanoff prepared the World Wide Web site and uploaded the materials for electronic access through site facilities at Rose-Hulman. During the academic years 1994-95 and 1995-96, faculty at Rose-Hulman at USMA located resources, developed significant problems, tested them in courses, summarized student reactions and solution strategies, and modified and extended problems. One high school student developer assisted (summer 1995) in evaluating solutions, assessed prerequisite knowledge needed for attempting these problems, and determined the level of effort and time needed for success.

PHILOSOPHY AND IMPLEMENTATION OF THE PROJECT

We believe students can grow when challenged and motivated. Moreover, students are motivated to learn mathematics when they see a reason for, most often an application of, the mathematics at hand. One can increase the complexity of the problem beyond the traditional text book problem through a number of ways:

[1] Preparation of the materials described in this paper is supported by the National Science Foundation under Division of Undergraduate Education grant DUE-9352849 and the Arvin Foundation of Columbus IN.

1. broadening the context through putting the mathematics in a new or veiled setting [3],
2. deepening the level of detail or analysis, e.g., asking for maximum acceleration levels on a messy function drawn from a realistic position function,
3. stepping up the mathematical level of the problem, e.g., going from linear to non-linear formulation, or
4. implementing more detail or differing strategies in topics minimally or superficially presented, e.g., optimizing a parameter in a model to be solved based on observed data.

To this end we developed mathematical problems which have the features of being technology-based, complex, and related to science and engineering. We believe, now more than ever, that it is imperative to have science and engineering faculty as a part of any problem development effort, for they bring the "realism" of their discipline, a knowledge of the subject matter, a representative spirit of inquiry, and a perspective on the mathematics itself which is shaped by years in their discipline – a perspective which lends problems their crucial credibility.

AVAILABILITY

All the problems developed (some 90 or more with more to come) for this project are available at a World Wide Web site:

http: //www.rose-hulman. edu/Class/CalculusProbs

For each problem there is general information about the author(s), the statement of the problem, keywords, teacher notes (issues related to the problem, prerequisites, time allotment - time management, expectations, future payoffs, extensions, references and sources), possible solutions, and issues in solution. Further, for each problem there is a complete Mathematical notebook with all solution materials, an ASCII version of the file, and an html file for reading on the web itself – all ready for downloading and execution or editing. We would welcome further contributions of problems and problem ideas and suggestions on improving problems already in place.

We present several problems here in the areas of (1) visualization, (2) optimization, (3) differential equation modeling from data, and (4) dynamics. Finally we wrap up the paper with some short descriptions of problems in kinematics. We suggest the reader go to our web site and see the variety of problems, choosing the ones suitable for local use.

ILLUSTRATION FROM VISUALIZATION

As an illustration of a complex, technology-based problem - not too far from mathematics – consider the following problem is in the area of visualization. It incorporates graphics and three-dimensional visualization as we use both graphical and analytical tools.

Consider the following problem. For the function

$$f(x, y) = (x^3 - 3x + 4)/(x^4 + 5y^4 + 20),$$

suppose your eye is precisely on the surface $z = f(x, y)$ at the point (2.8, .5, f(2.8, .5)). You look to the west, i.e. in the direction (roughly) (— 1,0, O). You see a mountain before you.

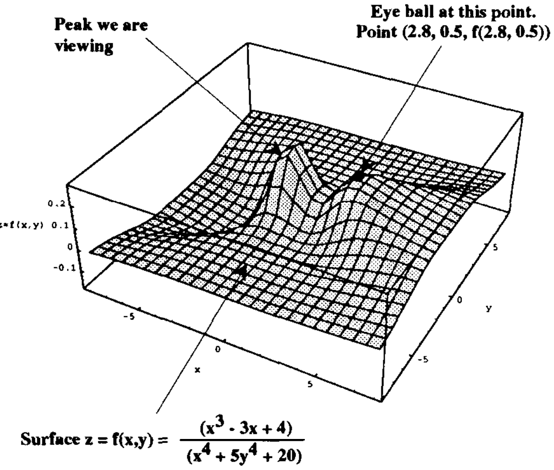

Figure 1. Illustration of context for visualization problem.

(a) Determine the point on the mountain which you can see which is nearest to you.

(b) Describe as best you can the points on the mountain which you can see from the point (2.8, .5, f(2.8, .5)).

(c) Determine the amount of surface area on the mountain which you can see from the point (2.8, .5, f[2.8, .5]).

When we first assigned this visualization problem to our calculus students in an integrated curriculum (see [2]) we spent three days in a multivariate calculus unit working on this problem. Students worked in small groups with the faculty circulating about, responding to queries, listening to ideas, making suggestions, and, from time to time, summarizing group progress for the class and asking groups to make progress reports during class time. Each student was responsible to write up an individual report. Students used gradients, equation solving, Mathematical programming, and numerical methods for determining points on the intersection of a plane and a surface. In solving this problem students used known concepts in new contexts and they worked through a complex process to assemble a solution to the visual problem before them.

We suggest to the reader that this problem is complex, does demand the use of computer technology, and is appropriate for science and technology students. Indeed, students used programming in Mathematical, multivariable calculus concepts such as gradient and tangent plane, and sophisticated equation solving routines, all while developing a keener sense of visualization as they grappled with exactly what one could see on one slope from a point on the opposite mountain.

OPTIMIZATION PROBLEM IN CHEMICAL KINETICS

We offer here an example of a complex problem in parameter estimation for kinetics modeling which leads to optimization.

A laboratory experiment is going on in the Projects Lab of your company. A colleague, a production chemist, comes to you for advice.

Compound A is heated to 120°C in order to produce compounds B and C. The temperature of the pot containing all of these compounds is kept at 120°C in order to keep the reaction going.

It is believed the reaction is a simple first order reaction where the reaction rate of converting compound A to B is k_1 and the reaction rate of converting compound B to C is k_2. The following data has been gathered

Time (rein)	A (mole)	B (mole)	C (mole)
o	1.00	0.00	0.000
2	0.88	0.12	0.003
6	0.69	0.29	0.030
10	0.53	0.42	0.050
20	0.28	0.56	0.16
30	0.15	0.57	0.28
50	0.043	0.46	0.50
70	0.012	0.33	0.66
90		0.22	0.78
120		0.12	0.88
150		0.06	0.94
200		0.02	0.98

The goal is to produce compound B for marketing, thus we seek a mathematical model as an aid in telling us the best time to stop the process and extract compound B for the market. But a measure of best is most profit or net return. We know that A costs $0.50 per mole and B sells for $3.50 per mole. Our production engineer friend has found that for compound C we can expect a return of $0.25 per mole from the recovered C.

Find the optimal shutoff time for this process in order to optimize the net return on this process. We make some suggestions.

1. Without using any "sophisticated" modelling techniques offer a plausible approach and suggest a method to solve such a problem, Look at the data!
2. Form a mathematical model. Be sure to state your assumptions - mathematical and chemical. Use this mathematical model to tell your corporate friend just when to shut off the process and just how much profit they can expect on this process.

Ideas for modeling consist of curve fitting, differential (rate) equations, or difference equations.

Results using non-linear least squares (a *raison d'être* for the use of technology) produce a plot of the fit, with parameter estimates on reaction rate constants in this case. See Figure 2.

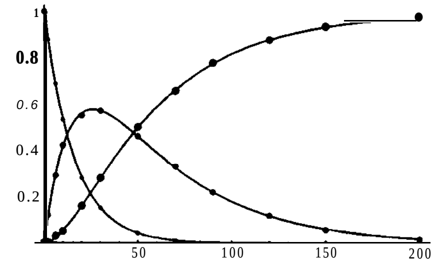

Figure 2. Plot resulting from application of non-linear parameter estimation.

We discuss post solution issues, modifications, and more problems which could arise in this context. For example, suppose your colleague comes back to you and says that you NOW have to consider the optimization in view of the energy costs to run the process and it is known that the energy costs to run this process are $0.005 /min. Formulate a model which will find the optimal shutoff time for this process in order to optimize the net return on this process. Compare your results when you consider energy to when you do not. Finally attempt to determine the sensitivity of yield and final net return in terms of changes in the energy cost. Namely offer a plot of net return and change in net return/unit change in price of energy (marginal net return) as a function of price of energy, say in the range [0, .040], i.e. from no costs to approximately 8 times the current costs of $0.005/min

DIFFERENTIAL EQUATION MODELING FROM DATA

A nice problem we offer in the collection is the following problem which demands data fitting to obtain a differential equation model for the spread of an oil slick.

An oil-slick spreads at sea. From time to time, but irregularly, a helicopter is dispatched. On each trip, it arrives over the slick, the pilot takes a picture, waits 10 minutes, takes another, and heads home. On each of seven trips the size (in area) of the slick is measured from both photographs. The data is offered below.

Area of Slick (square miles) Initial Observation	10 Minutes Later
1.047	1.136
2.005	2.085
3.348	3.415
5.719	5.762
7.273	7.301
8.410	8.426
9.117	9.126

1. Build a model for the growth of the oil-slick. We need a model we can use to predict the size of this slick beyond the observations and we see this as a part of a larger study of oil slick spread.
2. Confirm that your model is a reasonable one for predicting the growth of this oil slick.
3. Determine what your model predicts as the long term growth of this oil slick.
4. Discuss weaknesses, attention to reality, and assumptions you had to make in your model.

In this problem students want to plot data (e.g., size of oil slick) vs. time but that is not how the data is offered – the pilot does not keep a log! Eventually, and it does not take too long if you supply them with a piece of graph paper, students will start differencing the data and plot OilSize(t + 10) - OilSize(t) vs. OilSize(t). After fitting a "curve" to this data (see Figure 3) they will arrive at a reasonable differential equation,

OilSize'(t) = 0.00998536 – 0.000994001. OilSize(t)
OilSlick(O) = 1.047

which they can then solve and finally offer up a predictive model for OilSize(t).

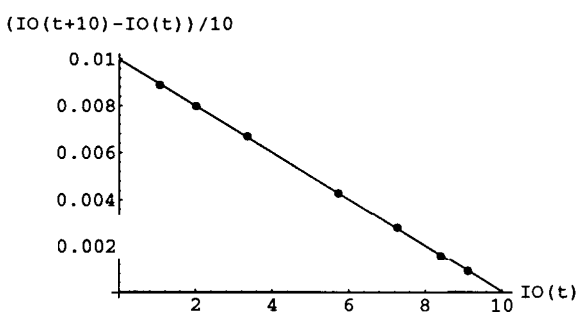

Figure 3. Plot of difference data to enable discovery of a differential equation model for oil slick growth.

ILLUSTRATION FROM DYNAMICS

An excellent source of problems is the set of books for future courses for your students. We have routinely taken and massaged problems from sophomore level statics and dynamics books or professor's handouts and identified them as such for our calculus students. The following comes from Problem 15.110 on page 742 of *Vector Mechanics for Engineers: Dynamics –Fifth Edition* [1] the very book used in the following year by our students.

A rod is attached to an overhead pivot at point A and hangs vertically. The angle at which the rod is hanging is being controlled by a disk of radius r centered at B attached to a yoke. The rod rests on the outer circumference of the disk at moving point C. As the yoke pushes up, the disk pushes the rod away from the vertical and as the yoke pushes down, the disk allows the rod to move closer to the vertical. The pivot of the rod, the center of the disk, and the yoke all lie on a vertical line AB.

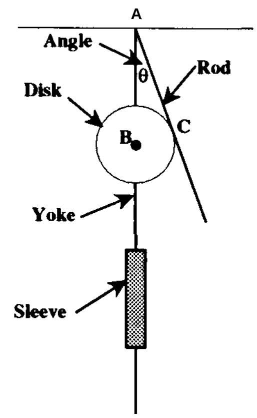

Figure 4. Rod yoke assembly for motion problem.

To ease analysis we create functions and coordinative the device as follows:
A =pivot point= (O, 20) dm (dm is a decimeter= 10 cm).
r =radius of disk= 8dm.
cc(t) =center of disk= (O, M sin(d)); M = 10 dm, w = 1 radian/see.

Examine the figure. The disk is initially centered at the origin and begins its motion upward, thus pushing the rod out.

We are interested in the angle between the vertical yoke and the rod $(\theta(t))$ as well as the point on the disk, C, which the rod contacts as the yoke moves up and down, *P(t)* = $(x(t), y(t))$.

1. Derive an expression that relates the position of the following: center of disk (B) to pivot (A), radius of the disk *(r)*, and the angle between the rod and the vertical (f?).
2. Describe the relationship between the vertical velocity of the yoke (i.e. the rate of change point C moves) and the rate of change in the distance between the pivot (A) and the center of the disk (B).
3. Derive an expression for the angular velocity of rod, i.e. the rate at which theta changes with respect to time t, in terms of the distance between the center of the disk and pivot (s) and the radius *(r)*.
4. Plot angle theta vs. time over the time interval [0, 4π] sec.
5. Plot angular velocity vs. time over the time interval [0, 4π] sec.
6. Plot angular velocity vs. distance from pivot to center of the disk over the time interval [0, 4π] sec. This is called a phase portrait and you should describe how this curve is traced out as time increases, e.g., where on the curve would you find a point corresponding to t = O, $\pi/2, \pi, 3\pi/2$, 27r, $\pi/4$? Recall that *s(t)* = *SO* – $M\sin(\omega t)$ is the distance from pivot to the center of the disk.
7. When is the angular velocity at a maximum size?

8. Examine the angular acceleration and determine when it is at a maximum.
9. Determine the point (P(t)) on the disk where the rod touches the disk and plot its trajectory.

This problem, as with all problems in which students are asked to model some simple motion, proves very challenging. But with the aid of computer algebra systems such as Mathematical or Maple students can try out their conjectures and use the plots as visual feedback of the motion to confirm their analyses.

A SAMPLE OF OTHER PROBLEM AREAS

In a problem about a sky diver under Issues in Solution we write,

> It is far more important that the students are allowed time to grapple with modeling issues instead of the method of solution. While the methods involved with solving intermediate parts of the problem are interesting in their own right, if the students are forced to dwell on the details in this problem, they will never get to the major issues of the problem. Thus, use of a computer algebra system such as Mathematical or Maple to solve the DE's quickly is advisable.

The problem itself reads:

The table below gives approximate terminal velocities of a skydiver:

Diver's Position	terminal velocity
Vertical With Hands at Side (feet pointing at ground):	278.8 ft/sec
Horizontal (spread eagle – body parallel to ground):	180.4 ft/sec
With Chute Open (diver vertical):	21.32 ft/sec

A sky diver weighs 130 pounds and her pack weighs an additional 25 pounds. She falls from a plane flying at 5000 feet.

1. Assuming that the drag due to wind resistance is proportional to her speed, determine 3 equations for separate conditions which account for her velocity in each case.
2. Utilize the equations found in (1) to determine how long it will take for the sky diver to hit the ground if she drops from 5000 feet vertically, at first with her body pointing downward, and then dropping with her body parallel in "spread eagle" fashion from 3000 feet down to 500 feet where she opens her chute.
3. Further modify your model to account for non-instantaneous changes between stages of flight.

In the area of fluid flow we have a rich area of problems essentially involving pipe closings which demand some hefty integral work.

1. We consider a pipe of fixed diameter (an open circle measured inside the pipe) along with valves (covers) large enough to close the pipe. In particular, consider the cases of a

   a) rectangle valve (large enough to plug the pipe)
   b) circular valve of the same radius as the pipe
   c) circular valve of a larger radius than the pipe

   The valve (either rectangular or circular) can be slid across the opening of the pipe to slow the flow of water or stop it all together. Let the center of the pipe be the origin, and let $A(x)$ be the cross sectional area of the pipe through which water can flow with x labeling the location of the edge of the valve as it is slides across the opening of the pipe. For each case, set up the horizontal axis along which the valve moves and label x. Then, determine $A(x)$.
2. Find the rate at which the open area of pipe changes as the distance changes (constantly) for each of the three cases from (1). What is the maximum rate of change of open pipe area as the valve moves at constant rate across for each case?
3. Consider each of the three valve coverings separately and answer this question. If a valve covering is closing at a constant rate (i.e, moved along the position axis at a constant rate), which valve allows the greatest flow and which allows the least flow of water? The water will flow greatest when the open pipe area is greatest.
4. An Inverse Problem: With a given shaped valve type (do this for each of the three-cases considered), find the rule of motion for the valve so that rate of change of flow through the pipe is constant.

Again we emphasize that a good source of problems is in engineering books themselves. The area of statics and dynamics provides a rich source of interesting problems. A text like *Vector Mechanics for Engineers: Statics and Dynamics* [1] provides direct ideas on creating mathematical models to describe motion and hence velocity and acceleration. This is in the area of kinematics. These types of sources serve to stimulate the complex mathematical problem formulator (the latter author) to create the following problem.

We consider the problem of dropping a ball onto an inclined plane (ramp) of slope and watching it bounce. Suppose the coefficient of of restitution is .5, applied only to the component of the velocity which is perpendicular to the plane of the ramp. We wonder what angle the ramp should be to give the bounce the maximum horizontal distance along the level ground.

Place the ball at position (10, 40), Place the ramp's base at (O, O) and assume the slope of the ramp is a positive value, *m*. Thus the ramp has equation $y = mx$. We drop the ball from a height of 40 meters above the ground (not necessarily above the ramp though). What slope m of the ramp will permit the ball to bounce farthest along the ground, i.e. maximize horizontal range?

ONE FINAL SEASONAL PROBLEM

Consider a baseball field with prescribed irregular outfield and fence configuration. Suppose a player can hit a home run which just clears the fence at one given point if the ball leaving the bat (2 feet off the ground) has initial velocity v_0 and launch angle at 45 degrees. Assume no wind resistance. Redesign the fences so that this player (and hence any player) has the same chance of hitting a homerun to anywhere in the outfield (i.e., in the outfield and fair!)

We design Hypothetical Stadium with the following outfield wall configuration. Home plate is at (O, O); left field foul pole is straight up the positive y-axis; right field foul pole is straight down the positive x-axis. Left field outfield fence runs parallel to first base line for 135 ft at a distance of 360 ft from homeplate. Right field outfield fence is a straight line from right field foul line at (409, O) to the point (300, 252). Center field outfield fence is a circular wall going through deepest center field point out at $(200\sqrt{2}, 200\sqrt{2})$, point LC = (135, 360) in left center, and point RC = (300, 252) in right center.

(a) Write equations which completely describe the outfield fences and plot them with homeplate at (O, O).

(b) Determine the farthest point of the outfield fence from home plate, measured along the ground to the base of the fence.

(c) Place a 10 ft high fence at this farthest point found in (b) and determine the initial velocity a hitter must impart to a ball hit 2 ft high over homeplate to get it to just clear the 10 ft high fence here. We shall assume there is no air resistance. Hint: It is known (and you should be able to prove it is true!) that a hitter should always try to launch the ball from the bat at a 45 degree angle with the horizontal for maximum range.

(d) Now determine for every other point at the base of the outfield fence just how high the fence must be so that this player can hit a homerun by launching a ball from his bat 2 feet above homeplate with exactly the same initial velocity at a 45 degree angle with the horizontal.

CONCLUSION

We have described an effort to create technology-based, complex problems in mathematics from science and engineering and a site where our efforts can be found. We heartily recommend stepping up the complexity of problems you offer your students, making use of the capabilities of computer algebra systems such as Mathematical, Maple, MathSoft, DERIVE, etc. to liberate your students to go beyond the limitations of paper and pencil in exploring their own capabilities, and trying your hand at developing such problem activities. It is great fun and very rewarding to see your students building rather detailed and deep solution strategies using technology for complex problems which they find real.

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BIOGRAPHICAL SKETCHES ---

Aaron Klebanoff teaches mathematics at Rose-Hulman Institute of Technology where he is continually exploring ways to best utilize the latest technologies in his classroom. He is a strong advocate of the current differential equations and calculus reform movements, and he enjoys investigating pedagogical ideas which help improve student learning.

He earned his PhD studying chaos in three species food chains and is currently studying various aspects of dynamical systems. His interests in chaos and fractals has led him to develop an undergraduate course on these topics which he is attempting to integrate with two other complementary courses taught in the physics and computer science departments.

Brian Winkel edits two journals *PRIMUS - Problems, Resources, and Issues in* Undergraduate *Studies* and *Cryptologia* in addition to continually trying to improve his own teaching techniques by including more problem-solving opportunities for students, cooperative learning, and use of computer technology.

Before coming to the United States Military Academy he taught at a liberal arts college, Albion College, and an engineering institute, Rose-Hulman Institute of Technology. While his background and PhD are in abstract algebra he has found challenge and opportunity for intellectual growth – his own, his colleagues', and his students' – in curricular innovation in science, engineering, and mathematics using complex problems and technology.