Combining Wavelets in a Digital Signal Processing Course

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Abstract

Wavelets is a relatively new topic in engineering and computer science. Some of its applications are in the area of digital signal processing. This paper suggests to integrate a session in a digital signal processing class at an undergraduate level that would expose students to some of the basic topics of the Wavelet Transform and its applications in the analysis of signals (e.g. detection of discontinuity, detection of self similarity and denoising of signals) as well as image compression. This would enhance the DSP class and enrich students view of the time - frequency relations.

Introduction

A session on the Wavelet Transform in a DSP course should present the topic in the context of time-frequency representation of signals. Probably the most natural point to start with is periodic signals, where the most natural tool for their representation would be the Fourier Series. It will provide the amplitudes of the frequency components of the signal, the fundamental frequency and its harmonics. In the case of non-periodic signals, we can apply the Fourier Integral. While it provides information about the frequency components comprising the signal, this transform does not provide information on the time-frequency relations in the signal. The Fourier Transform does not provide information that would let us associate certain events (e.g. abrupt changes, long term behavior of a signal) with certain points of time. This presentation can provide a motivation for the Wavelet Transform.

The Windowed Fourier Transform (WFT)

A partial solution to the above problem would be to partition the signal into sections (the width of which depends on the application), then apply the Fourier Transform to each piece. This is the Windowed Fourier Transform (WFT). Clearly questions regarding when a section should begin and
end will come up and without getting into the technical details it is reasonable to assume that smoothing each section before applying the Fourier Transform would provide better results regarding the frequency spectrum of each section. This would determine the type of window we use to extract sections of the signal. The Windowed Fourier Transform is a compromise in which we sacrifice precision, to provide information both in the time and the frequency domains.

The solution provided by the WFT to the problem of getting simultaneous information on the time-frequency aspects of a signal provides the direction of the Wavelet Transform. Clearly, to get information about the low frequency components one has to observe the signal for a longer period of time than that needed to get information about the high frequency components at a section. This brings us to the idea of a flexible window size rather than the fixed window used in the WFT. Clearly by choosing the size of the window we would be able to capture information on both low and high frequencies and relate them to the time axis.

**The Scaling Function of the Wavelet Transform**

The general mathematical framework of the Wavelet Transform is quite involved. There is a wide range of literature which present the topic at different levels of depth. Here we provide a short, descriptive and simplistic background without getting into any rigorous mathematical analysis of the subject. A good starting point to describe the mathematics of the wavelet transform is linear algebra: presentations of vectors in some n-dimensional vector space \( \mathbb{R}^n \).

Given a set of vectors which constitute a basis for this space, any vector can be presented as a linear combination of these basis vectors. We apply this idea to a set of data under consideration (whether the data is a sampled image or a biological signal like ECG). In wavelet analysis the basis functions consist of a single function, called the scaling function, \( \phi \). The Wavelet Transform uses a single scaling function to create a basis to present the signal. By being able to choose various scaling values, we can choose the size of the window we use to analyze the signal.

We illustrate the concept of the scaling function using the 'Haar' wavelet. This is the simplest basis to present a signal. A scaling function for the 'Haar' wavelet is the box function \( \phi(x) \):
\[ \phi(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

and its scaled, translated versions would be the set:

\[ \phi_i(x) = \phi(2^i x - i) \quad i = 0, 1, ..., 2^j - 1 \]

where in this case \( j \) represents the resolution of the basis. The effect of the scaling functions, when the signal is presented as a linear combination of these functions, is to average sections of the signal.

As an example, if \( j=2 \), the basis function would be the set of 4 box functions:

\[ \phi(2^4 x), \phi(2^4 x - 1), \phi(2^4 x - 2), \phi(2^4 x - 3) \]

Given a signal \( s(x) \), it can be approximated by these 4 functions as:

\[ s(x) = c_0 \phi(2^4 x) + c_1 \phi(2^4 x - 1) + c_2 \phi(2^4 x - 2) + c_3 \phi(2^4 x - 3) \]

Clearly, the coefficients provide information about the 'average' value of the signal at each of the sections.

By using a different value for \( j \), we can provide an 'average' description of the signal at different levels of precision. We can describe the signal at different levels of resolution. This would be analogous to viewing the signal with a magnifying glass at various levels of magnification.

The Wavelet Transform and Wavelet Functions

The higher the value of \( j \), the higher the resolution with which we present the signal. During this averaging process we lose some information. This information is provided by the wavelet function that is associated with the differences between consecutive values of the signal, which are lost during the averaging process done by the scaling function.

The set of wavelet functions are also derived from a single function by scaling and translation. Continuing with the 'Haar' wavelet, the wavelet function corresponding to the 'Haar' basis is the function
and a family of wavelet functions corresponding to a resolution level $j$, would be just a set of $j$ scaled and translated versions of $\psi(x)$, i.e.:

$$\psi^j(x) = \psi(2^j x - i) \quad i = 0, 1, ..., 2^j - 1$$

Each of the translated wavelet functions provide 'local' information on changes in the signal at different sections.

The wavelet transform of a signal is composed of these two sets of coefficients derived by applying the scaling functions and the wavelet functions to the data. The scaling function provides information on averages value of the signal at each section and the wavelet functions provide information on changes in the signal in each section. Together, they can provide all the information needed to reconstruct the signal.

Moreover, we can apply the WT transform at a high resolution, getting information on average values of the signal and information on changes in signal. Then we can apply a second iteration of the transform to the set of coefficients representing the average of the signal. Continue this process the original data will be presented at each level at a decreasing level of resolution (compared to the previous level). This process enables us to present the original signal at different levels of resolutions, called "multiresolution" and is very useful for certain applications.

Image Compression, Image Query and the Wavelet Transform

The 'Haar’ wavelet is the simplest wavelet. Other wavelets were developed (e.g. the famous "Daubechies Wavelets") with greater efficiency for certain applications. For example, in image compression, a good basis would be one that among other things, can present the image by a small set of nonzero coefficients. In other words we want a basis in which most of the coefficients are either zero or have small values so that if they are ignored in the reconstruction of an image from its wavelet coefficients, the degradation of the
image, when compared to the original, is very small. For certain Daubechies Wavelets, we can achieve compression ratios which are comparable to the best compression algorithms known.

In the area of image query, wavelets were found to be very useful too. Image query for large databases presents difficult computational and storage problems. Using wavelets, images can be represented by a small set of wavelet coefficients (the set is composed of the largest, in magnitude, of the wavelet coefficients of the image). Then, in the process of an image query, this small set of wavelet coefficients is used to compare the query image to the target image. For certain data bases this technique was found to be very powerful compared to traditional methods (e.g. using feature vectors and color histograms).

Filter Banks: Signal Processing and the Wavelet Transform

Filter banks combine subjects from signal processing and the Wavelet Transform. In the broader view, the wavelet analysis and synthesis correspond to the application of a filter bank where the signal is passed through two channel filter banks: The analysis filter bank, composed normally of a low pass filter, $H_0$, followed by a down sampler and a high pass filter, $H_1$, followed by a down sampler. This filter bank is followed by a synthesis bank composed of upsampling the signal in both channels followed by a pair of filters, $F_0$ and $F_1$ (see fig. 1). The filter bank provides a good example for discussion of aliasing since the down sampling of the signal creates an aliasing that has to be canceled by the pair of filters, $F_0$ and $F_1$. Ideal filters and the right choice of $F_0$ and $F_1$ would enable a perfect construction of the signal.

Other applications

In addition to image compression and image query, the WT transform were found useful in data analysis in a variety of areas. Noise removal, detection of discontinuities (which are not detectable in the time domain), detection of long term behavior of signals and self similarity are just a few examples. The WT seems to be a natural tool in the detection of self similarity (fractal structure) in an image. This is because the set of basis functions used to present the signal are derived from a single function (the scaling function) and are just scaled and translated versions of this single function. This means that a signal with self similarity property, should like the same when looked at different levels of resolution. Thus by an appropriate comparison of the coefficients at different levels of resolution we can get information about
the level of the self similarity of the signal.

**MATLAB Tools for WT**

The Wavelet toolbox offered by Mathworks, Inc. is an excellent pedagogical tool to present wavelets in a signal processing course. Once the students are familiar with the basic ideas of the WT, the learning curve in using the software is relatively short. Starting with some of the demonstration examples, students can learn how to apply the WT to a given signal for some representative applications such as detecting discontinuities, detecting long term evolution of a signal and image compression. The graphical interface provide a convenient tool to view the wavelet coefficients at different levels of the wavelet decomposition and the commands for analysis and synthesis of signals in the command line mode are simple and convenient. The examples contain both one dimensional and two dimensional(images) signals that would make some of the abstract subjects more tangible to students. Reviewing the examples provides a good background to continue with the analysis of signals synthesized by the students in MATLAB or imported out of MATLAB.
Bibliography


