A Geometrical Derivation of the Three Trigonometric Identities

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Abstract

Hearing the word trigonometry often produces a feeling of hopelessness in adults. Memory banks recall the acronym "SOHCAHTOA" (a factious Indian Word!) which when verbalized stimulates thoughts about the sides of a triangle. After completing a six-month course in trigonometry in high school and graduating from MIT and receiving a Ph.D. from Columbia University, I was convinced that I knew trigonometry cold! However, I now teach physics and mathematics to college students, who frequently do not get it!

Background of Class

This paper is a topic in a mathematics class, MAT201, Mathematical Analysis ll. The textbook used in this course is Caulter, Paul A & Michael A. "Technical Mathematics with Calculus Fourth Edition[1]. Students taking this course major in both Electrical Engineering Technology and Computer Science Technology. The course is taken in the second semester. The fall 2002 class consisted of 18 students, 15 male and 3 female. The student ages ranged from 19 to 53. One student is white, six are Hispanic, five students are black, and five are Asian.

Many of our students have deficits in mathematics. They find word problems very difficult. The scientific method starts by formulating a problem. How do you teach students to formulate problems? My approach is to have students construct a triangle and then step by step derive the trigonometric identities. With this hands on approach to mathematics students acquire an understanding of basic concepts rather than relying on rote memorization.

Introduction

Confucius said, "A journey of 1000 miles begins with a single step"

Acquiring an understanding of trigonometry should begin by constructing a right triangle. (Figure 1). Draw a straight line. Place a compass point on the line and mark off two points on opposite sides of the point. Move the compass point to the marked off points and draw an arc on top and an arc on the bottom. Next move the compass to the second marked off point and draw an arc on top and on the bottom. The two arcs intersect above and below the original line. Place a straight edge at each intersection and draw a line which is perpendicular to the original line. Next construct a right triangle and label each angle A, B, and C. It is convenient to label the smallest angle A, and the largest angle C. Label the side opposite angle A as a, the side opposite angle B as b and the side opposite angle C as c. Every triangle will have three angles (A, B, and C) and three sides (a, b, and c)

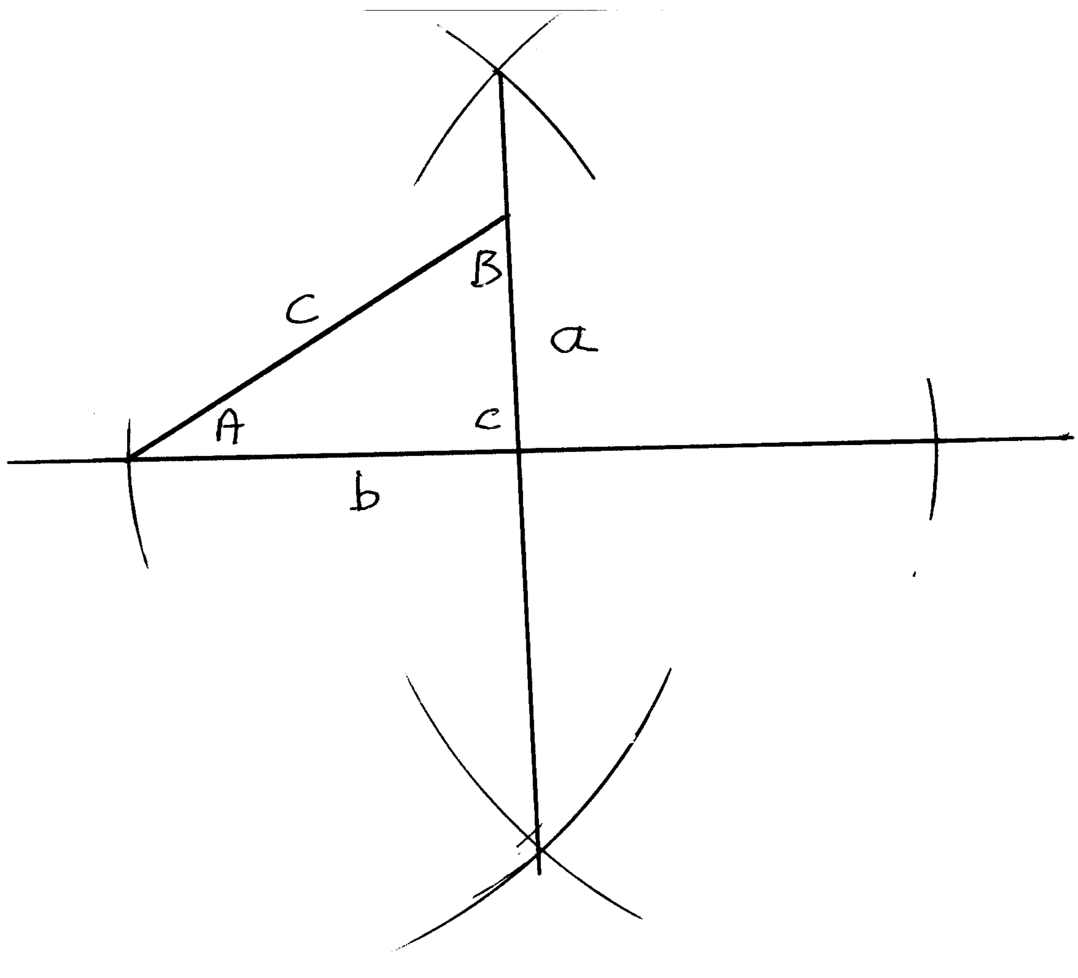


Figure 1

Angle	Side
A	a
B	b
C	c

$\sin A = \frac{a}{c}$ $\qquad$ $\csc A = \frac{c}{a}$

$\cos A = \frac{b}{c}$ $\qquad$ $\sec A = \frac{c}{b}$

$\tan A = \frac{a}{b}$ $\qquad$ $\cot A = \frac{b}{a}$

Students focus on the fact that a triangle has three sides (a, b, and c), and three angles (A, B and C). A ratio is a fraction which has a numerator and a denominator. Using the three sides of a right triangle, six ratios can be formed. See Table 1 below.

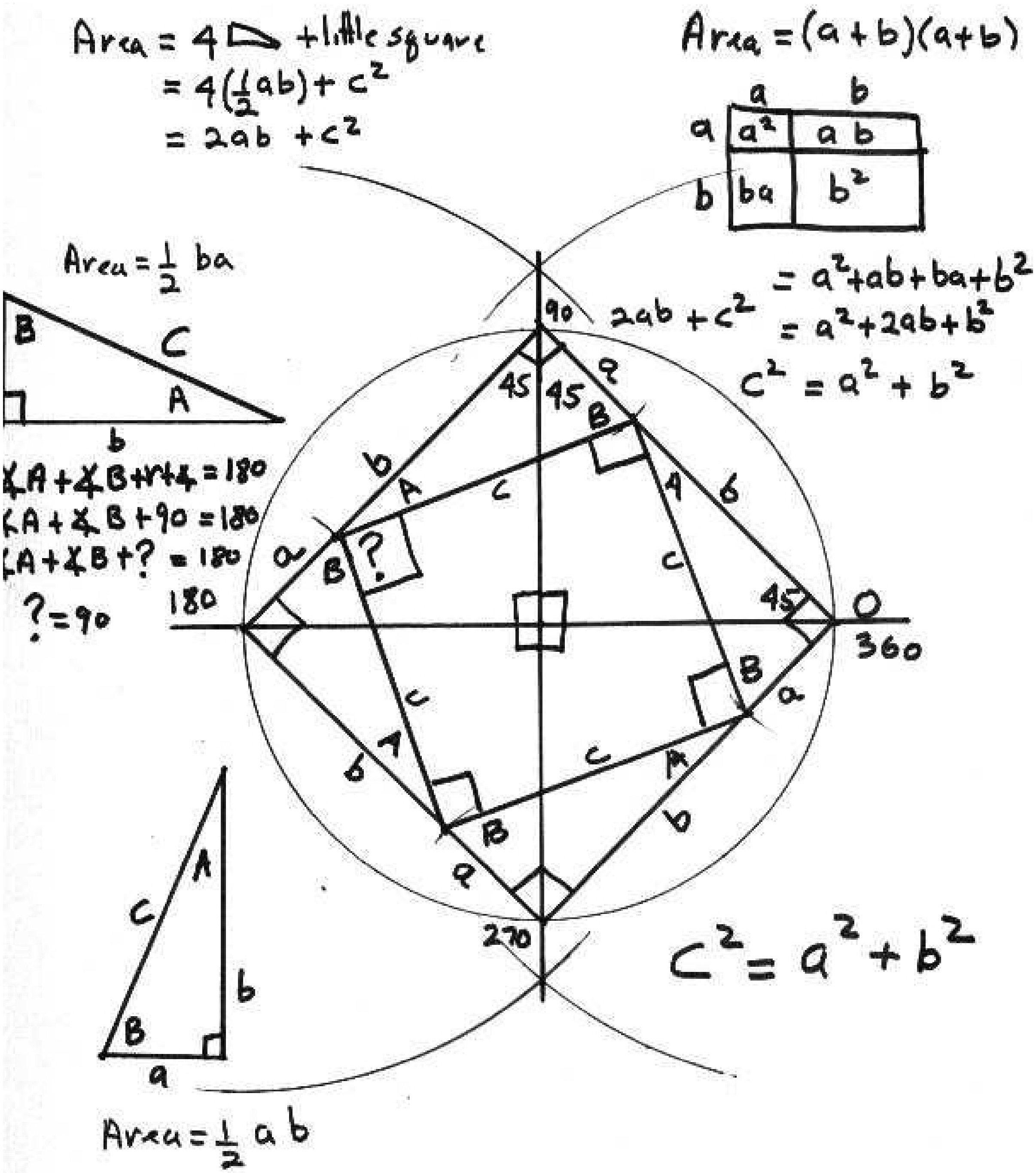
NAME of Ratio	SYMBOL	RATIO
sine A	sin A =	a/c
cosine A	cos A =	b/c
tangent A	tan A =	a/b
cosecant A	csc A =	c/a
cecant A	sec A =	c/b
cotangent A	cot A =	b/a

(Table 2

These six ratios enable students to acquire an understanding of the physical meaning of sine, cosine, tangent, cosecant, secant, and cotangent.

Students also study and construct **"**A GEOMETRICAL PROOF OF PYTHAGORAS' THEOREM" The proof is shown in Figure 3. which appeared in the paper "Teaching Critical Thinking" [(2)], American Society, Engineering Education, Albuquerque, New Mexico, June 2001

Figure 3
“A GEOMETRICAL PROOF OF PYTHAGORAS’ THEOREM”



Derivation of the first trigonometric identity

The derivation begins with the Pythagorean Theorem. Then by dividing the Pythagorean Theorem by the square of the hypotenuse c^2, the first of three trigonometric identities ($(\sin A)^2 + (\cos A)^2 = 1$) is derived.

$$a^2 + b^2 = c^2$$

Divide both sides of the equation by c^2

$$\frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2}$$

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

$$(a/c)^2 + (b/c)^2 = 1$$

From Figure 1 we see that the ratio $(a/c)^2$ is equal to the $(\sin A)^2$, and the ratio $(b/c)^2$ is equal to the $(\cos A)^2$ our equation becomes

$$(\sin A)^2 + (\cos A)^2 = 1$$

This is the first trigonometric identity. To develop a feeling for the identity I ask a rhetorical question. Is this correct? Let's try it out. Choose the angle of 30^0

$$(\sin A)^2 + (\cos A)^2 = 1$$
$$(\sin 30)^2 + (\cos 30)^2 = 1$$
$$(0.5)^2 + (0.8660\ldots\ldots..)^2 = 1$$
$$1 = 1$$

Perhaps we were lucky, let's try an angle of 237.67935!

$$(\sin A)^2 + (\cos A)^2 = 1$$
$$(\sin 237.67935)^2 + (\cos 237.67935)^2 = 1$$
$$(-0.845\ldots)^2 + (-0.535\ldots\ldots.)^2 = 1$$
$$(0.714\ldots\ldots.)^2 + (0.285\ldots..^2 = 1$$
$$1 = 1$$

At this point students realize that luck had nothing to do with the solution. The trigonometric identity holds for all angles!

Derivation of the second trigonometric identity

A similar derivation then involves dividing the Pythagorean Theorem by the square of one of the sides of the triangle, a^2, the second trigonometric identity, $1 + (\cot A)^2 = (\csc A)^2$ is derived.

$$a^2 + b^2 = c^2$$

divide both sides of the equation by a^2

$$\frac{a^2 + b^2}{a^2} = \frac{c^2}{a^2}$$

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$

$$(1) + (b/a)^2 = (c/a)^2$$

From Figure 1 we see that the ratio $(b/a)^2$ is equal to the $(\cot A),^2$ and the ratio $(c/a)^2$ is equal to the $(\csc A)^2$ Our equation becomes

$$1 + (\cot A)^2 = (\csc A)^2$$

Derivation of the third trigonometric identity

Finally, by dividing the Pythagorean Theorem by the square of the third side of the triangle, b^2 the third trigonometric identity ($\tan^2 A + 1 = \sec^2 A$) is derived.

$$a^2 + b^2 = c^2$$

divide both sides of the equation by b^2

$$\frac{a^2 + b^2}{b^2} = \frac{c^2}{b^2}$$

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

$$(a/b)^2 + (1)^2 = (c/b)^2$$

From Figure 1 we see that the ratio $(a/b)^2$ is equal to the $(\tan A)^2$, and the ratio $(c/b)^2$ is equal to the $(\sec A)^2$. Our equation becomes

$$(\tan A)^2 + 1 = (\sec A)^2$$

The three trigonometric identities are listed below

$$(\sin A)^2 + (\cos A)^2 = 1$$
$$1 + (\cot A)^2 = (\csc A)^2$$
$$(\tan A)^2 + 1 = (\sec A)^2$$

Outcomes and Assessments

Students learn this derivation and are able to easily reproduce it at will. The comprehension of students for trigonometry and trigonometric identities is greatly enhanced by this method. This comprehension is essential for the analysis of vectors in physics and engineering and for evaluating some mathematical integrals. Three students received an A, four received a B, ten received a C and one grade was incomplete.

This course MAT201 is a prerequisite for Physics I. Many quantities in physics are vector quantities. It is often necessary to resolve a vector into component parts. This is required for analyzing Newton's Laws in two orthogonal coordinate systems. The resolution of a vector F into components F * cos A and F* sinA is accomplished by the students who know the derivation.

We have decided to change the textbook for the course next semester. Students had a great deal of difficulty with Calter's book. The new textbook was written by a colleague of mine Dr. Teddy Wong[(3).] Dr. Wong's Magic Hexagon, which will be used in conjunction with a compendium by Prentice Hall.

Conclusion

With a solid trigonometric understanding students proceed with confidence on a path leading to the completion of their required courses, and graduation. Knowing the derivation and working through the steps for the three trigonometric identities empowers our students to approach the future with confidence and knowledge.

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Bibliography

1. Caulter, Paul A & Michael A. "Technical Mathematics with Calculus Fourth Edition" John Wiley & Sons, Inc. New York 2000

2. Pariser, Bertram <u>"Teaching Critical Thinking",</u> American Society Engineering Education, Albuquerque, New Mexico, June 2001

3. Wong, Teddy "Wong's Magic Hexagon" Pearson Custom Publishing, New York 2002

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