A Comprehensive Step-by-Step Approach for Introducing Design of Control System

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Abstract

Control Systems classes cover many new topics and concepts. By the time instructors begin to teach design of controllers, students have already learned some analysis and synthesis tools such as Root Locus, Routh-Hurwitz criteria for stability, Bode plot and its relation to stability, and Nyquist plot and its stability criterion. The problem is that at this point in time, these topics can still be fragmented and partially disconnected in students’ minds. Oftentimes, the relations between the topics are not fully clear.

When it comes to design of feedback control systems, we have repeatedly found out that, despite continuous attempts to improve our teaching, there were still some problems in students’ understanding. These include:

•Connecting the concept of a controller to real-life, and to sensing-based daily examples
•Understanding the true meaning of controller design and its implementation
•“Translating” the plant model and the design specifications to different control tools, and inter-relating them

In this paper, we report on work in progress of an intuitive and visual approach to teaching design of controllers in a closed loop control system using a specific comprehensive third order system. For example, all explanations use different colors consistently to show stability (green), instability (red), and marginally stable (orange) systems on related plots.

This paper presents several topics, such as clarifying the meaning and importance of controllers, through daily, story-telling-based examples. It uses a comprehensive approach for analyzing and understanding a plant (to be controlled) in open loop, the controller, and the final design. For example, multiple synthesis tools are used to analyze the plant’s open loop gain and show its effect on closed loop marginal stability; this shows the effect of the open loop gain on the Bode Plot (shift in gain expressed in dB), the Nyquist Plot (shrinking and expanding effect of the plot), the Root Locus plot (new locations of closed loop poles), and the first column of Routh Table.
In addition, this paper shows the effects of different controllers (P, I, D, PD, PI, and PID) and their relations to the desired performance. We intentionally show unsuccessful designs: this helps in explaining some pros and cons of different controllers. This is followed by a successful design of a controller. Lastly, we present multiple ways to observe and analyze the effect of the final controller design using multiple design tools, as well as MATLAB simulations. This also includes discussing design “rules of thumb” and how they are manifested in each tool.

It should be noted that the material presented in this paper is not meant to replace existing textbooks chapters. It is merely an add-on to better explain, learn, and comprehend the topic of design, and see the bigger picture. This is work in progress. However, we have tested the approach a few times and received a very positive feedback from students. A more comprehensive assessment approach is planned for the near future.

I. Introduction

Learning styles of many students are reshaping [1], [2]. Due to the increase in preference for visual media, and the fast growing use of information technology, instructors may notice that it is harder for students to understand difficult concepts when using traditional textbook-based explanations. Such a case is noted by Tyler DeWitt, a chemistry high school teacher and Ph.D. student at MIT [3]. He noticed that his students missed key concepts although they were attending well planned lectures and completing assignments. To remedy this, he engaged students with a different style of teaching that made the subject less intimidating and more fun.

Much like what DeWitt has noted, we have observed that there are students who have successfully completed the Control Systems 1 class, yet are still having trouble in grasping some key concepts. When it comes to design of control systems, we have repeatedly found out that, despite continuous attempts to improve our teaching, there were still some problems in students’ understanding. These include: connecting the concept of a controller to real-life examples, understanding the true meaning of controller design and its implementation, and “translating” the plant model and the design specifications to different control tools.

Some excellent controls textbooks present the material in bits and pieces and sometimes do not “connect the dots” for a better comprehension (see for example [4], [5], [6], and [7]). Essentially, students are presented with new information which relies on mental structures of prior knowledge. Without consistency between the structures and the new information, the new information will probably not be fully incorporated [8]. This inconsistency, coupled with the “rapid-fire” succession of equations thrown at students, is often overwhelming [9].

When it comes to conceptual understanding of control systems design, there seems to be a disconnect between traditional textbooks and students’ conceptual understanding. More recent books have been published on the premise of taking advantage of the growing trend of visual learning to create intuitive analogies [10], [11]. There are also many experiments in which this idea of “different teaching” is tested. For example, we have even seen encouraging preliminary results when teaching a Dynamics course using interactive video games [12], [13]. Given the
vast amount of innovations that make the web more available to people, we begin to see new developments spring forth from this new environment.

This paper focuses on a visual, intuitive, and comprehensive approach to teaching design of control systems. It presents several topics such as clarifying the meaning and importance of controllers through daily, story-telling-based examples. It introduces a comprehensive approach for analyzing and understanding the plant in open loop, the controller, and the final design. The paper “connects the dots” by relating different control tools so that students can get the “bigger picture” and comprehend the material better. For example, using one specific example, it details the effect of changing the open loop gain on the Bode Plot (i.e., shift in gain expressed in $dB$), the Nyquist Plot (i.e., shrinking and expanding effects of the plot), the Root Locus plot (i.e., new locations of closed loop poles), and the first column of Routh Table. The same multiple tools are used also to analyze the same system’s open loop gain and to show its effect on closed loop stability and stability margins.

In addition, this paper shows the effects of different controllers (P, I, D, PD, PI, and PID) and their relations to the desired performance of one specific system. The paper also shows the pros and cons of each controller, and why they can or can’t be used. In other words, it shows unsuccessful designs for a controller before designing a successful one. Lastly, the paper refers to multiple ways for analyzing the effect of the final controller design using multiple analytic and simulation (using MATLAB) design tools.

This paper is a work in progress. Further experimentation and assessment will be designed to test its effectiveness on students. Initial responses from students are very positive.

It should be noted that the material presented in this paper is not meant to replace existing textbooks chapters. It is merely an add-on aimed at better explaining, learning, and comprehending the topic of design.

II. Understanding the Meaning of a Controller

To operate effectively, a car needs a driver to sense, compare, decide, and then act. As we drive, we sense the environment using our vision, hearing, and touch. We then compare the environment to our desired destination and make a decision. By acting on decisions, a driver controls the vehicle, resulting in changes to speed and direction. The combination of a driver and a car becomes one system that moves towards its desired destination.

The main sensory information that we use for driving is visual. We constantly observe the environment while continually keeping in mind our desired route to the destination, and act accordingly. When we act, we affect the steering wheel, braking, or accelerating pedals, which in turn affect the speed and direction of the vehicle.
The process of sensing, making a decision, and acting on the decision can be represented using a block diagram, as seen in the following figure.

Figure 2.1 – Sensing and Deciding

The driver controls the car’s pedals (acceleration and braking) and the steering wheel, causing the car to change its speed and heading.

Figure 2.2 – Open Loop Car Block Diagram

The driver then looks ahead to see if any changes in the pedal and steering wheel are needed. For example, he/she may want to stay a certain distance from the car ahead. As the driver sees
the car in front become too close, he/she may step on the breaks or change lanes. Thus, the driver changes the car’s speed and heading and the process repeats.

**The Big Question**

How can we design a controller that meets the desired closed loop specifications?

![Feedback Diagram](Figure 2.4 – The Right Controller)

There is no unique controller that can give the desired closed loop behavior. In the above figure, a plant is given and a controller must be designed to result in an output that is the same or as close as possible to the desired input. In order to have a better grasp on choosing the right controller, let us first see some effects that a controller can have on a system. Controllers can be implemented in many ways, electrical, digital, pneumatic, etc.
III. Relating the Tools

In this paper, the relationships of design tools are shown in an attempt to connect the tools and help alleviate the fragmentation experienced by some students. In order to design controllers, engineers use different design tools. These tools are used to gauge the effect of controllers on closed loop behavior, usually using open loop information, to ensure that the design specifications are met. They provide information on the design from different points of view. Some of these tools, their purpose, and when to use them are collectively summarized in Figure 3.1 – Tools Chart.

Figure 3.1 – Tools Chart

Figure 3.1 shows some of the common uses of some design tools. In this section of the paper we detail the relationship between the tools. Specifically, we relate the Root Locus to the Routh Table, Bode Plot, and Nyquist Plot. In addition, we also relate the Nyquist Plot to the Bode Plot and, again, the Root Locus.
A. Relating Root Locus to the Routh Table

Given the following characteristic equation:

\[ 1 + \frac{K}{s \cdot (s + a) \cdot (s + b)} = 0 \]

We can use the Routh Table (which is not shown) to find the critical \( K \) to be:

\[ Critical \ K = K_{CR} = (a + b) \cdot ab \]

Then, we relate this to the Root Locus of the characteristic equation seen in Figure 3.2.

In Figure 3.2, the Root Locus plot is colored to signify the location of closed loop poles for a stable system (Green for small \( K \), \( K < K_{cr} \)), marginally stable system, (Orange for Critical \( K \), \( K = K_{cr} \)), and unstable system (Red for large \( K \), \( K > K_{cr} \)). This helps show to the student the points of stability using the colors as using black may not provide the same insight at first.
B. Relating Root Locus to Closed Loop Step Responses

Here we relate points on the Root Locus, i.e., points that relate to different values of $K$ to the corresponding closed loop step responses in the time domain. For illustration purposes, the arrows are pointing at only one pole’s branch, of the three, which are part of the Root Locus. This method of showing a particular $K$ on the Root Locus will be used for the remainder of this section.

![Root Locus Diagram](image)

Figure 3.3 – Root Locus and Time Domain (Step Response)

The following is a summary of the above qualitative visualization of the closed loop step response for different values of $K$.

- $K < K_{CR}$, stable closed loop system, shown in green
- Another $K < K_{CR}$ but larger. Stable closed loop system, shown in green
- $K = K_{CR}$, marginally stable closed loop system shown in orange, and
- $K > K_{CR}$, unstable closed loop system shown in red
Note the responses: When $K$ is small, there is no overshoot. Then, as $K$ increases, the response starts to have an overshoot. Then, oscillations appear when $K = K_{CR}$. Finally, for $K > K_{CR}$, the output becomes unbounded oscillations.

C. Relating Root Locus to the Open Loop Bode Plot

Each point on the Root Locus corresponds to a different Bode Plot (phase and gain as functions of $\omega$). For the Root Locus in the previous example, since the open loop transfer function is essentially the same for all points on the Root Locus, all Bode Plots will have the same phase. The only difference between the Bode Plots is a shift in amplitude (in dB). Note that all Bode Plots relate to the open loop $G(s)H(s)$ transfer functions, while the Root Locus shows the closed loop poles.

In each of the Bode plots there are two additional vertical lines, one dashed and the other dotted. The dashed line passes through the point where the open loop gain is 1 (i.e., 0 dB), and the dotted line passes through the point where the open loop phase is $-180^\circ$. The two lines can be used to obtain the phase and gain margins of the system. For illustration purposes, the arrows are pointing at only one pole’s branch, of the three, which are part of the Root Locus. This method of showing a particular $K$ on the Root Locus will be used for the remainder of this section.
Figure 3.4 is a pictorial summary of the above cases. The same green, orange and red “color code” system is used for stable, marginally stable and unstable system, respectively. An important insight to note is that the closed loop system is stable when the dashed line (for the 0 dB point) is on the left of the dotted line. In this case both Phase Margin and Gain Margin are positive. The closed loop system is marginally stable when the two lines overlap. In this case both Phase Margin and Gain Margin equal to zero. In the case where the closed loop system is unstable the dashed line is located to the right of the dotted line. In this case both phase margin and gain margin are negative.
D. Relating Root Locus to the Nyquist Plot

The following examples are MATLAB partial quantitative Nyquist plots, shown near the ‘-1 point’ in the $GH$-Domain. For illustration purposes, the arrows are pointing at only one pole’s branch, of the three, which are part of the Root Locus.

![Image of Root Locus and Nyquist Plot]

**Figure 3.5 – Root Locus and Nyquist Plot**

Again the same **green**, **orange** and **red** “color code” system is used for **stable**, **marginally stable** and **unstable system** respectively. It is important to note the relationship between the Nyquist plots and the “−1” point. In our case, if the “−1” point is outside of the Nyquist Plot (i.e., the plot is to the right of the “−1” point) then the system is stable, shown in **green**. When the Nyquist Plot crosses the “−1” point the closed loop system is marginally stable, shown in **orange**. In the case where the Nyquist Plot crosses the real axis to the left of the “−1” point the system becomes unstable, shown in **red**. Another insight that can be gained is that as the open loop gain $K$ increases, the Nyquist plot simply “expands.” Similarly, as the open loop gain $K$ decreases, the Nyquist Plot “shrinks.”
E. Relating Nyquist Plots to Root Locus

The following are Nyquist Plots for three different values of $K$.
- $K = 0.5 \cdot K_{CR}$
- $K = K_{CR}$
- $K = 2 \cdot K_{CR}$

Each of the infinite number of points seen on the Root Locus corresponds to a different Nyquist Plot. The plot shown in Figure 3.6 corresponds to three different points on the same Root Locus, giving us three separate Nyquist Plots. Another visual representation can be seen in Figure 3.7.
Figure 3.7 – All Three Nyquist Plots and Root Locus

From Figure 3.7, it can see that a point on the Root Locus corresponds to a whole Nyquist Plot. A smaller value of $K$ yields a “shrunken” Nyquist Plot. The effect can be seen in Figure 3.7 where the green Nyquist plot that does not encircle " $-1$ " Since the value of $K$ is less than the critical $K$, the system is stable. Similarly, a larger value of $K$ yields an “expanded” Nyquist Plot. In Figure 3.7, the red Nyquist Plot encircles " $-1$ " and indicates the system is unstable.
K. Relating Nyquist Plots to Step Response

In Figure 3.8, it can be seen with the **green** plot that for a small $K$, \( K < K_{cr} \), the step response settles at a steady state value. For a marginally stable system, denoted in **orange** for Critical $K$, \( K = K_{cr} \), the step response oscillates but does not increase in amplitude. In an unstable system which is plotted in **red** for a large $K$, \( K > K_{cr} \), the step response increases in amplitude.
IV. Understanding the Plant from Different Points of View

Designing a Controller

Design Specifications

1.) The “50” gain cannot be reduced (in other words, \( K_V \geq 50 \text{ sec}^{-1} \))

2.) Phase Margin \( \geq 45^\circ \)

What do these specifications mean?

The first specification, \( K_V \geq 50 \text{ sec}^{-1} \), means that the steady state error to a unit RAMP input should not exceed 0.02 or 1/50. As can be seen from the block diagram of the example, the plant has already one integrator (i.e., a pole at \( s = 0 \)), meaning that it is a type 1 system (assuming the controller has no poles or zeroes at \( s = 0 \)). Type 1 tracking system, i.e., a system that has one integrator in \( G(s) \), has a steady state error that is equal to zero for a STEP input.

The second specification, i.e., Phase Margin \( \geq 45^\circ \), means that at the \( G(j\omega) \) crossover frequency, \( \omega_C \) (the frequency at which the gain is 0 dB), the phase must be \(-135^\circ \) or higher (less negative).

Understanding the Given System: \( G_p(s) \)

Let us look at the given closed loop system. This is similar to what is given in textbooks but we show it in different points of view.

\[
\begin{align*}
R(s) & \rightarrow G_c(s) \rightarrow G_p(s) \rightarrow C(s) \\
\end{align*}
\]

**Figure 4.1 – Unity Feedback Closed Loop Block Diagram**

The closed loop transfer function is:

\[
\frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}
\]

From the denominator of the transfer function, we can obtain the characteristic equation: \( 1 + G_c(s)G_p(s) = 0 \). Using the characteristic equation, we can find the poles of the closed loop system. As mentioned earlier, the plant \( G_p(s) \) is given and cannot be changed. So, \( G_c(s) \) must be designed such that the overall system meets the desired specifications. Before we can design \( G_c(s) \), let’s further understand \( G_p(s) \).
By looking at the transfer function of the plant, \( G_p(s) \), we can see that it has a gain, \( K \), of 50 sec\(^{-1}\), one integrator (or a pole at \( s = 0 \)), and two identical poles at \( s = -20 \). Also, \( G_p(s) \) has no zeroes. In the next few sub-sections we look at the system from multiple points of view using the Routh Table, Root Locus, Nyquist Criterion, and Bode Plot.

**Using the Routh Table**

If we let \( G_C(s) = 1 \), then the characteristic equation becomes \( 1 + G_p(s) = 0 \). In other words, we have:

\[
1 + G_C(s)G_p(s) = 1 + \frac{50}{s(1 + s/20)^2} = 0
\]

The roots of this equation are the location of the closed loop poles. It is a 3\(^{rd}\) order equation so there are three roots. We can solve the equation analytically or numerically to find the location of the roots. However, this is not always necessary as sometimes we are interested in less information.

The Routh Table allows us to quickly find out if there are roots in the right hand side of the \( s \)-plane and, if there are, how many. This indicates stability or instability of the closed loop system. The following steps show how to obtain the Routh Table for our specific example.

\[
1 + \frac{50}{s(1 + s/20)^2} = 0
\]

The equation can be rewritten as:

\[
s(1 + s/20)^2 + 50 = 0
\]

\[
s^3 \frac{1}{400} + s^2 \frac{1}{10} + 1 \cdot s + 50 = 0
\]
From the characteristic equation above, the corresponding Routh Table is:

<table>
<thead>
<tr>
<th></th>
<th>$s^3$</th>
<th>$s^2$</th>
<th>$s^1$</th>
<th>$s^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^3$</td>
<td>$1$</td>
<td>$\frac{1}{400}$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$s^2$</td>
<td>$1$</td>
<td>$\frac{1}{10}$</td>
<td></td>
<td>$50$</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$\frac{1}{10} \cdot 1 - 50 \cdot \frac{1}{400}$</td>
<td>$\frac{1}{10}$</td>
<td>$-\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td>$-\frac{1}{4} \cdot 50 - \frac{1}{10} \cdot 0$</td>
<td>$\frac{1}{10}$</td>
<td>$50$</td>
<td></td>
</tr>
</tbody>
</table>

From the Routh Table, we can immediately see that there are two sign changes (from $\frac{1}{10}$ to $-\frac{1}{4}$ and from $-\frac{1}{4}$ to $50$) in the first column. This tells us that two roots of the characteristic equation are in the right-hand side of the $s$-plane. Furthermore, it means that two poles of the closed loop transfer function, $\frac{G_p(s)}{1+G_p(s)}$, are in the right-hand side of the $s$-plane, and therefore the closed loop system is unstable.

Just for curiosity, we can also find the gain for which the system poles are on the “verge” of instability. In other words, we can find the critical $K$. To do so, let:

$$G_p(s) = \frac{K}{s(1 + s/20)^2}$$

We will try to find the gain $K$ for which the system is on the “borderline” between stability and instability ($K = K_{CR}$).

The characteristic equation, with $G_C(s) = 1$ is:

$$1 + G_C(s)G_p(s) = 1 + \frac{K}{s(1 + s/20)^2} = 0$$

Using the Routh Table we can find that $K < 40$. Combining the positive $K$ constraint, we obtain:

$$0 < K < 40$$

For this range of $K$, the closed loop system is stable. In other words, all poles of the closed loop system are in the left hand side of the $s$-plane. When $K$ is equal or larger than 40, the closed loop system is unstable. For $0 < K < 40$ all poles of the closed loop system are in the left hand side of the $s$-plane. The critical $K$ is then $K_{CR} = 40$. 
When \( K = 40 \), the first column of the Routh Table is neither all positive nor does it have a negative coefficient. In this case the closed loop transfer function has 2 poles on the imaginary axis and the system is on the borderline of stability and instability. This borderline condition is also considered to be unstable.

**Plotting the Root Locus**

Recall that the Root Locus is the plot of the poles of the closed loop system as a function of \( k \). As we know, the closed loop poles can be found by using the characteristic equation for \( G_c(s) = 1 \) and \( G_p(s) = \frac{K}{s(1+s/20)^2} \):

We can rearrange the plant equation to be:

\[
G_p(s) = \frac{400K}{400s(1 + s/20)^2} = \frac{400K}{s(20 + s)^2} = \frac{k}{s(20 + s)^2}
\]

Where we let \( k = 400K \). Note that by rearranging the plant transfer function, the root locus gain, \( k \), is proportional to the open loop gain, \( K \). The characteristic equation then is:

\[
1 + G_c(s)G_p(s) = 1 + \frac{k}{s(20 + s)^2} = s(20 + s)^2 + k = 0
\]

\[
1 + G_c(s)G_p(s) = s^3 + 40s^2 + 400s + k = 0
\]

The Root Locus of the closed loop system is:

The Root Locus plot starts at the open loop poles and ends at the zeroes. In this case, there are no open loop zeroes so the poles will keep moving towards infinity near or on the asymptotes as the root locus open loop gain, \( k \), is increased. The location of the closed loop poles at \( K = 50 \), or \( k = 400 * 50 = 20000 \), (the gain of the plant) can be seen in the following:
Figure 4.2 – Root Locus Plot Pointing the Pole Location for $K = 50$

Note that two of the poles are in the right hand side of the $s$-plane, indicating instability of the closed loop system. In particular, for $K = 50$, these 2 closed loop poles are at $s = 0.9293 \pm j21.84$. This is expected since the Routh Table had two sign changes. We can also use the Root Locus to find the critical $K$. The poles when $K = K_{CR}$ can be seen in the following figure:
Note that for the critical $K$, $K_{CR} = 40$, two of the poles are on the imaginary axis, $j\omega$. These two poles are on the “borderline” between stability and instability (which, as noted earlier, is also considered instability). When $K > 40$, or $k > 16000$ for the root locus gain, the 2 closed loop poles are located in the right hand side of the $s$-plane.

**Using the Nyquist Criterion**

To continue the example, we will again use the characteristic equation for $G_C(s) = 1$ and the open loop gain is kept at $K = 50$.

\[
1 + G_C(s)G_p(s) = 1 + \frac{50}{s(1 + \frac{s}{20})^2} = 0
\]

there are no encircled poles in the right hand side of the $s$-plane, therefore $P = 0$. Now we map the contour C from the $s$-plane to the contour in the $GH$-Domain. Qualitatively, we obtain a Nyquist Plot with the following form.
Keep in mind that the Nyquist Plot above is qualitative and merely shows the form of the actual Nyquist Plot to be discussed later. As can be seen from the Nyquist plot above, the contour encircles the “-1” point twice. This means that $N_{CW} = 2$. Using the Nyquist Criterion we can obtain the number of zeroes of the characteristic equation in the right hand side of the $s$-plane.

$$N_{CW} = Z - P$$

$$2 = Z - 0$$

$$Z = 2$$

Having 2 zeroes of the characteristic equation in the right hand side of the $s$-plane means that there are two poles of the closed loop transfer function in the right hand side of the $s$-plane. This is (as expected) the same number of poles found by using the Routh Table and by using the Root Locus. We can see a more accurate Nyquist plot in the following figure where the plot is “zoomed in.”

Figure 4.4 – Qualitative Nyquist Plot for $K = 50$
Designing Using Bode Plot

To understand the given system using the Bode Plot, we must first convert the open loop transfer function from the $s$ parameter to $j\omega$. This can be done only for stable systems that have steady state, but we do allow for poles at $s = 0$. Note: It makes no sense to discuss the Bode Plots of unstable systems.

Again, this example uses $G_C(s) = 1$ and $K = 50$. So, we have the following:

$$G_C(j\omega)G_p(j\omega) = G(j\omega) = \frac{50}{j\omega(1 + j\omega/20)^2}$$

Using Bode Plot approximations, we can approximate the Bode Plot of the above open loop function (gain only) to become:

![Bode Diagram](image)

**Figure 4.5** – Bode Plot of $G(j\omega) = G_C(j\omega)G_p(j\omega)$
The phase at $\omega = 20 \text{ [rad/sec]}$ is $-180^\circ$. The Phase Margin can be obtained as follows:

$$PM = 180 + (\text{phase at crossover frequency})$$

We can read the crossover frequency from the Bode Plot. In this case, the crossover frequency is about $\omega_c = 22.3 \text{ [rad/sec]}$. Using this value, we can calculate the phase at the crossover frequency:

$$-90^\circ - 2 \tan^{-1} \left( \frac{\omega_c}{20} \right) = -90^\circ - 2 \tan^{-1} \left( \frac{22.3}{20} \right) = -90^\circ - 96.22^\circ = -186.22^\circ$$

The Phase Margin becomes:

$$PM = 180 + (-186.22^\circ) = -6.22^\circ$$

The Phase Margin is negative. This can also be seen from the figure above where the phase at the crossover frequency is more negative than $-180^\circ$. This is undesired because it indicates an unstable system. If we manage to reduce the gain at high frequencies without affecting the gain at low frequencies, then the cross over frequency will be smaller leading to a better Phase Margin. In addition, the effect of the two poles on the phase at the new $\omega_c$ will be less impactful. The Phase Margin can then be improved.
V. Designing the Controller

In this section, we intentionally show controllers that do not work with the design. This is meant to give students a feel for the trial and error process that can be experienced from a design project. Students in class tend to see the solution first which can lead to the confusion of why other controllers should not be used.

Some Comments Regarding Potential Controllers

This section outlines the effects of some controllers on the plant, when they might be used, and why they do not meet the specifications required for this design example. Again, the specifications are:

1.) The “50” gain cannot be reduced (in other words, $K_V \geq 50 \ sec^{-1}$)
2.) Phase Margin $\geq 45^\circ$

Proportional Controller

The Proportional Controller, or P Controller, can improve the phase to achieve the desired Phase Margin. For example, we can have a controller: $G_C(s) = \frac{1}{10}$. The phase will stay the same but the gain with be reduced and the crossover frequency will change (in our case to become smaller). This can improve the Phase Margin and stability. However, this will reduce the open loop gain of 50 to $50 \times \frac{1}{10} = 5$, which does not satisfy specification #1 of the design, i.e., to have an open loop gain of 50.

Let us look at the Bode Plot using a P Controller. In the following figure, the Bode Plot of the plant, $G_P(j\omega)$, is plotted in dark blue. The new Bode Plot including the P Controller is plotted in light blue.
Figure 5.1 – Bode Plot of $G(j\omega) = G_p(j\omega)/10$

The P Controller causes the Bode Plot to shift down by 20 dB while the phase is kept the same. The new crossover frequency is now $\omega_c = 4.73 \ [\text{rad/sec}]$. At this frequency, the phase is $-117^\circ$. So, the new Phase Margin is:

$$PM = 180 + (-117^\circ) = 63^\circ$$

The second specification, $PM > 45^\circ$, is met. However, the gain is too low so this P Controller cannot be used to meet the design specifications. Let us look at the effect of using a PI Controller.

**Proportional Integral Controller**

The Proportional Integral Controller or PI Controller is formed by summing the output of a Proportional Controller and an Integral Controller. Its transfer function is of the following form:

$$G_C(s) = K_p + \frac{K_I}{s} = \frac{K_p s + K_I}{s}$$

The PI controller can make the system stable but the phase margin will be lower than the desired by specification #2. For this example, we use a PI Controller with the following transfer function:

$$G_C(s) = \frac{1 + s/7}{s}$$
The effect of using the above PI controller can be seen in the following Bode Plot. Again, the Bode Plot of the plant, \( G_p(j\omega) \), is plotted in dark blue while the new Bode Plot including the PI Controller is plotted in light blue.

As we can see from the Bode Plot above, the crossover frequency is smaller than the case where the controller is just a gain of 1 (just the plant). In this case, the new crossover frequency with the PI Controller is \( \omega_c = 8.11 \, [rad/sec] \). The phase at this frequency is \(-174.9^\circ\) and so the Phase Margin is:

\[
PM = 180^\circ + (-174.9^\circ) = 5.1^\circ
\]

Although the Phase Margin is positive, it is far too small to meet the design specification of \( PM > 45^\circ \). This is due to the introduction of another integrator in the PI Controller which brought the low frequency phase down to \(-180^\circ\).

**Proportional Derivative Controller**

The Proportional Derivative Controller, or PD Controller, can be used to increase the phase and improve the Phase Margin. However, it increases the gain in high frequencies which may be undesirable. Observe the following Bode Plot of the open loop transfer function with a PD Controller of the form:

\[
G_c(s) = 1 + s/20
\]
The following figure shows the comparison of the open loop gain of the system with just the plant and when the PD Controller is used. The Bode Plot of just the plant, $G_p(j\omega)$, is in dark blue. The Bode Plot of the system with the PD Controller is shown in light blue.

By looking at the Bode Plot, we can see that the introduction of the zero from the PD Controller increased the gain at higher frequencies. The crossover frequency is also slightly higher than that of just the plant at $\omega_C = 28.6$ [rad/sec]. The zero introduced by the PD Controller also helped in increasing the phase at the crossover frequency to $-145^\circ$. This then yields a phase margin of:

$$PM = 180 + (-145^\circ) = 35^\circ$$

This PD Controller allowed for a design that kept the open loop gain at 50 but was not able to meet the desired Phase Margin of greater than 45°.
Lead Controller

Consider the following Lead Controller:

\[ G_C(s) = \frac{1 + \frac{s}{20}}{1 + \frac{s}{200}} \]

The Lead Controller can improve the phase due to the zero location which appears at a lower frequency than the pole. However, it will not achieve a Phase Margin of at least 45°. This can be seen in the following Bode Plot. It is to be noted that the dark blue Bode Plot is that of just the plant \( G_P(j\omega) \) and the light blue Bode Plot is that of the system with the Lead Controller.

![Bode Diagram](image)

**Figure 5.4 – Bode Plot of \( G(j\omega) = G_C(j\omega)G_P(j\omega) \) with Lead Controller**

In the figure above, the gain of the open loop transfer function with the Lead controller “splits off” from the gain of just the plant near \( \omega = 20 \text{ [rad/sec]} \). This is due to the zero introduced by the Lead Controller. The new crossover frequency, \( \omega_C = 28.5 \text{ [rad/sec]} \), is slightly higher from when only the plant is used. In addition, with the new controller, the phase at the crossover frequency has changed to \(-153^\circ\). This then yields a phase margin of:

\[ PM = 180 + (-153^\circ) = 27^\circ \]

Clearly the Phase Margin did improve but not enough to meet the desired specification of at least 45° so the Lead Controller cannot be used for this design.
Designing a Lag Controller

Designing Using Bode Plot

“Lag” controllers allow us to reduce the gain at high frequencies while keeping the gain at low frequencies. In our case, it makes sense to use it since the specification gain of at least 50 cannot be reduced by a simple scale factor. One possible lag controller to use is the following:

\[ G_c(s) = \frac{1 + s/1}{1 + s/0.1} \]

Let us look at its effect on the Bode Plot:

![Bode Diagram](image)

**Figure 5.5 – Approximated Bode Plot with Lag Controller**

In Figure 5.5, the purple plot is the approximated plot without the controller, and the blue plot shows the approximated bode plot with the lag controller. As can be seen from the figure above, the introduction of the Lag controller has shifted the crossover frequency. By simulating the actual bode plot, we can obtain the new crossover frequency, and see the effect on the phase as well.
Figure 5.6 – Simulated Bode Plot with Lag Controller

We can see from the Bode plot in Figure 5.6 that the new frequency when the phase is $-180^\circ$ is now $\omega_{180} = 19.1$ [rad/sec]. Also, the gain at this frequency is negative (in dB terms). The new crossover frequency is now $\omega_c = 4.82$ [rad/sec]. And, at the crossover frequency, the phase is now $-128^\circ$. This means that the new Phase Margin is:

$$ PM = 180 + (-128^\circ) = 52^\circ $$

Without the controller, the Phase Margin is $PM = -6.22^\circ$. Due to the introduction of the Lag Controller, the Phase Margin is now positive which indicates closed loop system stability and satisfies the Phase Margin Specification.

Note that the poles of the Lag Controller should be carefully picked to be “low enough” to avoid the effect of the negative phase of the controller on stability. If the controller is chosen to not have “low enough” frequencies there is a possibility of crossing the $-180^\circ$ line more than once (at some lower frequencies) and this can cause instability!
Using the Routh Table

first obtain the characteristic equation. Since $H(s) = 1$ the characteristic equation is simply:

$$1 + G_c(s)G_p(s) = 0$$

So, we can substitute the controller and plant transfer functions to obtain:

$$1 + \left( \frac{1 + \frac{s}{1}}{1 + \frac{s}{0.1}} \right) \left( \frac{50}{s(1 + \frac{s}{20})^2} \right) = 0$$

Following some algebraic manipulation, we get:

$$1 + \left( \frac{50s + 50}{0.025s^4 + 1.002s^3 + 10.1s^2 + s} \right) = 0$$

$$0.025s^4 + 1.002s^3 + 10.1s^2 + s + 50s + 50 = 0$$

$$0.025s^4 + 1.002s^3 + 10.1s^2 + 51s + 50 = 0$$
Using this form of the characteristic equation, we can form the Routh Table:

\[
\begin{array}{ccc}
  s^4 & 0.025 & 10.1 & 50 \\
  s^3 & 1.002 & & 51 \\
  s^2 & \frac{1.002 \cdot 10.1 - 51 \cdot 0.025}{1.002} & 8.828 & \frac{1.002 \cdot 50 - 0 \cdot 0.025}{1.002} = 50 \\
  s^1 & \frac{8.828 \cdot 51 - 50 \cdot 1.002}{8.828} = 45.325 & \frac{8.828 \cdot 0 - 0 \cdot 1.002}{8.828} = 0 \\
  s^0 & \frac{45.325 \cdot 50 - 0 \cdot 8.828}{45.325} = 50 & \\
\end{array}
\]

From the Routh Table above, we can see that the first column contains only positive values, i.e., there are no sign changes. This means that the closed loop system is stable.

**Using the Root Locus**

To find the Root Locus, the open loop gain is set to \( K \). So, we have the following open loop transfer function:

\[
G(s) = G_c(s)G_p(s) = \left( \frac{1 + \frac{s}{1}}{1 + \frac{s}{0.1}} \right) \left( \frac{K}{s(1 + \frac{s}{20})^2} \right)
\]

Using the \( G(s) \) above, we can use MATLAB to plot the following Root Locus in Figure 5.7.
As we can see, all closed loop poles are on the left hand side of the $s$-plane. This ensures that the closed loop system is stable for $K = 50$. As a final point of view, we will observe the effect of the Lag controller on the Nyquist Plot.

### Using the Nyquist Criterion

As we recall, the original Nyquist Plot (without the controller) proved to be unstable as “$-1$” is encircled twice by the contour in the $GH$-Domain. When adding the designed controller $G_C(s) = \frac{1 + \frac{s}{1}}{1 + \frac{s}{0.1}}$, we expect the closed loop system to become stable which is the same as the results obtained using the other tools. The open loop transfer function is as follows:

$$G(s)H(s) = \left(\frac{1 + \frac{s}{1}}{1 + \frac{s}{0.1}}\right)\left(\frac{50}{s\left(1 + \frac{s}{20}\right)^2}\right)$$

From the open loop transfer function above, we can see that the poles are at: $s = 0, -0.1, -20, -20$. This means that $G(s)H(s)$ has no poles on the right hand side of the $s$–plane so $P = 0$. The following is a qualitative Nyquist Plot of the system with the added controller.
From the figure above, we can see that “−1” is not encircled by the Nyquist Plot (this is also shown in the following zoomed-in simulated Nyquist plot). This means that $N_{CW} = 0$. Applying the Nyquist Criterion:

$$N_{CW} = Z - P$$

$$0 = Z - 0$$

$$Z = 0$$

The characteristic equation has no zeroes in the right hand side of the $s$-plane. This means that the closed loop transfer function has no poles in the right hand side of the $s$-plane and therefore the closed loop system is stable.
As can be seen from the figure above, the contour does not encircle “−1.” In addition, although it is not marked on the plot, the contour crosses the Real axis at $\omega_{180} = 19.1 \text{ [rad/sec]}$. At this point, the phase is $-180^\circ$. As expected, this is the same frequency as found earlier from the Bode Plot.

**Conclusion**

The material in this paper is an attempt to bridge the gap that we observed in student understanding of designing of feedback control systems. This is work in progress which, in part, was attempted in class. The initial feedback from students is very positive as they appreciated that this approach “connected the dots” for the different design tools. We were surprised to learn that the use of color that is green for stable, yellow, for marginal, and red for unstable were very appreciated by the students for understanding. The approach of relating the tools together graphically, turned out to be a great visual tool. We plan to have a detailed assessment of the approach that we hope to share in the near future.
References


