All Day Mathematics Workshops

Chandni Shah, Lindsey Van Wagenen
Department of Mathematics
Polytechnic University

Introduction

One of the challenges that many students face when making the transition from mathematics courses in high school to ones in college is learning to read and dissect a complex problem. Most students achieve success in high school by learning and applying standard methods to solving problems. When in college, the hardest thing for students to learn is the process of struggling with a problem over an extended period of time. Too many teachers, in a misguided effort to boost the students' confidence, avoid assigning problems that have no clear path to the solution. Worse, they only assign problems that are closely modeled on examples, for which step-by-step solutions have been provided. As a result, students learn that it is acceptable *not* to read a problem carefully, and to follow a previously learned procedure. Students develop the view that understanding what a problem is asking and why the procedure for obtaining the answer works is unnecessary and even a distraction. Most students trained in this way are prone to making mistakes due to memory lapses and often misapply formulas or procedures when faced with novel problems.

The Department of Freshman Mathematics (DFM) at Polytechnic University introduced *all day workshops* to address these issues. The goal of

these workshops is to help a student develop the computational and reasoning skills necessary to solve problems completely and without making mistakes. An equally important consequence is that the students also develop self-confidence in their ability to understand a problem and to solve it through deductive reasoning.

The workshops are focused on developing a student's skills in *doing* calculus. We constantly remind the students about the analogies of learning other skills, such as playing basketball or driving a car. A particularly good analogy is learning to become a carpenter. The traditional notion of developing skills in mathematics is through large quantities of computational drills. This is like trying to learn carpentry by hammering a lot of nails, drilling a lot of holes, and sawing a lot of wood. Few of us believe that this is the most effective way to learn to be a carpenter. Instead, it is more effective to develop these necessary skills in the process of building real objects, such as boxes, tables, and cabinets. At the same time that one develops proficiency in basic skills, one also develops the ability to properly design, measure, and assemble the object.

Mathematics is best learned in exactly the same way, by doing it, not by reading about it or listening to someone else talk about it. Moreover, just as a carpenter must learn to decide for herself which tool to use when, a student must learn how to decide which mathematical tools are appropriate in any given situation.

It is crucial that neither instructors nor students tolerate any deficiency or weakness in either the conceptual understanding or the computational skills needed to solve a problem. A carpenter who knows in principle how to build a cabinet, but is unable to saw straight or hammer nails properly is not a good carpenter. Imagine how dangerous it would be if in a city in which many car drivers knew in principle how to drive well but who constantly made mistakes. A student who understands the mathematical concepts necessary to solve a problem, but who makes mistakes in arithmetic or algebra does not know how to do mathematics.

The appropriate role for an instructor is therefore not to dictate procedures but, like a basketball coach, to point out mistakes and deficiencies, to offer

encouragement and reinforcement when correct decisions and actions are taken, and to help a student strengthen basic skills when necessary.

All Day Workshops

The Workshops have three crucial features:

(1) The workshops are mandatory and count as a significant part of students' grades.
(2) Students have almost an unlimited amount of time.
(3) Each student can obtain help from a variety of sources.

Eliminate any one of them and the workshops lose much of their effectiveness. The unlimited amount of time is particularly powerful because it allows the instructors to guide the students without telling them the specific steps to solving a given problem. Instead, one can review generalities, walk a student through simpler analogous problems, and ask leading questions. A student eventually learns how to solve the problem, but does most of the mental work herself. The instructor acts as a facilitator and a "spotter" who helps the student see the mistakes and errors.

Working with a group of other students helps a student see and explore different ideas and approaches that she might not have thought of by herself. Also, the student learns to critique different ideas. It is, however, important for instructors to help the more shy and less confident students express their thoughts and ideas.

Once a student grasps the key ideas of a problem, she is encouraged to explain it to other students struggling with the same problem. Having to teach what she has just learned forces a student to clarify in her own mind her understanding of the problem and its solution. It also provides a strong boost to the student's self-confidence.

Workshop description

In Fall 2000, the DFM implemented a $2 + 2$ curriculum. Each 4-credit semester course in the first year courses has been divided into two 2–credit half-semester courses. In seventh week each student is required to take the

final for her course. A student is allowed to proceed to the next 2-credit course, only if she passes the course. Each 2–credit course is offered during each half of the semester. Along with the new curriculum, the DFM replaced a weekly one-hour recitation with the "All Day Mathematics Workshop" that meets every Friday. Thus for each 2-credit course taken by a student, she participates in six Friday Workshops.

Schedule
Each workshop is held all day Friday simultaneously in four classrooms. Students are allowed to come any time during the day, stay as long as they want, and leave whenever they want.

Staff
The workshops are staffed by faculty, graduate students, and talented undergraduates. Each instructor spends from four to eight hours in the workshop.

Students
Students registered in any of the freshman courses ranging from Introduction to Precalculus to Calculus II are required to attend the workshop. Each student is free to work in any of the four available classrooms.

Worksheets
During each week, and for each of the seven two-credit courses, a worksheet containing three to four problems is developed by the DFM. These worksheets are designed by the DFM to address key weaknesses in the students' skills.

Examples of Problems

Many of our problems focus on a student's understanding of what a function is and her ability to properly interpret notation involving functions. In our experience this (and not the concepts of derivative or integral) is the biggest obstacle to a student's understanding of how to do calculus.

The first example introduces the students to the idea that mathematical formulas and the concept of the function are related to real-world phenomena.

Example from Introduction to Precalculus
Suppose $w = j(x)$ is the average daily quantity of water (in gallons) required by an oak tree of height x feet.

(1) What does the expression $j(25)$ represent? What about $j^{-1}(25)$?

(2) What does the following equation tell you about v: $j(v) = 50$.

(3) Oak trees are on average z feet high and that a tree of average height requires p gallons of water. Represent this fact in terms of j and then in terms of j^{-1}.

(4) Using the definitions of z and p from part (3), what do the following expressions represent? $j(2z)$, $2j(z)$, $j^{-1}(p + 10)$, $j^{-1}(p) + 10$

In the next example students are required to apply the standard product rule of differentiation, but also give a practical interpretation of their answer.

Example from Calculus I
The quantity, q, of a certain skateboard sold depends on the selling price, p, in dollars, so we write $q = f(p)$. You are given that $f(140) = 15,000$ and $f'(140)$=-100.

(1) What do $f(140) = 15,000$ and $f'(140)$=-100 tell you about the sales of skateboards?

(2) The total revenue, R, earned by the sales of skateboards is given by $R = pq$. Find $\left. \dfrac{dR}{dp} \right|_{p=140}$.

(3) What is the sign of $\left. \dfrac{dR}{dp} \right|_{p=140}$? If the skateboards are currently selling for \$140, what happens to the revenue if the price is increased to \$141?

The example below addresses students' difficulties understanding and using the Second Fundamental Theorem of Calculus. Most students seem quite comfortable with indefinite integrals but resist using definte integrals to solve initial value problems. The problem below was designed to get students accustomed to functions defined as definite integrals.

Parts (1) and (3) require students to use integration by parts. They also need to see the relationship between the functions f and G. Part (2) requires the use of substitution. In addition, the students need to recall properties of logarithms and the fact that the exponential and the logarithm functions are inverses of each other.

Finally part (4) requires the students to rewrite the given function as a composition of known functions and then recall the Chain Rule of differentiation to complete the problem.

Example from Calculus II

Let $G(x) = \int_0^x f(t)\,dt$, where f is a continuous function. Some of the values of G and its derivatives are given in the table below.

x	$G(x)$	$G'(x)$	$G''(x)$
-3	5	0.5	-1
-1	1.2	-3	2
0	0	-0.5	1
1	2	3	2
1.5	-2	0	0.3
2	1	7	0
4	4	-2	1

Evaluate the following. Write "NEI" if there is not enough information to give exact answer.

(1) $\int_0^1 x f''(x)\,dx.$

(2) $\int_{\ln(\sqrt{3})-\ln(\sqrt{2})}^{\ln(2)} e^{2s} f(e^{2s})\,ds.$

(3) $\int_0^2 x f'(x)\,dx.$

(4) $m'(4)$ where $m(x) = \int_x^{\sqrt{x}} f(t)\,dt$

The following example forces students to dispel some misconceptions. Usually students have difficulty interpreting inverse functions. Even after understanding the concept of an inverse function, there is the added difficulty of dealing with the restricted domain and range. In this problem they not only learn that it is not always true that $f^{-1}(f(x)) = x$ but in order to solve the problem they are required to understand reference angles and how the different trigonometric functions behave in different quadrants.

Example from Precalculus
Evaluate exactly. (Note: None of the expressions simplifies to 4.)

(1) $\sin^{-1}(\sin 4)$

(2) $\cos^{-1}(\cos 4)$

(3) $\tan^{-1}(\tan 4)$

(4) $\sin(\sin^{-1} 4)$

The following requires the ideas underlying the concept of Riemann sums, but is difficult to do by "plugging into a formula". We usually encourage students to do it "without using calculus".

Example from Calculus I
When an aircraft attempts to climb as rapidly as possible, its climb rate decreases with altitude. The table below shows performance data for a certain single-engine aircraft.

| Altitude (1000 ft)	0	1	2	3	4	5
Climb rate (ft/min)	925	875	830	780	730	685
Altitude (1000 ft)	5	6	7	8	9	10
Climb rate (ft/min)	685	635	585	535	490	440

(1) Calculate upper and lower estimates for the time required for this aircraft to climb from sea level to 10,000 ft.
(2) If climb rate data were available in increments of 500 ft, what would be the difference between a lower and upper estimate of climb time based on 20 subdivisions.

The example below tests a student's ability to interpret information given in a graph and to connect it with fundamental concepts in calculus.

Example from Calculus I
Each of the quantities below can be represented in the picture. For each quantity, state whether it is represented by a length, a slope or an area. Then using the letters on the picture, make clear exactly which length, slope or area represents it. [Note: The letters P, Q, R, etc. represent points.]
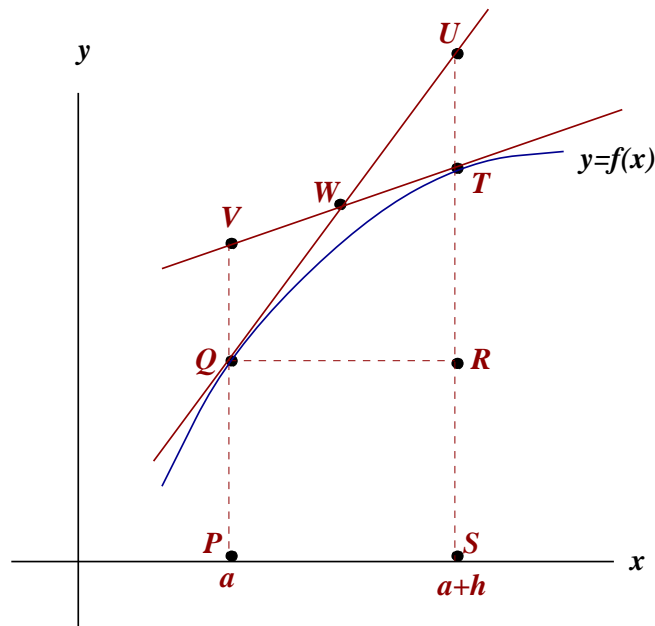
(1) $f'(a)h$

(2) $f(a)h$

(3) State whether the quantity $\dfrac{1}{h} \displaystyle\int_a^{a+h} f(x)\,dx$ is represented by a length or area in the picture. Draw the length or shade the area in the picture above.

(4) Same question as part (3) for $\displaystyle\int_a^{a+h} f(x)\,dx$.

How It Works in Practice

Typically, a student will stay until she has finished solving all of the problems. The student generally begins by trying to answer the problems herself. If unable to proceed with the answer, she solicits help from either a peer or an instructor. The instructor can guide the student in different ways depending on the student's understanding and ability. The instructor may suggest that the student use trial and error, review some of the material from the course or one of its prerequisites, or propose an analogous but simpler question. If a substantial group of students is confused by the same concept, then the instructor will present some general concepts relevant to the problem and then allow the group to work out the solution together. As the semester proceeds, the students become more proficient in the following:

a) Reading and dissecting a problem into what is given and what is to be solved.

b) Carefully selecting from possible solution strategies.

c) Trying strategies and recognizing when to reject one.

d) Detecting errors in reasoning and calculation while solving a problem.

e) Doing reality testing before, during, and after solving a problem.

f) Answering a problem correctly from beginning to end, including the arithmetic.

Why Friday?
Currently most of the departments at Polytechnic University offer lectures Monday through Thursdays. Thus students have a sufficiently light load on Fridays. Typically a student attends a Physics, Chemistry or Engineering lab on Fridays. Although, it is not possible to choose say a specific three-hour block on Friday, if the schedule is flexible enough, each student is able to devote sufficient time to complete the worksheet.

Conclusion

The unlimited time for doing mathematics provided by the all day workshops is ideal, since students are compelled to struggle with problems on their own, while under supervision. An alert instructor offers just enough encouragement, suggestions, and corrections to keep the student working. Since an instructor limits the amount of help given to any specific student at a given moment, the instructor is able to supervise a large number of students of varying talents simultaneously.

Mathematics, like carpentry, basketball, or driving, is learned by doing it, not by watching someone else doing it or talking about it. The give–and–take and the debate that occurs during these workshops is often quite powerful for both the students who participate actively and the ones who simply listen carefully and think quietly.

The workshops have been so successful that the department is planning to implement them for the sophomore level math courses such as linear algebra, differential equations and multivariable calculus. The only question is whether they can be implemented for the entire undergraduate and graduate mathematics curriculum.

Biographical Information

CHANDNI SHAH

Dr. Chandni Shah is a Lecturer of Mathematics at Polytechnic University. She also serves as the Director of Freshman Mathematics. Dr. Shah received her B.S. and M.S. degrees in Mathematics from University of Illinois at Chicago and a Ph.D. in Mathematics from the University of Texas at Austin.

LINDSEY VAN WAGENEN

Dr. Lindsey Van Wagenen is an Instructor of Mathematics at Polytechnic University. She also serves as the Associate Director of Freshman Mathematics. Dr. Van Wagenen received her B.A. in Mathematics and Physics from the University of Oregon, an M.S. in Physics from Brown University and a Ph.D. in Applied Physics from Columbia University.