A Time Domain Interpretation of the Bode Plot For Instructional Purposes

Monty J. Smith, Lakshmi Sudhaker
Department of Mechanical Engineering
College of Applied Science
Arkansas Tech University, Russellville, AR

Abstract

In this note, the classical single input single output map (transfer function) as represented in the frequency domain, will be equivalently described in the time domain with gain and phase characteristics explicitly identified. This enables the educator to represent input-output maps in a purely time domain perspective, and clarify the complications introduced by using the Laplace operator and the frequency domain when analyzing system gain and phase portraits (Bode plots). A simple under damped externally driven bounded input bounded output stable differential equation will be used in this study. Comparisons in the time and frequency domain will be shown with examples, and possible assets for future classroom presentations will be discussed.

1. Introduction

Modeling and simulation of mechanical and electrical systems (in a dynamic sense) often begin with a simplified low order linear model representing disturbance phenomena from a localized viewpoint. There is a strong correlation between understanding the vibration aspects of an open loop low order system (from combined electronics and the structural aspects) and the effects of the same system when exogenous disturbances produce unwanted effects in a controlled closed loop system. Some of the most elementary approaches for this type analysis involve time domain simulation (time response due to disturbances) and additionally frequency domain analysis using Bode plots and intrinsic evaluation of the spectral contents of the model.

In this paper, we will consider a typical second order linear model with a forced disturbance (one mode of a complicated array of dynamics from an assortment of electronics and mechanics) to understand in a very simple format the fundamental aspects of modeling and evaluation in the time versus frequency domain (spectral analysis). For example, consider the following solar panel that is pitch controlled via a slide actuator, and wind gusts capable of disturbing the orientation of the panel in Figure 1.

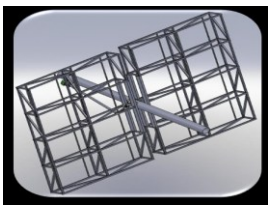

Figure 1 Structural Array / Solar Panel

If only local disturbance phenomena is being initially considered, then the first bending mode (without any sensor or actuator dynamics being considered) can be represented in its most elementary linear form as follows,

$$\ddot{x} + 2\zeta w_n \dot{x} + w_n{}^2 x = f \tag{1}$$

where $x \in \Re^1$ are coordinates in state space representing solar panel angles (deflection) off the vertical axis (the symbols representing differentiation $\dot{x} = dx/dt$, etc., has been used here), f is an external wind gust, the pair $\{\zeta, w_n\}$ are assumed positive constants representing the damping ratio ($\zeta << 1$) and natural frequency (radians) of the bending mode of the solar panel.

Since this presentation is focused on the analogies between the time and frequency domains, there are conditions on the external disturbance f that restrict the analysis based on Bode plots, and gain and phase margins. Since most signals can be projected onto a series of harmonic time domain operators, i.e., a series of sine and cosine functions (without presenting further mathematical details, the function resides in a Hardy space), we will assume that the function f is of the form $w_n^2 \sin(wt)$ where w is a selected input driving frequency in the measure of radians. In summary, the following time domain second order differential equation driven by various external harmonic driving functions will be considered throughout this paper.

$$\ddot{x} + 2\zeta w_n \dot{x} + w_n{}^2 x = w_n^2 \sin(wt) \tag{2}$$

When w is the only variable in the previous equation(ζ and w_n are constants), then an output to input function can be described in peak to peak terms, i.e., if the input is defined as $u(t) = w_n^2 \sin(wt)$, then it obviously has a peak value of w_n^2 on a time scale plot. In addition, since it can be shown that the solution to the output state is $x(t) = M \cdot \sin(wt + \phi)$ where M and ϕ are functions of the driving frequency w, then the output state has a peak value of M and the ratio of output to input is exactly M / w_n^2. The phase shift between peaks from input to output is also paramount, and both the magnitude and phase will be clearly presented in the time domain and the analogous frequency domain (transfer function and Bode plots) for a full understanding the analogies between the frequency and time domains.

In order to present the information needed to fully understand both domains (time and frequency) and the effective meaning in terms of the Bode plot, the main focus of the paper has been divided into 5 sections. Section 2 and section 3 present the time and frequency domain solutions to the differential equation being considered (in that order), and discusses the implications of the

resulting solutions. Details of the Bode plot and its original meaning will then be presented, and clear comparisons will be established between the time domain perspective and corresponding frequency domain. Finally, section 4 provides some examples that support the mathematical solutions in a graphical display, and section 5 concludes this paper with some comments on tools for educational possibilities in the classroom environment.

2. Time Domain Solution

In this section, we will briefly review the general solution to the externally driven differential equation in (2) with assumed initial conditions $x(t=0)=x_0$ and $\dot{x}(t=0)=\dot{x}_0$, and discuss the portion of the solution that's of interest in this paper.

Recall from the standard mathematical format (see ref [2] for details) that a general solution $x(t)$ to (2) can be described by two parts; a homogeneous solution purely derived by the initial conditions independent of the forcing function (i.e., setting the forcing function $u(t)=w_n^2\sin(wt)$ to zero), and a particular solution that satisfies (2) by assuming the given forcing function is known (i.e., $u(t)=0$). In fact, since the differential equation is locally Lipschitz (see [2], in essence the derivative is bounded), the solution is also unique. For convenience, we will use the constant $w_d=w_n\sqrt{1-\varsigma^2}$, where the constant under the radical is real due to the assumption that ς is positive and less than unity. In this format, the homogeneous solution is of the form

$$x_h(t)=e^{(-\varsigma w_n t)}(C\cos(w_d t)+D\sin(w_d t)) \tag{3}$$

where the constants $C=x(0)$ and $D=(\dot{x}(0)+\varsigma w_n x(0))/w_d$ are determined from the substitution of the initial conditions on the state $x(t)$ and the derivative $\dot{x}(t)$ at time $t=0$. In addition, the particular solution is the superposition of the cosine and sine functions (to account for phase) with constants A and B as follows,

$$\begin{aligned} x_p(t) &= (A\cos(wt)+B\sin(wt)) \\ &= \sqrt{A^2+B^2}\left\{\begin{matrix} A/\sqrt{A^2+B^2}\cos(wt) \\ +B/\sqrt{A^2+B^2}\sin(wt)\end{matrix}\right\} \end{aligned} \tag{4}$$

with A and B described as follows, $A=-2\varsigma w_n^3 w/((w_n^2-w^2)^2+(2\varsigma w_n w)^2)$ $B=w_n^2(w_n^2-w^2)/((w_n^2-w^2)^2+(2\varsigma w_n w)^2)$.

Finally, setting

$\sin(\phi)=A/\sqrt{A^2+B^2}$, $\cos(\phi)=B/\sqrt{A^2+B^2}$

we obtain

$$x_p(t)=\sqrt{A^2+B^2}\{\sin(wt+\phi)\}. \tag{5}$$

An explicit formula for ϕ is given by $\phi=-\tan^{-1}(2\varsigma w w_n/(w_n^2-w^2))$, and the leading peak value coefficient can be explicitly expressed as

$$\sqrt{A^2+B^2}=\frac{w_n^2}{\sqrt{(w_n^2-w^2)^2+(2\varsigma w w_n)^2}}. \tag{6}$$

Note that the frequency at which the particular solution is given consists of the same driving frequency w in the external forcing function (this is not a surprise since the solution should be periodic and of the same frequency as the driving function). Finally, by superposition of both components $x_h(t)$ and $x_p(t)$, we have obtained the complete solution $x(t)$, i.e., $x(t) = x_h(t) + x_p(t)$ due to initial conditions and the external driving function.

To conclude this section, notice that if viewing $x(t)$ as $t \to \infty$, the homogeneous portion of the solution vanishes (due to the A matrix being Hurwitz [3]), and we obtain

$$\lim_{t\to\infty} x(t) = x_p(t) \cdot$$

The particular solution (or the steady state response as time progresses towards infinity) is the solution of interest in this paper. In the next section, the Laplace operator will be used to obtain the frequency domain input-output map, and finally both the time and frequency domain solutions will both be compared descriptively, and how they relate using the Bode plot (and disregarding initial conditions on the state).

3. Frequency Domain Solution

To compare the solution of (2) in the frequency domain, the Laplace operator (denoted L) can be applied to both sides of the differential equation in (??) (and disregarding any terms associated with initial conditions at time zero, i.e., $t = 0$), we obtain $G(s)X(s) = L(\ddot{x} + 2\zeta w_n \dot{x} + w_n{}^2 x) = L(w_n^2 \sin(wt)) = U(s)$ and appropriate substitution (s=wj) we have the gain and phase functions

$$|G(jw)| = \frac{w_n^2}{\sqrt{(w_n^2 - w^2)^2 + (2\varsigma w w_n)^2}} \tag{7}$$

$$\phi = -\tan^{-1}((2\zeta w w_n)/(w_n^2 - w^2))$$

Notice these functions are the same as the time domain solutions obtained in section 2. In essence, we have provided identical gain and phase functions when considering input to output mapping from two different perspectives, all as a function of w. Since the actual Bode magnitude function is determined from taking the log and multiplying by a constant (20 log(|G(jw)|), the function in (xx) can be easily converted by a simple evaluation on a log scale.

4. Time / Frequency Domain Correlations

In this section, the driving sinusoidal function and the output position component of the second order system (as given in (2)) will both be plotted for multiple assigned values of w, and gain and phase will be carefully defined in display format to further support our understanding of the transfer function from a time domain perspective (plots will be provided on a log scale in the time domain for consistency with the scale used on Bode plots).

Since both ζ and w_n are constants throughout this paper, we will conveniently choose

$\zeta = \sqrt{3}/2$ and $w_n = 2\pi$ (which results in a damped natural frequency of π cycles per second) for display of an explicit solution for any given forcing function $u(t) = w_n^2 \sin(wt)$ with a set of solutions considered due to a span of the forcing frequency w.

Since we want to ultimately consider a span of external driving frequencies w for our comparative study, the analytical solution for one point at $w = \pi$ radians per second has been conveniently chosen in order to provide the transfer function value (one point on the Bode plot, gain and phase) as a simple example for clarity. Insertion of these numbers into (2), we have the following externally driven second order system,

$$\ddot{x} + 2\sqrt{3}\pi\,\dot{x} + 4\pi^2 x = 4\pi^2 \sin(\pi t). \tag{8}$$

For this system, the driving function and position have been derived and given explicitly as functions of time,

$$f(t) = 4\pi^2 \sin(\pi t),\ x_p(t) = (4/\sqrt{21})\{\sin(\pi t + \phi)\}. \tag{9}$$

A more convenient way to look at the dynamical system in (8) is by normalizing the forcing function to obtain

$$(\ddot{x} + 2\sqrt{3}\pi\,\dot{x})/(4\pi^2) + x = \sin(\pi t) \tag{10}$$

and the resulting newly defined forcing function and position, f' , and x_p, respectively, are expressed as

$$f'(t) = \sin(\pi t),\ x_p(t) = (4/\sqrt{21})\{\sin(\pi t + \phi)\}. \tag{11}$$

From the equations in (10), we can conveniently express the input-output relationship in time (effectively, the Bode equivalent) by finding the maximum of the output signal x_p at any given time, divided by the maximum of the input signal f', independent of any particular time point in either sequence. The resulting gain (denoted by Mag as a function of w) and phase in units of decibels and degrees, respectively, for the point chosen (i.e., $w = \pi$) is

$$\begin{aligned} Mag(\pi) &= 20 * \log((4/\sqrt{21})/1) \cong -2.72 \\ \phi &= -\tan^{-1}(2/(\sqrt{3})) \cong -49.11^{\circ} \end{aligned} \tag{12}$$

A sequence of points have been selected for evaluating the magnitude and phase, and listed for varying w in the set $[\pi, 30\pi]$ at 5π intervals as listed in the following Table 1.

Table 1 Gain-Phase Versus Frequency

w (rad's)	Magnitude (db)	Phase (degs)
pi	-2.72	-49.11
5 pi	-38.35	-140.48
10 pi	-64.79	-160.16
15 pi	-80.78	-166.77
20 pi	-92.20	-170.08
25 pi	-101.10	-172.06
30 pi	-108.40	-173.38

For comparative purposes, a Bode plot has been constructed using the Mathworks© software for magnitude and phase (constructed based on the Laplace transform setting), and the numbers in Table 1 are consistent with the plots in figure 1 for the selected frequencies.

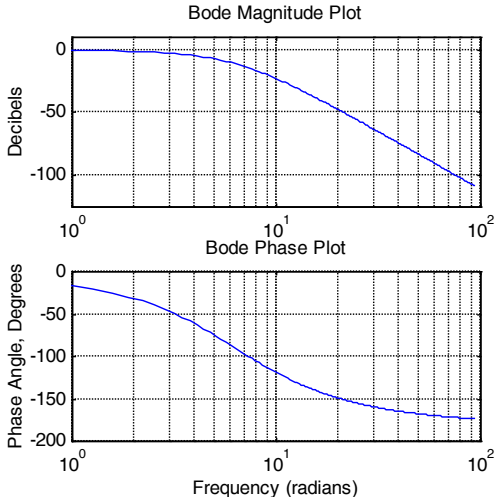

Figure 1 Bode Plot

This concludes our comparative discussion of the time and frequency domain functions and their corresponding Bode plot, with a simple second order system used for the analytical computations. We will conclude this paper with some thoughts about possible improved educational alternatives, when the frequency domain is present in the discussion (Laplace transforms and the Bode plot).

5. Conclusions

A time based equivalent development for describing the transfer function and Bode plot has been provided. The analysis was based purely on a stable system, and the technique emphasized the gain and phase portion in the time domain, and its' effective equivalent in the frequency domain. This has the potential for giving the instructor another method for describing transfer functions and their original intent, before introducing the more complicated tools used in the frequency domain (Bode plots, Nyquist plots, etc.).

References

(1) T. Kailath, "Linear Systems," Prentice-Hall, 1980.
(2) H. Khalil, "Nonlinear Systems," Pearson Prentice Hall, 3rd edition, 2007.
(3) K. Ogata, "Modern Control Engineering," Prentice-Hall, 5th edition, 2009.