A Solution of the Euler-Bernoulli Flexible Rotating Arm

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1 Abstract

The Euler-Bernoulli model provides a reliable method for calculating the deflection behavior of a single-link, flexible rotating arm. Previous work [4] has demonstrated a method of solving the Euler-Bernoulli equation for the displacement of the end effector, using variational methods. Other work [3, 5, 6, 8] has explored the feedback control of a flexible arm. The purpose of this paper is to derive the solution to the Euler-Bernoulli equation using Duhamel's principle as an illustrative alternative or even potential exercise for undergraduate students who have limited exposure to variational methods.

2 Introduction

The planar one-degree-of-freedom flexible arm is a canonical problem for students and researchers investigating novel feedback control algorithms, as well as PDE numerical and theoretical solution methods. The purpose of this paper is to show that it can also serve as a pedagogical tool for undergraduate PDE instructors. Generally, researchers have approached the problem via either energy methods (e.g. Hamilton's principle [7]) which are generally intractable to undergraduates, or by eigenfunction expansion. Eigenfunction expansion is an approach tractable to undergraduates; however, researchers traditionally do not handle the nonhomogeneous boundary conditions in manner consistent with typical undergraduate instruction. Here, we outline the solution in a way that parallels typical PDE instruction regarding nonhomogeneous boundary conditions, namely, to convert the nonhomogeneous boundary problem into a homogeneous boundary problem by adding an appropriate (nonhomogeneous) forcing term to the field equation. This idea is motivated by Duhamel's principle and can often be seen in textbooks, illustrated using the heat equation [1].

Following [4], the Euler-Bernoulli model is used to model the dynamics of the flexible beam. This model yields the following partial differential equation and boundary conditions.

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$$y^{(4)}(x,t) + \frac{\rho}{EI}\ddot{y}(x,t) = 0, \qquad 0 < x < L,\ t > 0, \tag{1a}$$
$$y(0,t) = 0, \qquad t > 0, \tag{1b}$$
$$y''(L,t) = 0, \qquad t > 0, \tag{1c}$$
$$J\ddot{y}'(0,t) - EIy''(0,t) = \tau(t), \quad t > 0, \tag{1d}$$
$$m\ddot{y}(L,t) - EIy'''(L,t) = 0 \qquad t > 0, \tag{1e}$$

where $y(x,t)$ is the deflection of the beam and the other parameters are listed in Table 1. Boundary condition (1d) is non-homogeneous, so we assume the solution can be decomposed [1] into the following two parts:

$$y(x,t) = w(x,t) + v(x,t) \tag{2}$$

where $w(x,t)$ is the solution to a second boundary-value-problem (BVP), and $v(x,t)$ is a function that we introduce in order to homogenize the boundary conditions of that BVP. Substituting (2) into (1) yields the following:

$$w^{(4)}(x,t) + \frac{\rho}{EI}\ddot{w}(x,t) = -v^{(4)}(x,t) - \frac{\rho}{EI}\ddot{v}(x,t), \qquad 0 < x < L,\ t > 0, \tag{3a}$$
$$w(0,t) + v(0,t) = 0, \qquad t > 0, \tag{3b}$$
$$w''(L,t) + v''(L,t) = 0, \qquad t > 0, \tag{3c}$$
$$J\left(\ddot{w}'(0,t) + \ddot{v}'(0,t)\right) - EI\left(w''(0,t) + v''(0,t)\right) = \tau(t), \qquad t > 0, \tag{3d}$$
$$m\left(\ddot{w}(L,t) + \ddot{v}(L,t)\right) - EI\left(w'''(L,t) + v'''(L,t)\right) = 0, \qquad t > 0, \tag{3e}$$

If we define

$$f(x,t) := -v^{(4)}(x,t) - \frac{\rho}{EI}\ddot{v}(x,t) \tag{4}$$

and choose the following conditions for $v(x,t)$:

$$v(0,t) = 0 \tag{5a}$$
$$v''(L,t) = 0 \tag{5b}$$
$$J\ddot{v}'(0,t) - EIv''(0,t) = \tau(t) \tag{5c}$$
$$m\ddot{v}'(L,t) - EIv'''(L,t) = 0 \tag{5d}$$

then the boundary value problem in (3) is transformed into

$$w^{(4)}(x,t) + \frac{\rho}{EI}\ddot{w}(x,t) = f(x,t), \quad 0 < x < L,\ t > 0, \tag{6a}$$
$$w(0,t) = 0, \qquad t > 0, \tag{6b}$$
$$w''(L,t) = 0, \qquad t > 0, \tag{6c}$$
$$J\ddot{w}'(0,t) - EIw''(0,t) = 0, \qquad t > 0, \tag{6d}$$
$$m\ddot{w}(L,t) - EIw'''(L,t) = 0, \qquad t > 0, \tag{6e}$$

which is a partial differential equation with homogeneous boundary conditions, but with a forcing function $f(x,t)$. Thus, the non-homogeneity is transferred from the boundary conditions to the field equation.

Beam Length L (cm)	40
Beam Height H (cm)	2.3
Beam Thickness T (cm)	0.18
Mass of End Effector m (g)	12.3
Linear Mass Density ρ (kg/m)	7.84×10^{-2}
Young's Modulus E (N/m^2)	70×10^9
Cross-Sectional Moment of Inertia I (m^4)	3.1491×10^{-12}
Motor Hub Moment of Inertia J (kg-m^2)	0.01

Table 1: Parameters of the flexible arm [8]

3 The Forcing Function

The forcing function is a function of the hub torque, $\tau(t)$, and some polynomial in x. To calculate $f(x,t)$, we first find $v(x,t)$. The only restriction on $v(x,t)$ is that it must satisfy the boundary conditions in (5). We assume that $v(x,t)$ is separable:

$$v(x,t) = g(x)\tau(t). \tag{7}$$

where $\tau(t)$ is the hub torque and we choose $g(x)$ as

$$g(x) = c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0, \tag{8}$$

where the coefficients c_n are found by translating the conditions in (5) into the following conditions for g:

$$g(0) = 0 \tag{9a}$$
$$g''(L) = 0 \tag{9b}$$
$$Jg'(0)\ddot{\tau}(t) - \left[EIg''(0) + 1\right]\tau(t) = 0 \tag{9c}$$
$$mg(L)\ddot{\tau}(t) - EIg'''(L)\tau(t) = 0 \tag{9d}$$

From (9c) and (9d), we also have that

$$g'(0) = 0 \tag{9e}$$
$$EIg''(0) + 1 = 0 \tag{9f}$$
$$g(L) = 0 \tag{9g}$$
$$g'''(L) = 0 \tag{9h}$$

Normally this would not be mathematically justifiable, but from a physical perspective we know that we have arbitrary control of τ and $\ddot{\tau}$. Therefore there is no *fixed* relationship between them and so it follows that the coefficients in (9c) and (9d) must be 0. Application of these boundary conditions yields a solution for $g(x)$, plotted in Figure 1.

4 The Eigenfunctions

The method of eigenfunction expansion begins by finding the eigenfunctions for the unforced (homogeneous) system [2], i.e. equation (6a) with $f(x,t) = 0$:

$$w^{(4)}(x,t) + \frac{\rho}{EI}\ddot{w}(x,t) = 0. \tag{10}$$

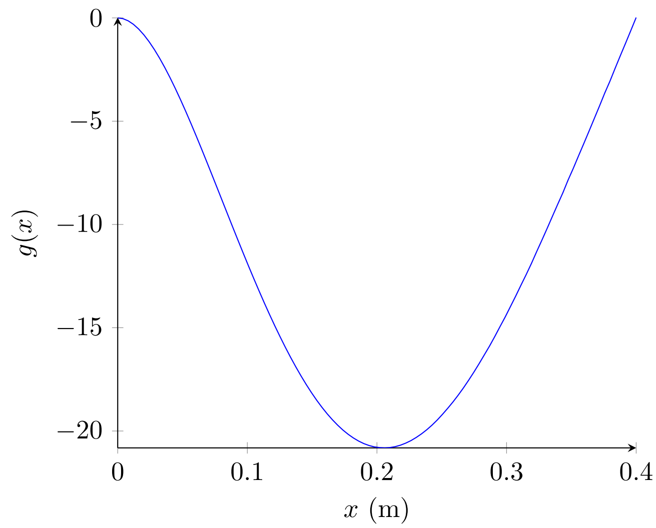

Figure 1: The function $g(x)$ represents a distributed forcing function that would displace the beam into the same shape as the application of a boundary hub torque.

We assume that the solution $w(x,t)$ is separable and has the form

$$w(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t). \tag{11}$$

Substituting this into the PDE for a particular n and rearranging gives

$$\frac{EIX_n^{(4)}(x)}{\rho X_n(x)} = -\frac{\ddot{T}_n(t)}{T_n(t)} = \omega_n^2. \tag{12}$$

which yields the following ODE:

$$X_n^{(4)} - \beta_n^4 X_n = 0, \tag{13}$$

where

$$\beta_n^4 = \frac{\rho\omega_n^2}{EI}. \tag{14}$$

We assume that the eigenfunctions $X_n(x)$ have the following form

$$X_n(x) = a_n \cos\beta_n x + b_n \sin\beta_n x + c_n \cosh\beta_n x + d_n \sinh\beta_n x \tag{15}$$

and we translate the boundary conditions from (6) into the following:

$$X(0) = 0 \tag{16a}$$
$$X''(L) = 0 \tag{16b}$$
$$J\omega^2 X'(0) + EIX''(0) = 0 \tag{16c}$$
$$m\omega^2 X(L) + EIX'''(L) = 0 \tag{16d}$$

Taking the first three derivatives of (15) and using (16) yields a system of four equations, represented by the following matrix equation:

$$\mathbf{Z}(\beta)\begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

where

$$\mathbf{Z}(\beta) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -\cos\beta L & -\sin\beta L & \cosh\beta L & \sinh\beta L \\ -1 & J\beta^3/\rho & 1 & J\beta^3/\rho \\ \sin\beta L + \frac{m}{\rho}\beta\cos\beta L & -\cos\beta L + \frac{m}{\rho}\beta\sin\beta L & \sinh\beta L + \frac{m}{\rho}\beta\cosh\beta L & \cosh\beta L + \frac{m}{\rho}\beta\sinh\beta L \end{bmatrix} \tag{18}$$

In order to solve for a_n, b_n, c_n and d_n, we find β for which $\mathbf{Z}(\beta_n)$ is singular. Ignoring the trivial solution, we assume a_n, b_n, c_n, d_n non-zero. For each β_n, the coefficients a_n, b_n, c_n, d_n can be computed numerically by finding the null-space of $\mathbf{Z}(\beta_n)$.

4.1 When $\beta = 0$

When $\beta = 0$, we have a different form for $X_0(x)$. Since we have that

$$X_0^{(4)} = 0, \tag{19}$$

we assume that $X_0(x)$ is a 3rd order polynomial.

$$X_0(x) = ax^3 + bx^2 + cx + d. \tag{20}$$

Again, we use the boundary conditions in (16) (with $\omega = 0$) and get that $a = b = d = 0$ and c is a free variable. Without loss of generality we will let $c = 1$. Then the eigenfunction for $\beta = 0$ is

$$X_0(x) = x. \tag{21}$$

5 Solution Via Orthogonality

Henceforth, we assume that $f(x,t) \neq 0$, so it is no longer true that $\ddot{T} + \omega^2 T = 0$. From [4] we have the following orthogonality condition on the eigenfunctions:

$$\int_0^L \rho X_r(x) X_n(x)\mathrm{d}x + mX_r(L)X_n(L) + JX_r'(0)X_n'(0) = \begin{cases} 0 & n \neq r \\ M_r & n = r \end{cases} \tag{22}$$

and it follows from (11) that

$$\int_0^L w(x,t)\rho X_r(x) = T_r(t)M_r - \sum_{n=0}^{\infty} T_n(t) m X_r(L) X_n(L) - \sum_{n=0}^{\infty} T_n(t) J X_r'(0) X_n'(0) \tag{23}$$

Recall that the field equation is

$$w^{(4)}(x,t) + \frac{\rho}{EI}\ddot{w}(x,t) = f(x,t). \tag{24}$$

Multiplying each side by $X_r(x)$ and integrating gives us

$$\int_0^L X_r(x)[w^{(4)}(x,t) + \frac{\rho}{EI}\ddot{w}(x,t)]\mathrm{d}x = \int_0^L X_r(x)f(x,t)\mathrm{d}x. \tag{25}$$

Rewrite (25) as

$$\int_0^L EIX_r(x)w^{(4)}(x,t)\mathrm{d}x + \int_0^L \rho X_r(x)\ddot{w}(x,t)\mathrm{d}x = \int_0^L EIX_r(x)f(x,t)\mathrm{d}x. \tag{26}$$

Using integration by parts four times on the first term and using boundary conditions from (6) and (16), we get

$$\begin{aligned} &mX_r(L)\ddot{w}(L,t) + JX_r(0)\ddot{w}'(0,t) + J\omega^2 X_r'(0)w'(0,t) + m\omega^2 X(L)w(L,t) + \\ &\qquad \int_0^L EIX_r^{(4)}w(x,t)\mathrm{d}x + \int_0^L \rho X_r(x)\ddot{w}(x,t)\mathrm{d}x = \int_0^L EIX_r(x)f(x,t)\mathrm{d}x \end{aligned} \tag{27}$$

where we have from (23) that

$$\int_0^L EIX_r^{(4)}w(x,t)\mathrm{d}x = \omega_r^2 \int_0^L \rho X_r w(x,t)\mathrm{d}x \tag{28}$$

$$= \omega_r^2 \left[T_r M_r - \sum_{n=0}^{\infty} T_n m X_r(L)X_n(L) - \sum_{n=0}^{\infty} T_n J X_r'(0)X_n'(0)\right] \tag{29}$$

and

$$\int_0^L \rho X_r(x)\ddot{w}(x,t)\mathrm{d}x = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^L \rho X_r w(x,t)\mathrm{d}x \tag{30}$$

$$= \ddot{T}_r M_r - \sum_{n=0}^{\infty} \ddot{T}_n m X_r(L)X_n(L) - \sum_{n=0}^{\infty} \ddot{T}_n J X_r'(0)X_n'(0) \tag{31}$$

We see that substituting (11), (29) and (31) into (27) results in

$$\left(\ddot{T}_r(t) + \omega^2 T_r\right) M_r = \int_0^L EIX_r(x)f(x,t)\mathrm{d}x. \tag{32}$$

Now, substituting in (4) and (7) gives us

$$\left(\ddot{T}_r(t) + \omega^2 T_r\right) M_r = -\tau(t)EI\int_0^L X_r(x)g^{(4)}(x)\mathrm{d}x - \ddot{\tau}(t)\rho\int_0^L X_r(x)g(x)\mathrm{d}x. \tag{33}$$

If we perform integration by parts four times on

$$EI\int_0^L X_r(x)g^{(4)}(x)\mathrm{d}x, \tag{34}$$

using the boundary conditions from (9) and (16), we get

$$\omega_r^2\int_0^L \rho X_r(x)g(x)\mathrm{d}x - X_r'(0). \tag{35}$$

If we then define

$$A_r = \int_0^L \rho X_r(x) g(x) \mathrm{d}x, \tag{36}$$

we can simplify (33) to the following:

$$\left(\ddot{T}_r(t) + \omega_r^2 T_r(t)\right) M_r = -\tau(t)\left(\omega_r^2 A_r - X_r'(0)\right) - \ddot{\tau}(t) A_r. \tag{37}$$

Taking the Laplace transform and rearranging terms gives us

$$T_r(s) = -\tau(s)\left(\frac{A_r}{M_r} - \frac{X_r'(0)}{M_r(s^2+\omega_r^2)}\right). \tag{38}$$

Making use of (11) and (2) gives us

$$y(x,s) = g(x)\tau(s) - \tau(s)\sum_{r=0}^{\infty} X_r(x)\left(\frac{A_r}{M_r} - \frac{X_r'(0)}{M_r(s^2+\omega_r^2)}\right) \tag{39}$$

Proceeding formally, if we expand $g(x)$ using the eigenfunctions $X_r(x)$ as its basis, we can define g_r such that

$$g(x) = \sum_{r=0}^{\infty} g_r X_r(x). \tag{40}$$

Then

$$g_r = \frac{\int_0^L \rho X_r(x) g(x) \mathrm{d}x}{\int_0^L \rho X_r^2(x)\mathrm{d}x + m X_r^2(L) + J\left(X_r'(0)\right)^2} = \frac{A_r}{M_r} \tag{41}$$

Using (41), equation (39) can now be written as

$$y(x,s) = \tau(s)\left(g(x) - \sum_{r=0}^{\infty} g_r X_r(x)\right) + \tau(s)\sum_{r=0}^{\infty}\frac{X_r(x)X_r'(0)}{M_r(s^2+\omega_r^2)} \tag{42}$$

and (40) implies that

$$\frac{y(x,s)}{\tau(s)} = \sum_{r=0}^{\infty}\frac{X_r(x)X_r'(0)}{M_r(s^2+\omega_r^2)} \tag{43}$$

Since $X_0(x) = x$,

$$\frac{y(x,s)}{\tau(s)} = \frac{x}{M_0 s^2} + \sum_{r=1}^{\infty}\frac{X_r(x)X_r'(0)}{M_r(s^2+\omega_r^2)} \tag{44}$$

6 Simulation

Putting the solution back into the time domain gives us

$$y(x,t) = \tau(t)\left(\frac{x}{M_0}t + \sum_{r=1}^{\infty}\frac{X_r(x)X_r'(0)}{M_r}\frac{\sin(\omega_r t)}{\omega_r}\right). \tag{45}$$

We use the following coordinate transformation [4]:

$$y(x,t) = u(x,t) + x\phi(t) \tag{46}$$

where

$$\phi(t) = y'(0,t) \tag{47}$$

and

$$u(x,t) = y(x,t) - x\phi(t) \tag{48}$$

The objective here is to solve for the beam deflection $u(x,t)$ for a range of t-values, and then rotate these solutions about the origin to account for rigid body motion. Note that $x\phi(t)$ is an arc length and the beam deflection $u(x,t)$ is a straight line that approximates an arc length for small deflections. See Figure 2. Performing this transformation gives us

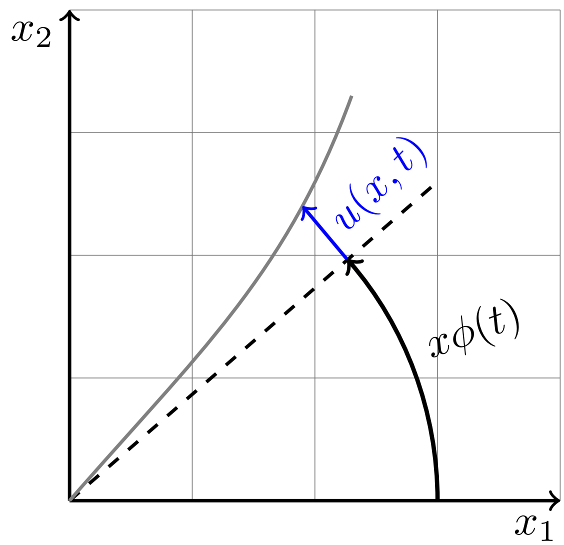

Figure 2: The coordinate transformation

$$u(x,t) = \tau(t)\left(\sum_{r=1}^{\infty} \frac{\sin(\omega_r t)}{\omega_r}\frac{X_r'(0)}{M_r}\left(X_r(x) - xX_r'(0)\right)\right) \tag{49}$$

Figure 3 shows $u(x,t)$ plotted with constant step input τ for several values of t. In this simulation, only the first vibrational mode is represented for visual clarity.

7 Summary

In this paper, we illustrate a modal solution method for the Euler-Bernoulli flexible rotating arm that can serve as a pedagogical reference for undergraduates studying partial differential equations. The method's distinctive contribution is to convert the nonhomogeneous boundary conditions into homogeneous boundary conditions by adding an appropriate forcing function to the field equation.

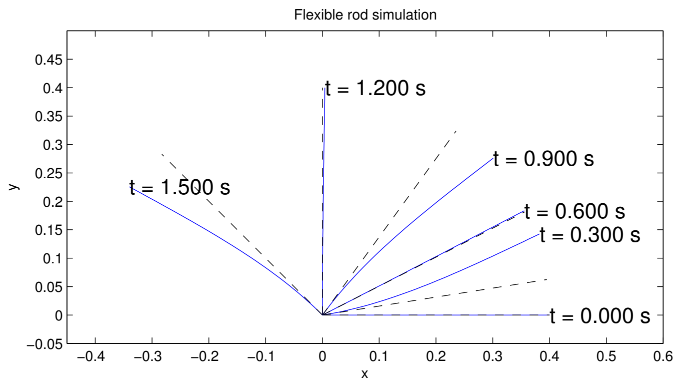

Figure 3: Step response with $\tau(t) = 1$ for $t \geq 0$.

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