

**AC 2010-182: A FRAMEWORK FOR TEACHING NONLINEAR OP-AMP
CIRCUITS TO JUNIOR UNDERGRADUATE ELECTRICAL ENGINEERING
STUDENTS**

Bharathwaj Muthuswamy, Milwaukee School of Engineering

Joerg Mossbrucker, Milwaukee School of Engineering

A Framework for Teaching Nonlinear Operational Amplifier Circuits to Junior Undergraduate Electrical Engineering Students

Abstract

In this work, we propose a framework for teaching nonlinear operational amplifier (op-amp) circuits. This course would be for junior electrical engineering students who have a working knowledge of linear circuit theory and are starting the study of op-amp circuits. The framework involves mathematically understanding a nonlinear op-amp circuit, simulating the circuit and implementing the circuit in the laboratory. The students compare and study the results from all three approaches. The goal of this framework is to teach a few basic but very powerful concepts which can be used to analyze practical nonlinear op-amp circuits. This paper describes the framework followed by an application to the design, simulation and implementation of a negative impedance converter.

1 Introduction

The main objective of this paper is to present an approach (i.e. framework) for understanding nonlinear op-amp circuits. Although other frameworks have been proposed in the past¹, the defining feature of our framework is that it emphasizes all three fundamental aspects of engineering - design, simulation and implementation - with respect to nonlinear op-amp circuits. The purpose of teaching nonlinear op-amp circuits is to help the students gain a better understanding of op-amp functionality. We will be using this framework as a part of the nonlinear op-amp course for junior undergraduate electrical engineering students in Spring 2010.

The proposed approach (main concepts are highlighted in *italics*) is:

1. Explain the basic op-amp model by *mathematically highlighting the differences between positive and negative feedback*.
2. *Mathematically* derive an $i - v$ graph for a simple one-port op-amp circuit (the *Negative Impedance Converter or NIC*) by using the basic op-amp model from the previous step.
3. *Simulate* the NIC to justify mathematical calculations.
4. *Implement* the NIC and compare the results from steps 2, 3 and 4.
5. *Illustrate applications* of the NIC to oscillator design by introducing the concept of *dynamic route*².

The organization of this paper follows the framework proposed above. Mathematical details are left, as usual, to the appendices.

2 Review of the Opamp Model

Recall the op-amp symbol, shown in Fig. 1:

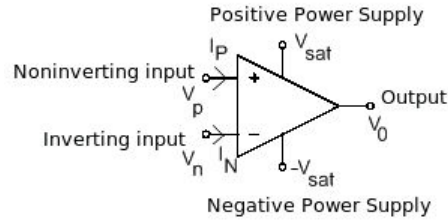


Figure 1: The circuit symbol for an op-amp

A mathematical model of the operational amplifier is given below:

$$v_0 = \begin{cases} +V_{sat} & v_p \geq v_n \\ A(v_p - v_n) & -V_{sat} < v_0 < +V_{sat} \\ -V_{sat} & v_p \leq v_n \end{cases} \quad (1)$$

Here, we have assumed that $V_{sat} > 0$. A is the open-loop voltage gain and is typically at least $200,000^2$. Thus, in the linear region of operation for the op-amp ($-V_{sat} < v_0 < +V_{sat}$), $v_p \approx v_n$. The positive saturation region is when $v_p \geq v_n$ ($v_0 = +V_{sat}$). Similarly negative saturation is when $v_p \leq v_n$ ($v_0 = -V_{sat}$). The op-amp model is picturized in the voltage transfer characteristic in Fig. 2.

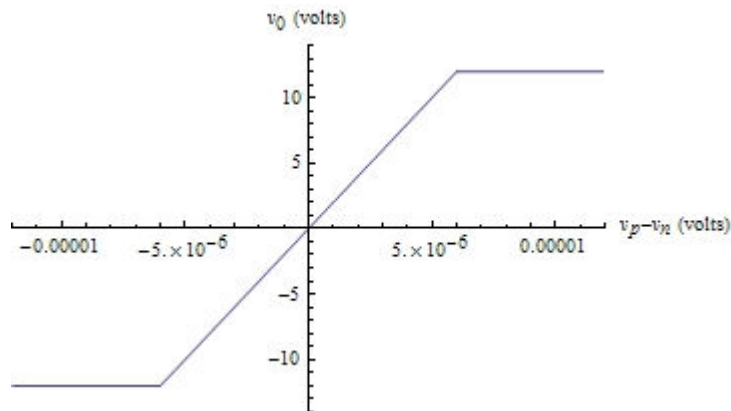


Figure 2: The voltage transfer characterstic of an op-amp.

Also, $I_p \approx I_n = 0$. Notice that the current at the output terminal is usually non-zero.

2.1 Difference between Negative and Positive Feedback

Consider the circuit shown in Fig. 3. This is the familiar voltage follower. Assume $V_{sat} = 12V$. The complete voltage transfer characteristic is shown in Fig. 4.

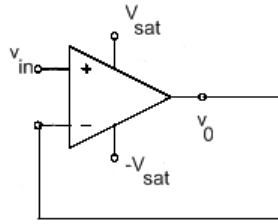


Figure 3: A negative feedback circuit (voltage follower)

The circuit in Fig. 3 is said to have negative feedback because the output voltage is fed back to the inverting input. What happens if we interchange the inverting and noninverting terminals as shown in Fig. 5? By inspection, we can see that $v_0 = v_{in}$ if $|v_{in}| < V_{sat}$. Hence, in the linear region, the transfer characteristic of this positive feedback circuit is identical to the negative feedback circuit in Fig. 3.

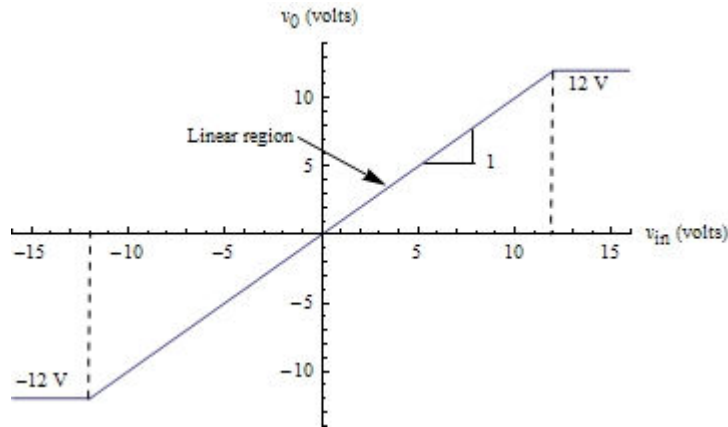


Figure 4: The voltage transfer characteristic of the follower. Contrast with Fig. 2.

In practice, however, the two circuits do not behave in the same way. To understand this behaviour, we need to complete the voltage transfer characteristic of the op-amp in positive feedback configuration. Refer to Fig. 5. If we sketch the voltage transfer characteristic for the positive feedback configuration, we get Fig. 6. The derivation is given in Appendix A.

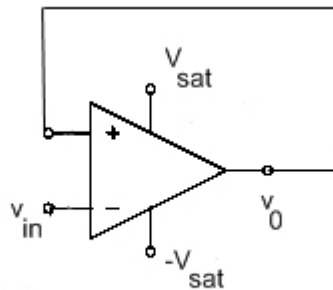


Figure 5: The op-amp in positive feedback configuration

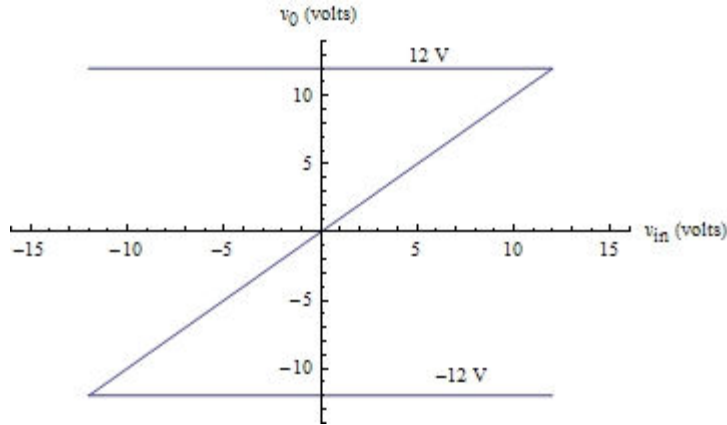


Figure 6: The voltage transfer characteristic for positive feedback.

Notice that the voltage transfer characteristics for the follower and the op-amp in positive feedback are completely different. For the positive feedback case, even if the op-amp is operating in the linear region, there are three distinct output voltages for each value of v_{in} . Using a more realistic op-amp model augmented by a capacitor, it can be shown that the operating points on the middle segment in Fig. 6 are unstable². Having unstable operating points means that even if the initial voltage $v_{in}(0)$ lies on this segment, the voltage will quickly move into the positive saturation region if $v_{in}(0) > 0$ or into the negative saturation region if $v_{in} < 0$.

We can also give an intuitive explanation of the unstable behaviour by referring back to the op-amp characteristics shown in Fig. 2, where in the linear region, we have:

$$v_o = A(v_p - v_n) \quad (2)$$

Eq. 2 shows that the output voltage v_o decreases (respectively increases) whenever the potential v_n at the inverting terminal increases (respectively decreases). Since physical signals propagate only at a finite velocity, changes in input voltage are not felt instantaneously at the output terminal, but at some moments (say 1 ns) later. Suppose $v_p - v_n = 1 \mu V$ in Fig. 3 and Fig. 5. In the case of Fig 3, $v_{in} - v_o = 1 V$ and in Fig. 5, $v_o - v_{in} = 1 V$. Lets say v_{in} is slightly increased in both circuits. In the case of the negative feedback circuit in Fig. 3, Eq. 2 implies that $v_o = A(v_{in} - v_o)$, thus v_o initially increases. But, this implies that $v_{in} - v_o$ decreases a short time later. Hence, the operating point will return to the original position. Notice that the exact opposite happens in the positive feedback circuit, the slightest disturbance in v_{in} gets amplified greatly because of the large value of A . Hence, the output rapidly moves towards saturation.

This fact that op-amps in positive feedback can operate in both saturation and linear regions can be used to synthesize negative resistances. We will show such a circuit in the next section.

3 Negative Impedance Converter (NIC)

3.1 Theory

Consider the circuit shown in Fig. 7.

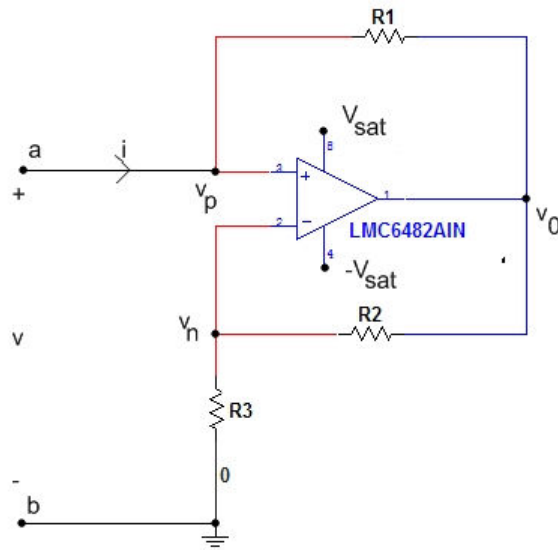


Figure 7: The NIC, the op-amp used in simulation is the LMC6482.

We will derive the $i - v$ characteristic of the circuit above as seen from terminals a-b. The $i - v$ graph can be derived by considering the three operating regions of the op-amp, just like the case of the positive and negative feedback op-amp circuits. Refer to Appendix B for the derivation. A plot of the $i - v$ graph, with the following component values: $R1 = 1k\Omega, R2 = 1k\Omega, R3 = 1k\Omega, V_{sat} = 5V$, is shown in Fig. 8

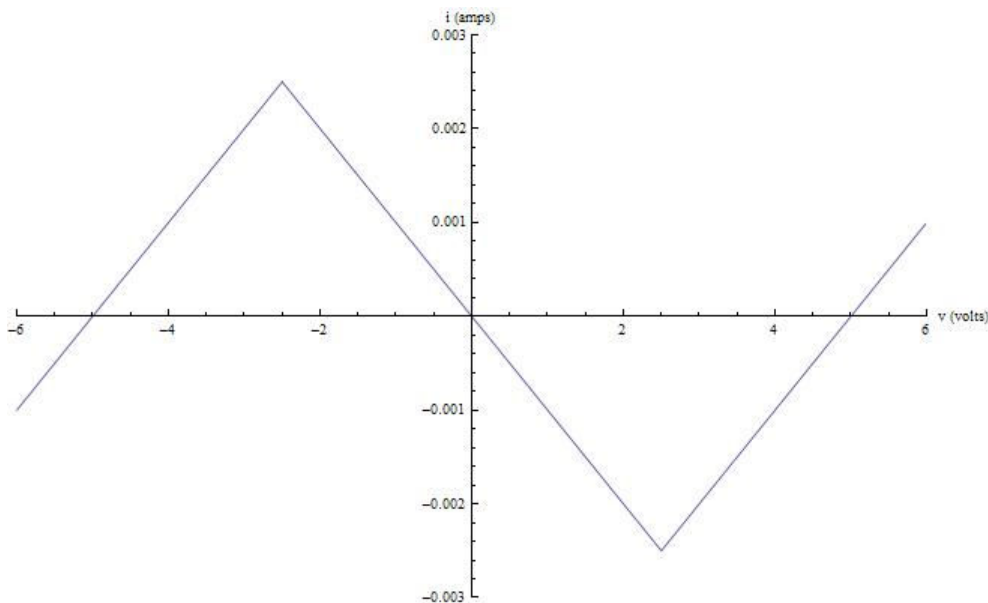


Figure 8: Plot of the Negative Impedance Converter $i - v$ graph

Notice that the slope of the $i - v$ graph in the linear region is negative (since all resistances are positive). Thus, we have managed to synthesize a negative resistor. Next, we will simulate and then realize this circuit on a breadboard.

3.2 Simulation

Fig. 9 shows the MultiSim simulation schematic. The MultiSim simulation is setup to run a DC sweep on source V3. The current to be plotted is the current through source V4. We have the dummy source since MultiSim treats currents as flowing into the positive terminal of voltage sources. If we used the current flowing through V3, then the current will be in the opposite direction to our reference current.

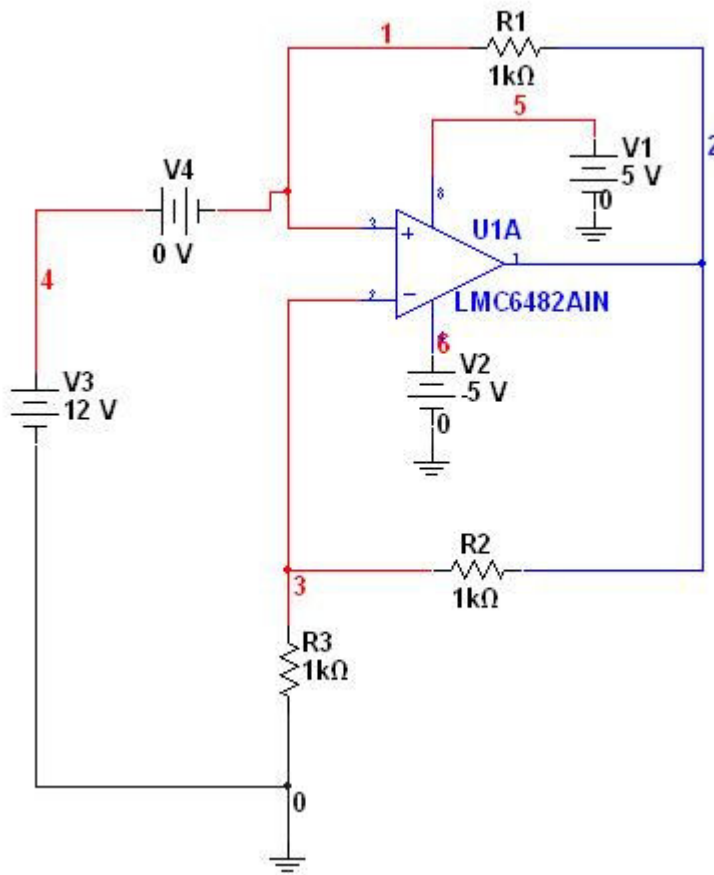


Figure 9: The Negative Impedance Converter in MultiSim 10

3.3 Comparing Mathematical Derivation, Simulation and Impementation

Once we implement the circuit from Fig. 9 on a breadboard, we can use SignalExpress to obtain the experimental $i-v$ graph from the physical circuit. The theoretical plot, the simulation plot and the experimental plots have been superimposed in Fig. 10 on the same set of axis for comparison.

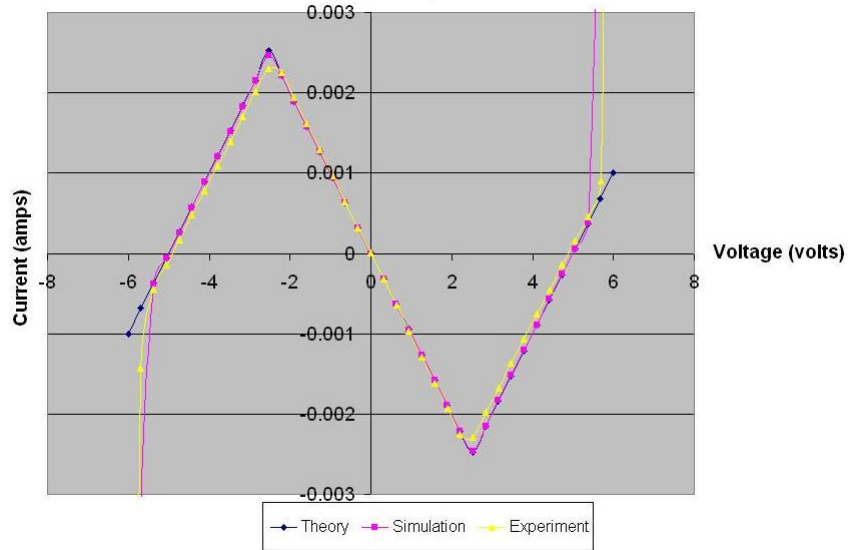


Figure 10: Comparing the three NIC plots via MS Excel

Notice that realistic op-amp effects of the simulation are evident in the values of v outside the positive and negative power supply values. We will also see these effects in the physical circuit. These realistic effects obviously cannot be predicted by our theoretical model. Nevertheless, there is excellent agreement between the three models for input voltages within the range of the power supplies.

The point to be noted from this section is that there is nothing inherently complicated about nonlinear op-amp circuits with positive feedback, as long as the op-amp rules are applied correctly. Moreover these circuits can be used to synthesize negative resistors, which in turn can be used for implementing oscillators. This is the topic of the next section.

4 Application: Oscillator

In this section, we will use the negative impedance converter to design an oscillator. The concept is to view the $i - v$ we derived as a block box and then to study the effects of adding a dynamic element at the input-terminals, refer to Fig. 11. The goal is to determine $i(t)$ and $v(t)$. The reasons for why this circuit functions as an oscillator will become clear after this section.

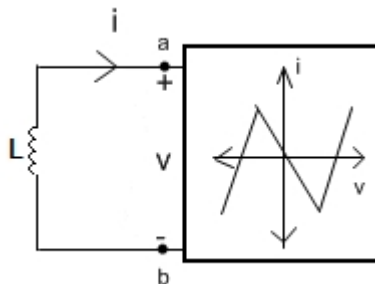


Figure 11: Adding an inductor at terminals ab results in an oscillator.

4.1 Theory

Consider terminals ab in Fig. 11. Fig. 12 shows the complete op-amp circuit.

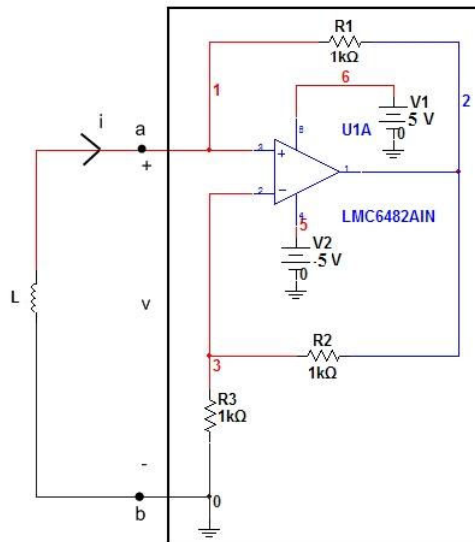


Figure 12: The oscillator showing the details of the NIC.

From the terminals of the inductor, we have:

$$v = -L \frac{di}{dt} \quad (3)$$

Notice the negative sign in the equation above. This is because of the passive sign convention. But, since (v, i) are also associated with the two-terminal device, these variables are related by the $i - v$ graph. We will use this fact to determine $i(t)$ and $v(t)$. In order to determine these variables, notice that the $i - v$ graph of the op-amp circuit is piecewise-linear. Thus, the solution $(v(t), i(t))$ can be found by determining first the specific “route” and “direction”, henceforth called the *dynamic route*². Therefore, the steps involved in determining $(v(t), i(t))$ are summarized below. For detailed analysis of the oscillator, please refer to Appendix C. In the analysis, simulation and implementation of the oscillator, we used $L = 100 \text{ mH}$.

- **Step 1: Determine Dynamic Route.**
- **Step 2: Determine Equilibrium Points.**
- **Step 3: Complete Dynamic Route.**
- **Step 4: Determining the analytic expressions for $v(t)$ and $i(t)$.**

4.2 Simulation

The MultiSim simulation schematic is shown in Fig. 12. We performed a transient simulation of this system with an initial current of 0.1 mA through the inductor.

4.3 Comparing Mathematical Derivation, Simulation and Implementation

A comparison of the output voltages of the op-amp from the theoretical calculations, simulation and physical circuit implementation is shown below. Notice that we see phase differences between the three waveforms. The source of this difference is obvious because the oscillators are not synchronized.

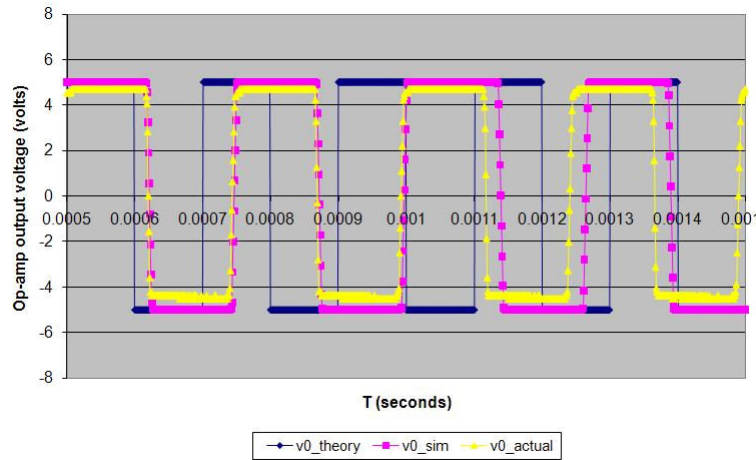


Figure 13: Comparing theory, simulation and experiment via MS Exce

5 Conclusions and Future Work

In this paper, we proposed an approach (framework) for understanding nonlinear op-amp circuits. The highlights of the framework are:

- *Clearly* explain the behaviour of op-amp circuits in positive feedback.
- *Emphasize* the theory, simulation and experiment approach in engineering.
- *Understand* the differences in results from the three approaches.

Based on point 3 above and the results from the oscillator section, a natural continuation of this paper would be to synchronize the oscillators from the simulation and physical circuit. Another extension would be to use the framework in this paper to analyze other negative impedance converters and oscillators (for instance, oscillators based on capacitors instead of inductors).

Bibliography

1. Chua, L. O. *A Unified Approach for Teaching Basic Nonlinear Electronic Circuit to Sophomores*. Proceedings of the IEEE, **59**(6), pp. 880 - 886. June, 1971.
2. Chua, L. O., Desoer, C. A. and Kuh, E. S. *Linear and Nonlinear Circuits*. McGraw-Hill book Company. 1987.

Appendix A

In this appendix, we derive the voltage transfer characteristic (Fig. 6) for the positive feedback configuration (Fig. 5).

- **Case 1: Positive Saturation** In positive saturation, Eqn. 1 simplifies to:

$$\begin{aligned}v_0 &= +V_{sat} \quad v_p \geq v_n \\ &= 12 \text{ V} \quad v_0 \geq v_{in} \\ &= 12 \text{ V} \quad 12 \text{ V} \geq v_{in}\end{aligned}$$

Thus,

$$v_0 = 12 \text{ V} \quad v_{in} \leq 12 \text{ V} \quad (4)$$

- **Case 2: Negative Saturation** In negative saturation, Eqn. 1 simplifies to:

$$\begin{aligned}v_0 &= -V_{sat} \quad v_p \leq v_n \\ &= -12 \text{ V} \quad v_0 \leq v_{in} \\ &= -12 \text{ V} \quad -12 \text{ V} \leq v_{in}\end{aligned}$$

Hence,

$$v_0 = -12 \text{ V} \quad v_{in} \geq -12 \text{ V} \quad (5)$$

Appendix B

In this appendix, we will derive the graph in Fig. 8. Refer to Fig. 7 for the relevant circuit.

- **Case 1: Op-amp is in the linear region of operation:**

$$\begin{aligned}v_p &= v_n \\ &= \frac{R3}{R3 + R2} v_0 \quad (\text{voltage divider at } v_n)\end{aligned}$$

Notice that we can apply voltage divider at v_n since current into inverting op-amp terminal is zero. Also, $v_p = v$. Thus, we have:

$$v_0 = \frac{R3 + R2}{R3} v \quad (6)$$

Applying Kirchoff's current law at v_p (current into the noninverting terminal of the op-amp is approximately 0):

$$i = \frac{v_p - v_0}{R1} = \frac{v - v_0}{R1} \quad (7)$$

Substituting v_0 from Eq. 6 in Eq. 7 and simplifying, we get the $i - v$ relationship in the linear region of operation:

$$\boxed{i = \frac{-R2}{R3} \frac{v_1}{R1}} \quad (8)$$

- **Case 2: Op-amp is in positive saturation:** First, we need to determine the range of v for which the op-amp is in positive saturation. We know from Eq. 1 that if the op-amp is in positive saturation:

$$\begin{aligned} v_p &> v_n \\ v &> \frac{R3}{R3 + R2} v_0 \end{aligned}$$

But, in positive saturation $v_0 = V_{sat}$ (we assume that $V_{sat} > 0$). Thus, the op-amp is in positive saturation when

$$v > \frac{R3}{R3 + R2} V_{sat} \quad (9)$$

The $i - v$ relationship when the op-amp is in positive saturation can be easily obtained by setting $v_0 = V_{sat}$ in Eq. 7:

$$i = \frac{v - V_{sat}}{R1} \quad (10)$$

- **Case 3: Op-amp is in negative saturation:** Similar to the previous case, we find that the op-amp is in negative saturation when:

$$v < \frac{-R3}{R3 + R2} V_{sat} \quad (11)$$

The $i - v$ relationship when the op-amp is in negative saturation is obtained by setting $v_0 = -V_{sat}$ in Eq. 7:

$$i = \frac{v + V_{sat}}{R1} \quad (12)$$

Using the following values for the components: $R1 = 1k\Omega, R2 = 1k\Omega, R3 = 1k\Omega, V_{sat} = 5V$, we have the equations below.

$$\begin{aligned} \text{Linear Region : } i &= \frac{-v}{1k\Omega} \quad -2.5V \leq v \leq 2.5V \\ \text{Positive Saturation : } i &= \frac{v - 5V}{1k\Omega} \quad v > 2.5V \\ \text{Negative Saturation : } i &= \frac{v + 5V}{1k\Omega} \quad v < -2.5V \end{aligned}$$

The $i - v$ graph is shown in Fig. 8.

Appendix C

In this appendix, we will derive the oscillator *dynamic route* and expressions $v(t)$ and $i(t)$ for the circuit in Fig. 12. We will expand the steps proposed in the theory section of the oscillator.

- **Step 1: Determine Dynamic Route.** First, we will determine the direction taken by variables (v, i) . This can be determined from Eq. 3. We get the following cases ($L > 0$):

Condition	Direction
$v > 0$	$i'(t) < 0$
$v < 0$	$i'(t) > 0$

Table 1: Equations for determining the dynamic route

The equations in Table 5 above can be used to determine the *dynamic route*, indicated by the arrows in Fig. 14. The sign of v and $i'(t)$ determine the direction of flow in each piecewise-linear segment. For example, in segment $Q'_1 Q_0$ in Fig. 14, $v < 0$, thus $i'(t) > 0$. This means that if we start at a point on the $i - v$ curve in segment $Q'_1 Q_0$, the dynamics of the inductor causes an *increase* in current (for $t > 0$). Hence, the arrows are “pointing up”. We can derive similar results for the other segments, resulting in the *dynamic route* in Fig. 14.

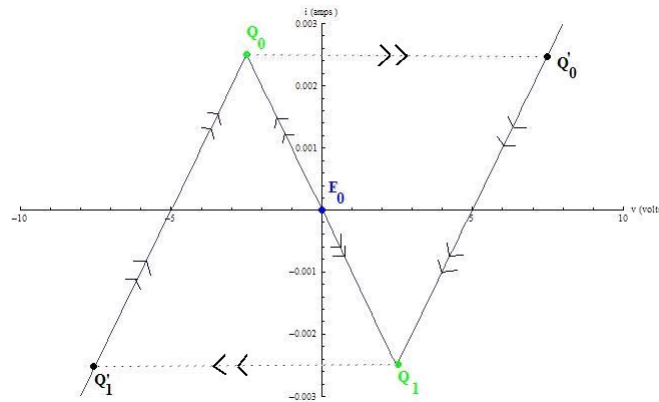


Figure 14: The dynamic route for our circuit indicated by the arrows.

- **Step 2: Determine Equilibrium Points.** One case not covered by Table 1 is what happens when $i'(t) = 0$. This is the mathematical definition of an *equilibrium point*. From Eq. 3, we can see that $v = 0$ at equilibrium. In other words, the equilibrium points are along the y -axis in Fig. 14. In this case, we have only one equilibrium point, $(0,0)$ marked by E_0 .

Notice that $v = 0$ at equilibrium implies that the inductor can be modeled as a short-circuit. This agrees with our definition of an inductor being modeled as a short-circuit at steady-state.

- **Step 3: Complete Dynamic Route - Impasse Points.** One question that remains in our problem is what happens when $(v, i) = (-2.5 V, 2.5 mA)$ and $(v, i) = (2.5 V, -2.5 mA)$ in Fig. 14? Note that these points are not equilibrium points, therefore the variables have to keep changing. Also, the arrowheads are oppositely directed, so it is impossible to continue drawing the dynamic route once the (v, i) reach Q_0 or Q_1 . In other words, an *impasse* is reached whenever the solution reaches Q_0 or Q_1 ².

We can now use the dynamic route to understand how the circuit functions as an oscillator. Suppose at $t = 0, v(0) \approx 0 V$. Since we are close to E_0 and the equilibrium point E_0 is unstable, the circuit variables (voltage and current) move either along segment $E_0 Q_1$

or segment E_0Q_0 . The actual route will depend on $v(0)$ and $i(0)$. But notice that the dynamic route implies the circuit variables quickly move away from E_0 . Once Q_0 (or Q_1) is reached, according to the dynamic route, the circuit variables jump to Q'_0 (or Q'_1). The dynamic route then only moves along $Q'_0Q_1 \rightarrow Q'_1Q_0$ (or $Q'_1Q_0 \rightarrow Q'_0Q_1$). This corresponds to the positive and negative saturation regions of the op-amp, hence the output voltage of the op-amp oscillates between $\pm V_{sat}$.

- **Step 4: Determining the analytic expressions for $v(t)$ and $i(t)$** Assuming we initially move along E_0Q_0 , we will derive the equation for $i(t)$. The equation for $v(t)$ can be determined using Eq. 3, this is left as an exercise for the reader.

In order to determine $i(t)$, notice that we can simplify the Negative Impedance Converter circuit in Q'_1Q_0 , Q_0Q_1 and $Q_1Q'_0$ into simply a resistance as shown in Fig. 15. Although in Q'_1Q_0 and $Q_1Q'_0$, the actual Thevenin equivalent is represented by a non-zero voltage source in series with the equivalent resistance, we will incorporate the effects of the voltage source in the initial conditions. This is possible because, as shown in Fig. 15, we simply have an RL circuit.

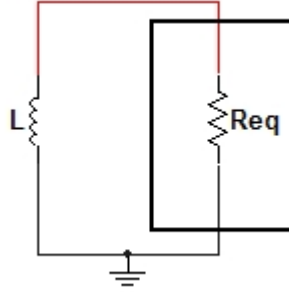


Figure 15: The simplified RL circuit for analyzing the oscillator.

We will analyze the oscillator in two cases, the first case will deal with region E_0Q_0 and the next case will deal with region Q'_0Q_1 . We will not analyze the dynamics in region Q'_1Q_0 because the dynamics will be equivalent to the one in region Q'_0Q_1 . Also, we will use $L = 100 \text{ mH}$ for our calculations.

– **Case 1: $i(t)$ in Region E_0Q_0**

We will determine $i(t)$. In this region, the Negative Impedance Converter $i - v$ relationship is given by Eq. 8. Note that this region of operation corresponds to the *linear* region of the op-amp ($v_p \approx v_n$). Using the parameters for the given circuit, the $i - v$ relationship is given by $i = \frac{-v}{1 \text{ k}\Omega}$. Thus, $R = -1 \text{ k}\Omega$. Since the dynamics of the system is represented by a first order RL circuit, $i(t)$ is:

$$i(t) = i_f + (i_i - i_f) \cdot e^{-\frac{t}{\tau}} \quad (13)$$

$\tau \equiv \frac{L}{R}$, thus in our case $\tau = \frac{L}{1000}$. Therefore, Eq. 13 becomes:

$$i(t) = i_f + (i_i - i_f) \cdot e^{-\frac{1000t}{L}} \quad (14)$$

Notice that for $t > 0$, $i(t) \rightarrow \pm\infty$. This agrees with our initial observation that the system is *unstable* in E_0Q_0 . Also note that as $t \rightarrow -\infty$, $i(t) \rightarrow i_f$. In other words,

i_f is a *virtual* equilibrium point². Using Fig. 14, $t \rightarrow -\infty$ corresponds to going against the dynamic route. In other words, $i_f = 0A$. Note that this extrapolation to the y-axis ($v = 0$) agrees with the concept of an inductor being a short circuit at equilibrium. i_i is our initial current. We assumed that $i_i \equiv \varepsilon$ is small and positive. Thus, our expression for $i(t)$ in E_0Q_0 is:

$$i(t) = \varepsilon \cdot e^{\frac{1000t}{L}} \quad \varepsilon > 0 \quad (15)$$

Suppose $\varepsilon = 0.1 \text{ mA}$. We also assumed that $L = 100 \text{ mH}$. We can then determine the time t_s at which the circuit switches to the Q'_0Q_1 region since $i(t_s) = 2.5 \text{ mA}$. Thus:

$$2.5 = 0.1 \cdot e^{\frac{1000t_s}{0.1}} \quad (16)$$

In other words, $t_s \approx 0.32 \text{ ms}$. Note that since t_s is so small, it is very difficult to observe $i(t)$ in practice. That is, when we turn on the circuit, $i(t)$ goes out of the negative resistance region of our $i - v$ graph very quickly and never returns (refer to the dynamic route in Fig. 14). In other words $i(t)$ and $v(t)$ will be confined to Q'_0Q_1 and Q'_1Q_0 after the circuit goes out of the negative resistance region Q_0Q_1 .

– **Case 2: $i(t)$ in Region Q'_0Q_1** In this region, $i(t)$ is given by:

$$i(t) = i_f + (i_i - i_f) \cdot e^{\frac{-(t-t_s)}{\tau}} \quad (17)$$

Here, $\tau = \frac{L}{R} > 0$, since $R > 0$. $i_i = 2.5 \text{ mA}$. i_f can be obtained by $t \rightarrow \infty$. Similar to the previous analysis in the linear region, extrapolating the $i - v$ graph to the y-axis gives i_f since at equilibrium the inductor is a short circuit. $i_f = -5 \text{ mA}$. Thus, the Eq. 17 becomes:

$$i(t) = -5 + 7.5 \cdot e^{\frac{-(t-0.32 \text{ ms})}{0.1 \text{ ms}}} \text{ mA} \quad (18)$$

Suppose t_x is defined as the time the circuit variables spend in Q'_0Q_1 . t_x can be easily determined from the equation above since $i(t_x + 0.32 \text{ ms}) = -2.5 \text{ mA}$. Notice that the offset of 0.32 ms is necessary because of our definition of t_x . Using the equation above, we get:

$$t_x = 0.1 \text{ ms} \quad (19)$$

Thus, the period of oscillation in our circuit is:

$$\boxed{T_p = 0.2 \text{ ms}} \quad (20)$$