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ABSTRACT

The formulation and explicit integration of the stiffness matrix for the two-node one-dimensional washer element are examined. An example problem is presented to illustrate the effectiveness of using various numerical integration methods for obtaining the element stiffness matrix when nearly singular integrations (for elements very close to the axis) are involved. Numerical examples are given for the two-node washer axisymmetric elasticity element. Errors in nodal displacements from the finite element solutions are compared for different integration methods. Integrating nearly singular axisymmetric washer elements the authors find that using a few sampling points with regular Gauss quadrature is inadequate and recommend new guidelines for numerically integrating these elements' stiffness matrices.

1. INTRODUCTION

Axisymmetric finite element (FE) models may be used to represent three-dimensional (3-D) structures exhibiting symmetry about a central axis of rotation. For conventional axisymmetric elements to be acceptable for modeling a structure, the body's geometry, loading, boundary conditions, and material properties must all be independent of the θ coordinate. Three common types axisymmetric elasticity elements include the two-node washer, the three-node triangle, and the general (distorted) four-node quadrilateral. Structures commonly modeled using axisymmetric elasticity elements include thick-walled pressure vessels, soil masses subjected to circular footing loads, and flywheels rotating at constant angular velocities.

The stiffness matrix for general axisymmetric elasticity elements is of the following form[1]:

$$\underline{K}_E = \iiint_V \underline{B}^T \underline{D} \, \underline{B} \, r \, dr \, d\theta \, dz \qquad (1)$$

where $\underline{B}$ is the kinematic matrix relating element strains to element nodal displacements ($\underline{\varepsilon} = \underline{B} \, \underline{u}_E$), $\underline{D}$ is the material law (Hooke's law in this case) relating element stresses to element strains ($\underline{\sigma} = \underline{D} \, \underline{\varepsilon}$), superscript T denotes the transpose operation, and V is the element volume. In axisymmetric elements the hoop strain ε_r is not constant; it is a function of 1/r that varies with radial position in the element. For this reason, $\underline{B}$ contains 1/r terms, as does the $\underline{B}^T \underline{D} \, \underline{B} \, r$ product.

Prior to proceeding further with the axisymmetric problem, the designation of integrals as regular, singular[2], or nearly singular[3,4] must be explained. A regular (or proper) integral has finite lower and upper limits of integration, and its integrand remains finite over these limits. A singular (or improper) integral may have at least one of its integration limits as infinite (±∞) or

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its integrand may become infinite at one or more points within finite limits of integration. A nearly singular (or quasi-singular) integral has an integrand that behaves very strongly or changes rapidly near one of the integration limits but does not diverge. In a strict mathematical sense, nearly singular integrals are no different from regular integrals; however, nearly singular integrals are not always handled correctly by numerical integration (as is discussed later).

Since the $\underline{B}^T\ \underline{D}\ \underline{B}$ r product is integrated over the radial coordinate, the integral in (1) becomes improper as r approaches zero. Figure 1 shows how the integral classifications discussed above arise for axisymmetric elements. For elements on the axis of rotation, the integral is singular or improper; for elements very close to the axis of rotation, the lower limit of integration for the radial coordinate is almost zero, and the integral is nearly singular (since 1/r behaves very strongly near r = 0). For elements farther from the axis, the integral is regular or proper.

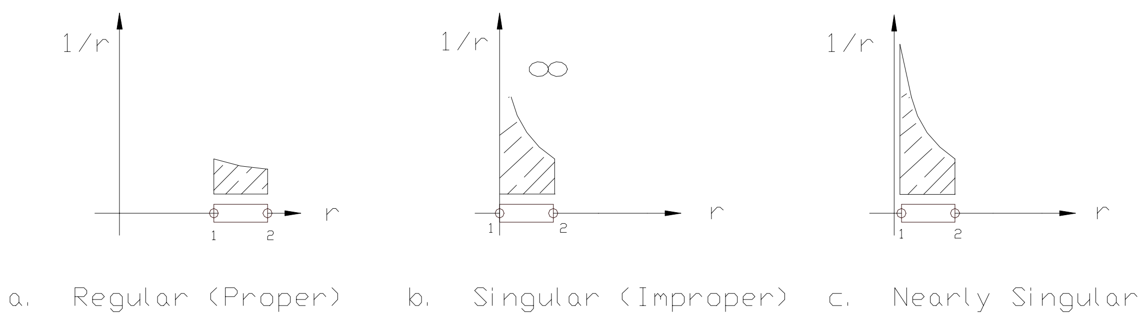


Figure 1: Integral Classification Based on Element Location.

Expression (1) may be evaluated numerically for any shape of element. The most commonly recommended numerical integration technique in FE analyses is Gauss (or Gauss-Legendre) quadrature[1,5,6], in which the integrand is evaluated and multiplied by a weight at pre-selected sampling points. These weighted values are summed to yield, in general, an approximation of the integral. A Gauss quadrature rule using n sampling points integrates exactly a polynomial of up to order 2n-1. Gauss quadrature is inexact for the integrand in (1), which is not a polynomial because of the 1/r terms. In isoparametric formulations, the standard practice is to use a 2-point quadrature rule for two-node line elements and a 2 x 2 quadrature rule for four-node quadrilateral elements[5,6]. However, the authors believe these conventions were developed for one-dimensional (1-D), linear bar and two-dimensional (2-D), plane stress or plane strain cases, respectively, and not specifically for axisymmetric elements, which are unique because of the 1/r terms. Expression (1) for axisymmetric elements far from the axis may be integrated very accurately using Gauss quadrature since the 1/r function changes slowly as r varies for large values of r and may be accurately approximated by a polynomial.

Standard Gauss quadrature also often yields incorrect results for nearly singular integrals, especially when only a few sampling points are used[3]. Therefore, the usual 2-point (1-D) or 2 x 2 (2-D) Gauss quadrature conventions may be insufficient for evaluating the integral in (1) for nearly singular elements. However, no FE textbooks present special rules for addressing the nearly singular case for axisymmetric formulations; the conventional Gauss quadrature rules are recommended for all elements, regardless of proximity to the axis of rotation. Only Cook[7] suggests using more sampling points in the standard Gauss quadrature routine for elements close to the axis of rotation. However, this reference does not specify how many additional points should be used or how close the elements must be to the axis to merit the use of these extra points.

Because conventional Gauss quadrature often yields erroneous results for singular and nearly singular integrations, various modified quadrature rules have been developed for evaluating these kinds of integrals. This paper examines the use of one such modified Gauss quadrature rule, developed by Telles[3,4], employing a cubic transformation that shifts the sampling points closer to the singularity.

2. AXISYMMETRIC WASHER ELEMENT

The axisymmetric washer element, obtained from Problem 10.6 in Cook et al.[6], is shaped like a thin, flat metal washer. This element is examined in detail because of the simplicity of its stiffness matrix, $\underline{K}_E$ (which has only four terms). The following subsections define the washer element, present this element's integral formulation of $\underline{K}_E$, and discuss the application of explicit and numerical methods to obtain the integrated $\underline{K}_E$.

2.1 Element Definition

The cross-section of the axisymmetric washer element, shown in Figure 2, includes two nodes, with nodes 1 and 2 located at radial coordinates r_1 and r_2, respectively. The complete element is this cross-section rotated about the z-axis. Each node has a single translational degree of freedom (DOF) in the radial direction, denoted as u_1 and u_2 for nodes 1 and 2, respectively. The assumptions for this element include the following: geometrically symmetric about the z-axis; loadings and boundary restraints symmetric about the z-axis; element of constant thickness t; element is thin, i.e., no axial normal stress and no shear stresses; material is isotropic, homogeneous, and behaves in linear elastic manner.

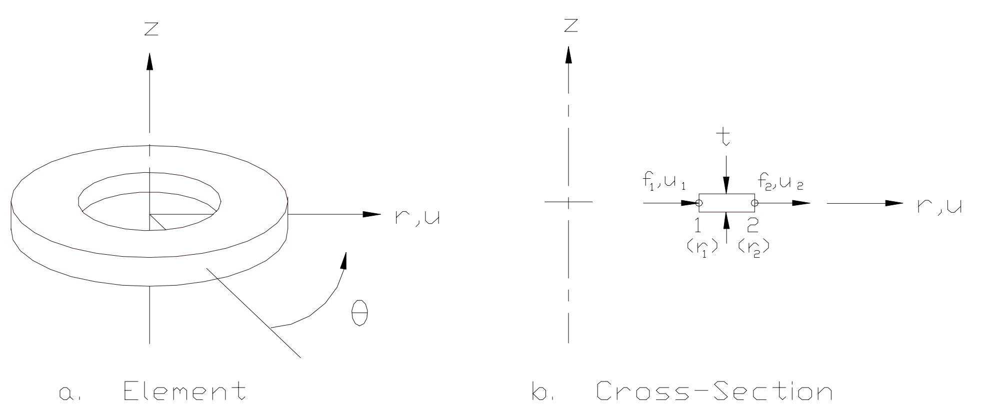


Figure 2: Washer Element and Cross-Section.

2.2 Element Stiffness Matrix Formulation

A linear radial displacement function, u(r), is assumed for this element. The strains are the radial ε_r and hoop ε_θ, and the stresses are radial σ_r and hoop σ_θ. The element stiffness matrix for the washer element is given as

$$\underline{K}_E = \frac{2\pi Et}{(r_2-r_1)^2(1-v^2)}\int_{r_1}^{r_2}\begin{bmatrix} 2(1+v)(r-r_2)+\frac{r_2^2}{r} & -2(1+v)r+(1+v)(r_1+r_2)-\frac{r_1r_2}{r} \\ -2(1+v)r+(1+v)(r_1+r_2)-\frac{r_1r_2}{r} & 2(1+v)(r-r_1)+\frac{r_1^2}{r}\end{bmatrix} dr \qquad (2)$$

2.3 Explicit Integration of $\underline{K}_E$ for the Washer Element

Each term in the $\underline{B}^T$ $\underline{D}$ $\underline{B}$ r integrand in (2) contains terms with r in the denominator. As the lower limit r_1 approaches zero (or as the element location shifts closer to the axis of rotation), these terms behave very strongly, and the integral becomes nearly singular, for an element very close to the axis, or singular, for an element on the axis (refer to Figure 1). For the *non-singular* case (includes nearly singular), expression (2) may be integrated explicitly to yield the following:

$$\underline{K}_E = \frac{2\pi\ Et}{(r_2-r_1)^2(1-v^2)}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22}\end{bmatrix} \qquad (3a)$$

where

$$k_{11} = (1+v)(r_2^2-r_1^2)-2(1+v)(r_2^2-r_1r_2)+r_2^2\ln\frac{r_2}{r_1} \qquad k_{12} = k_{21} = -r_1r_2\ln\frac{r_2}{r_1} \qquad (3b)$$

$$k_{22} = (1+v)(r_2^2-r_1^2)+2(1+v)(r_1^2-r_1r_2)+r_1^2\ln\frac{r_2}{r_1}$$

2.4 Numerical Integration of $\underline{K}_E$ for the Washer Element

For expression (2) to be integrated numerically using Gauss quadrature, the integral is written in terms of the natural coordinate, s, rather than the axisymmetric coordinate r, using the following linear transformation from r to s:

$$r = \underline{N}\ \underline{r}_E = \begin{bmatrix}\frac{1}{2}(1-s) & \frac{1}{2}(1+s)\end{bmatrix}\begin{Bmatrix} r_1 \\ r_2\end{Bmatrix} \qquad (4)$$

The Jacobian for this transformation is

$$J = \frac{dr}{ds} = \frac{r_2-r_1}{2} \qquad (5)$$

The limits of integration in the natural coordinate system become ± 1. To use Gauss quadrature to obtain $\underline{K}_E$ for the washer element, (4) and (5) are substituted into (2), and the integral becomes (not expanded)

$$\underline{K}_E = 2\pi\ t\int_{-1}^{1} \underline{B}^T\ \underline{D}\ \underline{B}\ r\ J\ ds = 2\pi\sum_{i-1}^{n} \underline{B}^T(s_i)\ \underline{D}\ \underline{B}(s_i)\ r(s_i)\ J\ w_i \qquad (6)$$

where J is given by (4), n denotes the number of Gauss points, i signifies the Gauss point and w is the weight.

2.5 Application of the Telles Transformation for the Washer Element

The cubic transformation by Telles[3,4] improves upon the accuracy of standard Gauss quadrature for nearly singular integrals by shifting the sampling points closer to the location of the singularity, thus accounting for the strong behavior of the integrand in the proximity of the singular point. When the transformation is used for the washer element, the sampling points move closer to r = 0 as the integral becomes more nearly singular. For regular integrals, the transformation degenerates to conventional Gauss quadrature, with the usual sampling points; therefore, the transformation technique may be safely applied to all elements in a mesh, whether regular, singular, or nearly singular. To apply this transformation technique, expressions (4) and (5) are substituted into the expanded $\underline{B}^T \underline{D}\, \underline{B}$ r product from (2). The entries in the $\underline{B}^T \underline{D}\, \underline{B}$ r product are then divided into constant terms, terms linear in r, and terms of the order 1/r. The constant and linear terms are integrated using conventional Gauss quadrature, each evaluated at the usual Gauss-Legendre sampling points. The 1/r terms are evaluated at the transformed (Telles) sampling points and are then multiplied by the Jacobian of the cubic transformation (different from the Jacobian in (4)). The location of the singularity in the s-domain (required to employ the transformation) is found by setting r = 0 in (3). The authors caution that the transformed sampling points found using the singularity at r = 0 may not be applied to every term in every entry of the $\underline{B}^T \underline{D}\, \underline{B}$ r matrix. Only the terms of order 1/r are undefined at the singularity point and can be evaluated at the transformed points. Thus, a major drawback in applying this technique is that the transformation must be used selectively on certain terms in the expanded $\underline{B}^T \underline{D}\, \underline{B}$ r product. For the washer element, the three unique entries (k_{11}, k_{12}, and k_{22}) in the $\underline{B}^T \underline{D}\, \underline{B}$ r product are fairly short, and applying the rule is not too difficult. However, symbolically expanding the $\underline{B}^T \underline{D}\, \underline{B}$ r integrand for more complicated elements (such as the 8 x 8 integrand for the four-node quadrilateral) and applying the rule to all the 1/r terms is much more tedious. In this study, when the Telles transformation was used, all terms in the integrand, whether requiring the conventional Gauss or the transformed sampling points, were evaluated using the same number of sampling points (the same order of quadrature).

3. EXAMPLES AND RESULTS

Figure 3 shows the axisymmetric problem used to judge the effectiveness of employing the conventional Gauss quadrature rules to obtain the element stiffness matrix for nearly singular elements. The problem consists of a flywheel (or disk) rotating at constant angular velocity ω. The disk's cross-section has length L and uniform small thickness t. Numerical values for all constants in the model--t, L, ω, and material properties--are included in Figure 3. A disk with a small hole (or pinhole) was used to model the nearly singular case. The hole size was varied in terms of element lengths in the radial coordinate; the smaller the hole, the more nearly singular the stiffness matrix integrand for the innermost element in the FE mesh. Gravity forces in the axial direction were neglected, leaving the centrifugal forces in the radial direction as the only forces acting on the body. The element body force vector was evaluated exactly for this study. The FE analyses were performed using the MATLAB software package, with codes based on programs from Kwon and Bang[8]. Only errors in nodal displacements in the FE solutions were considered since once the nodal displacements are found, the stiffness matrices are no longer used. Errors in strains and stresses were not considered because these are determined directly from the nodal displacements; additional errors not reflected in the nodal displacements due to the inaccurate integration of $\underline{K}_E$ are not introduced during strain and stress calculations.

The boundary condition of zero radial displacement at the innermost node is not applicable for nearly singular elements, so the problems with numerical integration of $\underline{K}_E$ are not so easily solved. The goal for the study of the nearly singular washer element was to determine a *universal* quadrature rule to apply to every element in the FE mesh, based on the distance of the innermost element to the axis of rotation. This distance is normalized in terms of the ratio r_m/l_E,

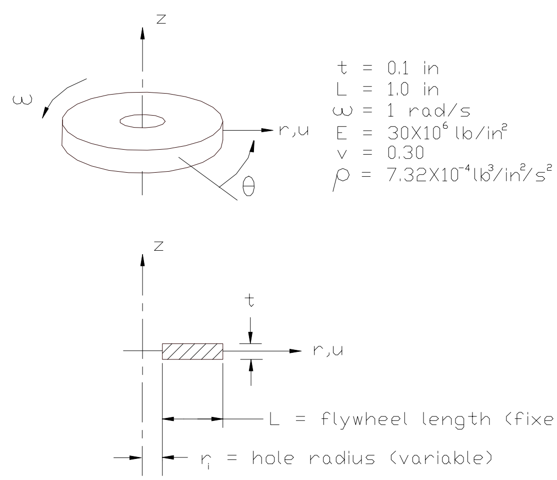


Figure 3: Rotating Flywheel Problem.

where r_m is the radial coordinate (in the global coordinate system) of the centroid of the element (the midpoint of the innermost element for the washer) and l_E is the length of the element (the integration domain). Figure 4 defines r_m and l_E for meshes of one, two, four, and eight equal-sized washer elements. A r_m/l_E value of 0.5 denotes a singular element, with a node at r = 0. Nearly singular elements have r_m/l_E values slightly greater than 0.5. The minimum r_m/l_E value considered for this study (of the nearly singular case) was 0.51, corresponding to a disk having a hole with a radius of 1/100 of the length of an element. The authors believe that stronger nearly singular cases ($0.5 < r_m/l_E < 0.51$) are unlikely to occur in practice. Errors in the nodal displacement u_1 at the innermost node were determined using both standard Gauss quadrature and the Telles quadrature rule (applied only to the 1/r terms, with conventional Gauss quadrature applied to the other terms, as described in Section 2.5) for integrating $\underline{K}_E$. Relative percentage errors were calculated in comparison to the FE solution obtained for u_1 when explicit integration was used to evaluate $\underline{K}_E$ (the explicit integral is given in Equations (3)). Errors in nodal displacements were most profound at the innermost node, and were less significant for nodes farther from the z-axis, hence the focus on error in u_1.

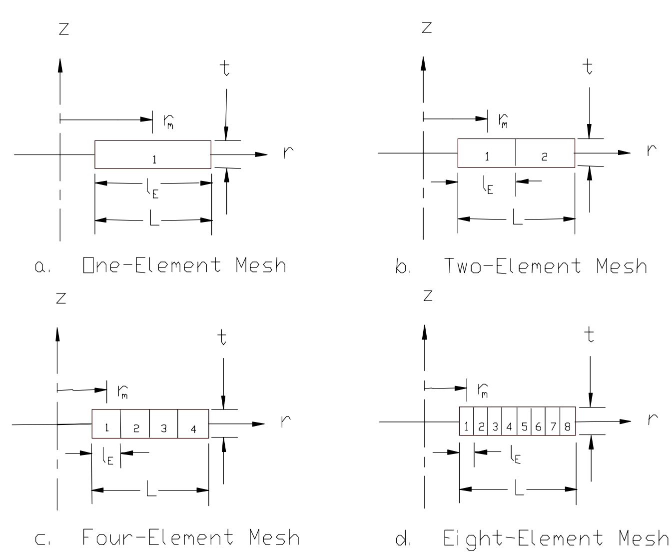
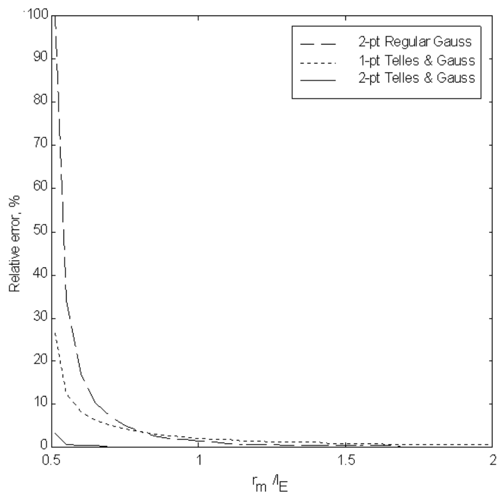
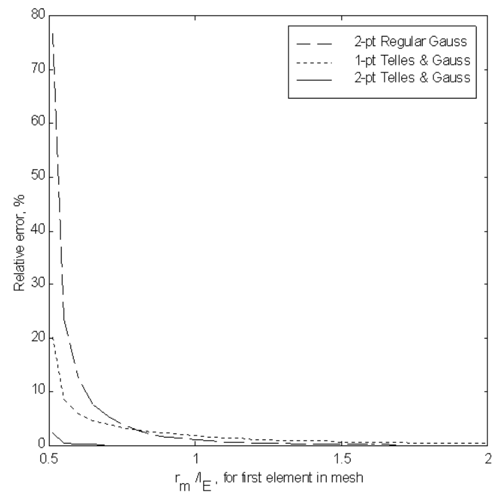


Figure 4: Washer Element Models of Flywheel.

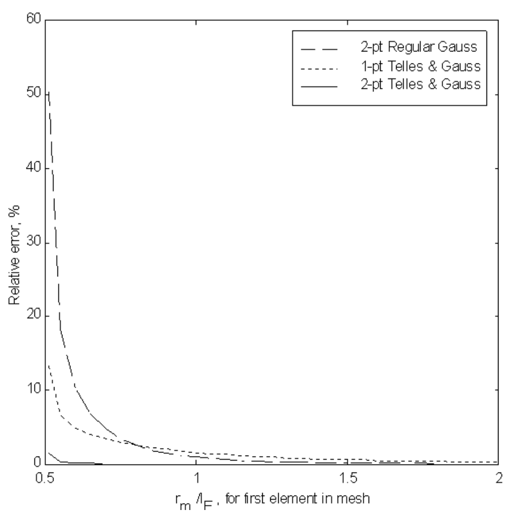
Figures 5a-d compare, for meshes of one, two, four, and eight elements, the error in u_1 when the usual 2-point regular Gauss rule[5,6] and the 1- and 2-point Telles rules are used to integrate $\underline{K}_E$. The lower limit of the ordinate for Figures 5a-d is 0.51; at this lower limit, e_u is nearly 100% for the 2-point Gauss rule for the one-element mesh. These figures show that the conventional 2-point Gauss rule is inadequate for nearly singular elements, and that using the Telles transformation results in much higher accuracy for low ranges of r_m/l_E. The 2-point rule by Telles offers significant improvement over the conventional 2-point Gauss rule for $r_m/l_E < 1.5$. Even the 1-point Telles rule betters the 2-point Gauss rule for $r_m/l_E < 0.75$.



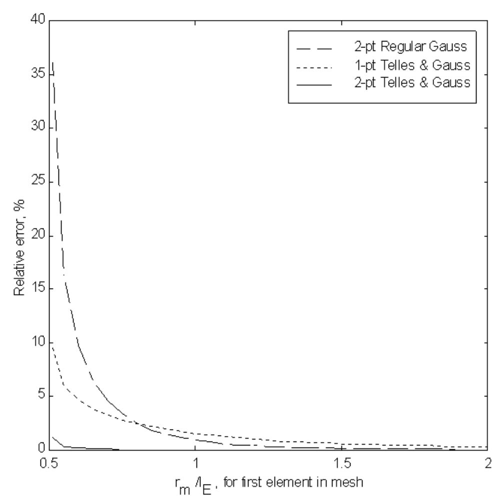
a: One-Element Mesh.



b: Two-Element Mesh.



c: Four-Element Mesh.

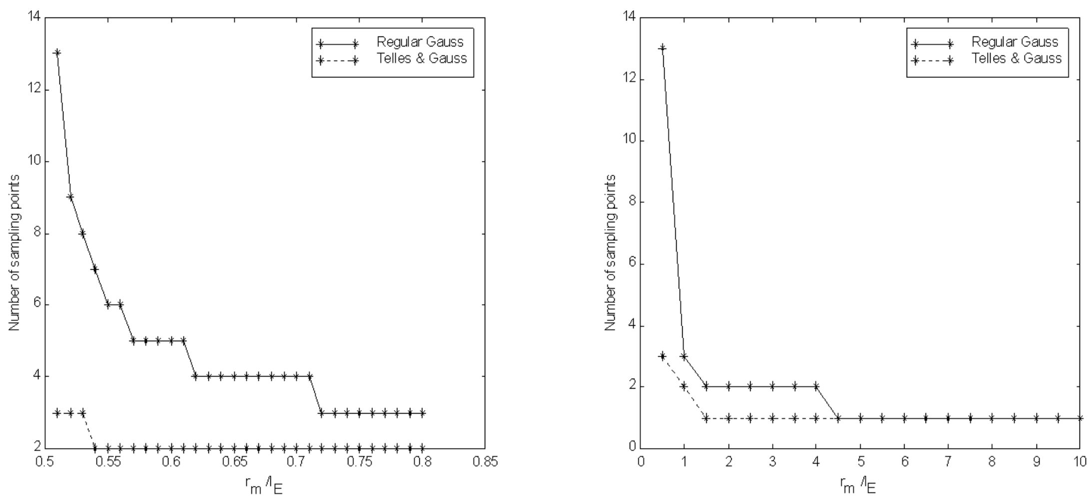


d: Eight-Element Mesh.

Figure 5: Displacement Relative Percentage Error (e_u) for u_1.

Figure 6a shows the number of sampling points required by the conventional Gauss method and by the Telles method to achieve less than 1% error in u_1 in a one-element mesh, for $0.51 \le r_m/l_E \le 0.80$. The number of sampling points required for an accurate value of u_1 increases as the r_m/l_E ratio decreases, i.e., as the hole shrinks, the stiffness matrix integrals become more nearly singular. When this ratio is 0.51 (the lower limit of the figure), standard Gauss integration requires 13 points for $e_u < 1\%$, but the Telles method requires only 3 points. In Figure 6b the upper limit of the ordinate of Figure 6a is extended to $r_m/l_E = 10$.

Based on Figures 6a and b, the authors recommend the quadrature rules presented in Table 1. For the washer element, the r_m/l_E value of the innermost (most nearly singular) element is used to select the integration rule to be applied universally to all elements in the mesh. Note that the suggestions in Table 1 are conservative since they are based on a one-element mesh. Also be aware that use of Table 1 for a different problem does not guarantee less than 1% error in the FE



a: $0.51 \le r_m/l_E \le 0.80$ b: $0.51 \le r_m/l_E \le 10$

Figure 6: Number of Sampling Points for e_u<1%, One-Element Mesh.

solution; the 1% value applies only when comparing the FE solutions for explicit and numerical integration of $\underline{K}_E$ for this specific flywheel problem. Nevertheless, adhering to these guidelines for strongly nearly singular washer elements in other axisymmetric problems should provide much higher accuracy than the conventional 1- or 2-point regular Gauss quadrature rules. The 1-point Telles rule is not recommended since it is less accurate than the 2-point Gauss rule for $1.50 \le r_m/l_E \le 4.50$. Recall that the Telles rule may be applied safely to all elements in the mesh since the modified rule degenerates to conventional Gauss quadrature for elements far from the axis of rotation. If equal-sized elements are used, nearly singular washer elements benefiting from the Telles transformation will not occur in a model of a solid disk. This is because the r_m/l_E value of the second element from the axis of rotation is 1.5, which requires only a 2-point regular Gauss integration for acceptable accuracy, as suggested in Table 1. Should the FE analyst/programmer prefer not to employ the Telles transformation technique, a large number of Gauss sampling points is required for acceptable accuracy for r_m/l_E values very close to 0.50. Table 2 presents reliable integration orders for washer elements when only regular Gauss quadrature is used, based on the results shown in Figures 6a and 6b. The quadrature rules suggested in this table may be applied separately to each element in a mesh (based on that element's r_m/l_E value). This was done in order to cut down on the computational effort that would be required if a large number of sampling points were used for integrating every element stiffness matrix.

Table 1. Numerical Integration Orders in Evaluation of Stiffness Matrices of Regular and Nearly Singular Washer Elements.

Location of Innermost Element	**Reliable Integration Order***
$0.51 \le r_m/l_E < 0.54$	3-point Telles & Gauss
$0.54 \le r_m/l_E < 1.50$	2-point Telles & Gauss
$1.50 \le r_m/l_E < 4.50$	2-point Regular Gauss
$r_m/l_E \ge 4.50$	1-point Regular Gauss

*Note: Same rule to be applied to all elements in a mesh.

Table 2. Numerical Integration Orders in Evaluation of Stiffness Matrices of Regular and Nearly Singular Washer Elements, Applying Conventional Gauss Quadrature Exclusively.

Location of Innermost Element	**Reliable Integration Order***
$0.51 \le r_m/l_E < 0.52$	13-point Regular Gauss
$0.52 \le r_m/l_E < 0.53$	9-point Regular Gauss
$0.53 \le r_m/l_E < 0.54$	8-point Regular Gauss
$0.54 \le r_m/l_E < 0.55$	7-point Regular Gauss
$0.55 \le r_m/l_E < 0.57$	6-point Regular Gauss
$0.57 \le r_m/l_E < 0.62$	5-point Regular Gauss
$0.62 \le r_m/l_E < 0.72$	4-point Regular Gauss
$0.72 \le r_m/l_E < 1.50$	3-point Regular Gauss
$1.50 \le r_m/l_E < 4.50$	2-point Regular Gauss
$r_m/l_E \ge 4.50$	1-point Regular Gauss

*Note: Rule may be applied separately to each element in a mesh.

4. CONCLUSIONS

The authors found that the using the conventional 2-point Gauss quadrature for integrating the element stiffness matrix for nearly singular washer elements (with a node close to the axis of rotation) yields unacceptable accuracy and suggest two approaches. The first integration approach is based on using a modified quadrature rule by Telles. Table 1 presents recommended universal quadrature rules to be applied for all elements in a washer element mesh, based on the closeness of the innermost element to the axis of rotation and the length of the element in the radial direction. The second integration approach is based on using the conventional Gauss quadrature. Table 2 gives the recommended rules to be applied selectively to each element in the mesh based on that element's distance from the axis of rotation and length. Because the washer element can only be used to model simple, 1-D axisymmetric problems, this type of element is of limited use to the practicing engineer. Therefore, this study needs to be extended to other, more practical elements such as the four-node quadrilateral.

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