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Alternative Modality of Delivery for the Exponential and Logarithmic Functions

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Alternative Modality of Delivery for the Exponential and Logarithmic Functions

Making engineering education more understandable to students can be difficult due to the demanding mathematical requirements that the major requires. One specific area of great difficulty for students is understanding the exponential and logarithmic functions. One part of the problem is that the logarithmic functions are not presented with an actual formula. They are only known as two keys on a calculator “ln” and “log”. A second problem is that they are represented in math books by the equation $y = log_b x$. Note that this notation does not reinforce standard function notation y = f(x). Secondly it also does not reinforce the fact that a logarithm is an exponent from the exponential function. It could be written as $x = log_b y$. The role of the inverse function is also not reinforced with standard college algebra math book notation. The inverse functions role is to change an (x,y) coordinate from the exponential function $y = b^x$ to (y,x) via the logarithmic function.

Logarithmic functions are classified as transcendental functions. A transcendental function cannot be written as a finite combination of algebraic expressions. This fact in most cases eliminates the equation from ever being seen by students. Students need an equation to gain a somewhat hands-on experience. This paper does not aim to eliminate the above-mentioned calculator keys from calculations but wishes to have students use the actual equation for a few examples. For simplicity only the first three terms from the selected natural logarithm function in its infinite formula will be used. These calculated values will then be compared to a value obtained with the “ln” calculator key.

The second major problem is the notation used for these functions in college algebra math books. The familiar function notation that students are accustomed to seeing in previous algebra courses is lost when in the standard logarithmic function notation. Students don’t even realize that the familiar f in f(x) is being replaced by “ln” in $y = lnx$. Also note that the standard notation lnx does not include parentheses such as y = ln(x) which is part of the standard function notation.

This paper is not written from a research perspective. There was no collected student data from surveys as to the effectiveness of this alternative approach. This paper is composed of the supplemental chapter needed for anyone interested in using this different teaching approach. This chapter will start with the introduction of the exponential functions

$$y = f(x) = b^x$$

and will show the transition to the inverse logarithmic functions.

The inverse (logarithmic) notation will be represented in the following form:

$$x = f(y) = \log_b y$$

The traditional inverse notation form will not be used which is the following:

$$y = f^{-1}(x) = \log_b x$$

In addition, the base will only be represented with the variable "b". It will not be replaced with x or any other variable when solving exponential or logarithmic equations. This supplemental chapter aims to reinforce the identity of the variables x, y, and b and not to interchange them.

Both the exponential function with base e, $y = f(x) = e^x$, and the natural logarithm $x = f(y) = \log_e(y) = \ln(y)$ will be graphed. The latter will be plotted with x on the vertical axis and y on the horizontal axis.

<u>Reasoning Behind the Supplemental Chapter</u>

The supplemental chapter is intended to present the exponential functions and their inverse (logarithmic) functions in a non-standard way to help students understand them. Four main changes in the delivery of the material are defined below.

Change 1) To show students the actual formula that the "ln" and "log" calculator keys represent.

The Power Series Expansion for the natural logarithm is given below.

$$\mathrm{x} = \log_e(y) = \ln(y) = 2A + \frac{2}{3}A^3 + \frac{2}{5}A^5 + \frac{2}{7}A^7 + \frac{2}{9}A^9 + \ldots$$

$$Where\ A = \frac{y - 1}{y + 1}\ and\ y > 0$$

The above equation was written with the y variable as the independent variable and x as the dependent variable. This is to reinforce to students the role of the inverse function.

To simplify the manual calculations, only the first three terms of the Power Series will be used.

$$x = \ln(y) \approx 2A + \frac{2}{3}A^3 + \frac{2}{5}A^5$$

$$Where\ A = \frac{y\ -\ 1}{y + 1}\ and\ y > 0$$

Having students perform a few calculations with the above formula will hopefully take the mystery out of the calculator “ln” and “log” keys. This will give them a hands-on feel of working with an actual equation.

To have a Power Series Expansion formula for “log” we will first use the Change of Base formula.

$$x = \log_{10}(y) = \log(y) = \left(\frac{1}{\ln 10}\right)(\ln(y))$$

$$x = \log_{10}(y) = \log(y) \approx \left(\frac{1}{\ln(10)}\right)(2A + \frac{2}{3}A^3 + \frac{2}{5}A^5)$$

$$Where\ A = \frac{y\ -\ 1}{y + 1}\ and\ y > 0$$

Change 2) To reinforce the role of the Inverse function.

Note in the above formulas for the natural and base 10 logarithms the independent variable was “y” not “x”. The standard notation is given as follows for the inverse function.

$$Standard\ Inverse\ Function\ Notation\ y = f^{-1}(x) = \log_b(x)$$

This supplemental chapter will use the following notation for the logarithmic (Inverse) function.

$$x = f(y) = \log_b(y)$$

This notation is used to reinforce that the inverse function takes in the y-values and returns the x-values from the exponential function.

Change 3) To reinforce function notation with the logarithmic function.

Traditional notation for the logarithmic function is as follows:

$$y = log_b x$$

The supplemental chapter will use the following notion.

$$y = f(x) = log_b(x)$$

The simple addition of parentheses around the independent variable. Here students can see that $"f"\ is\ being\ replaced\ by\ "log_b"$. The traditional notation is taking f(x) that students are familiar with and changing it to f x with no parentheses.

This change can help reduce errors such as the following:

$$\frac{ln8}{ln4} = ln2$$

Change 4) To limit the number of variables used and not to interchange them.

When solving traditional logarithmic equations in any algebra textbook, the variables b, and y become x as in the following 2 equations.

$$2 = \log_x 9$$

$$5/2 = \log_4(x)$$

To retain the variable identities, the equations will be written in the following form.

$$2 = \log_b 9$$

$$5/2 = \log_4(y)$$

The Supplemental Chapter for Students

The Exponential Function

The exponential function or the "**variable exponent function**" has the following form:

$$y = f(x) = b^x$$

$$x = the\ exponent$$

The base b can be any real number greater than zero. Two cases exist for the base b.

Case 1)

Base b is a **Proper Fraction** or is a real number in the following open interval:

$$(0,1)$$

Typical examples are the following:

$$y = f(x) = (1/2)^x$$

$$In\ this\ case\ base\ b = \frac{1}{2}$$

$$y = f(x) = (7/8)^x$$

$$In\ this\ case\ base\ b = \frac{7}{8}$$

A typical graph of the Case 1 exponential function is given in Figure 1 shown below.

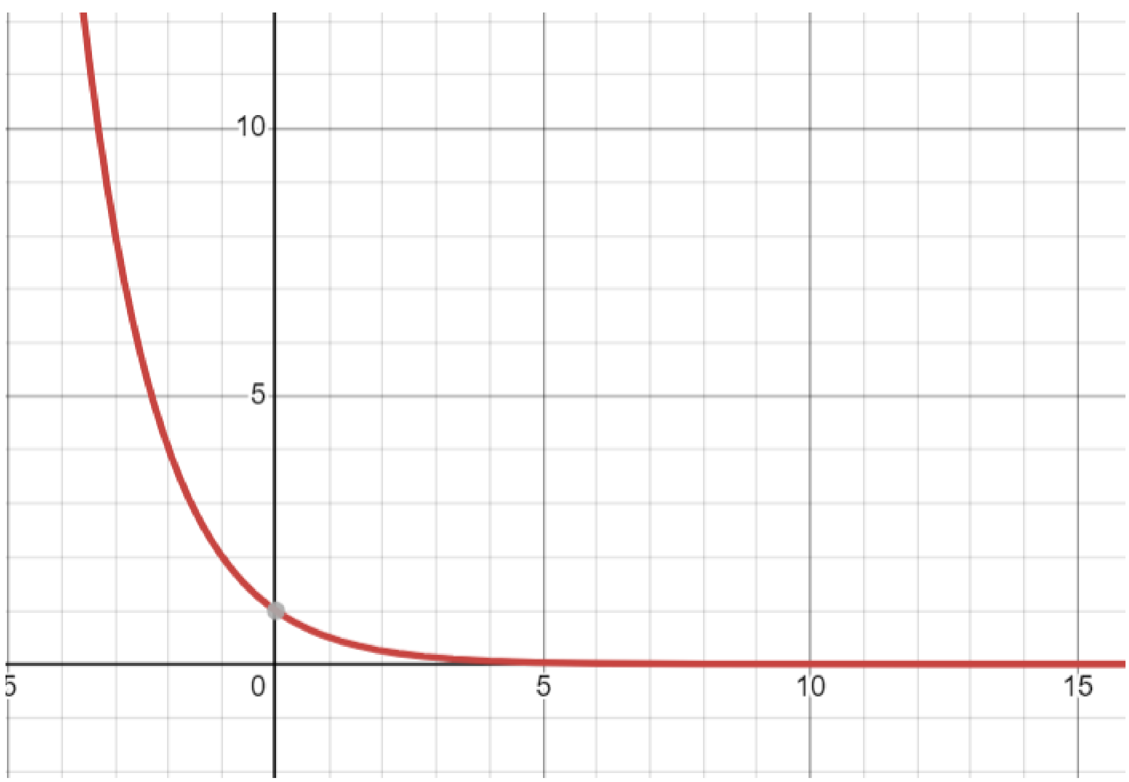


Figure 1

The y-values of the graph approach positive infinity as x approaches negative infinity. As the x-values approach positive infinity the y-values approach zero. The positive x-axis is a horizontal asymptote. Note the y-intercept of (0,1)

Case 2)

Base b is a **Whole Number** greater than 1 or an **Improper Fraction** in the following open interval:

$$(1, \infty)$$

Typical examples are the following:

$$y = f(x) = (2)^x$$

$$In\ this\ case\ base\ b\ (Whole\ Number) = \frac{2}{1}$$

$$y = f(x) = (9/7)^x$$

$$In\ this\ case\ base\ b = \frac{9}{7}$$

You could use b = 1, but 1 to any power is always 1, so the graph is just the horizontal line y = 1.

Two special bases in Case 2 will be highlighted in this chapter, base e, and base 10. The number e is an irrational number and can be approximated as follows, e = 2.718. The resulting exponential equations are as follows:

$$y = f(x) = (e)^x$$

$$y = f(x) = (10)^x$$

A typical graph of the Case 2 exponential function is given in Figure 2 shown below.

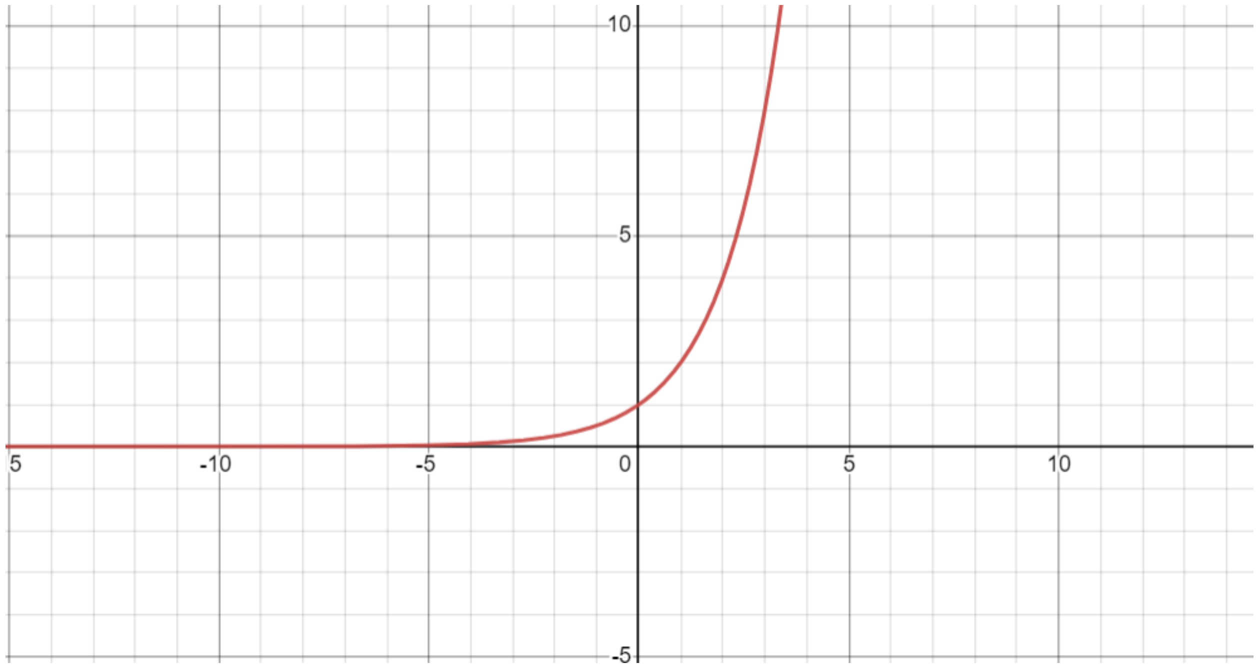


Figure 2

The y-values of the graph approach positive infinity as x approaches positive infinity. As the x-values approach negative infinity the y-values approach zero. The negative x-axis is a horizontal asymptote. Note the y-intercept of (0,1)

The exponential function is classified as a one-to-one function. This means that a horizontal line drawn anywhere through the graph of the function crosses the graph at exactly one and only one point. An example of this is given in Figure 3 shown below.

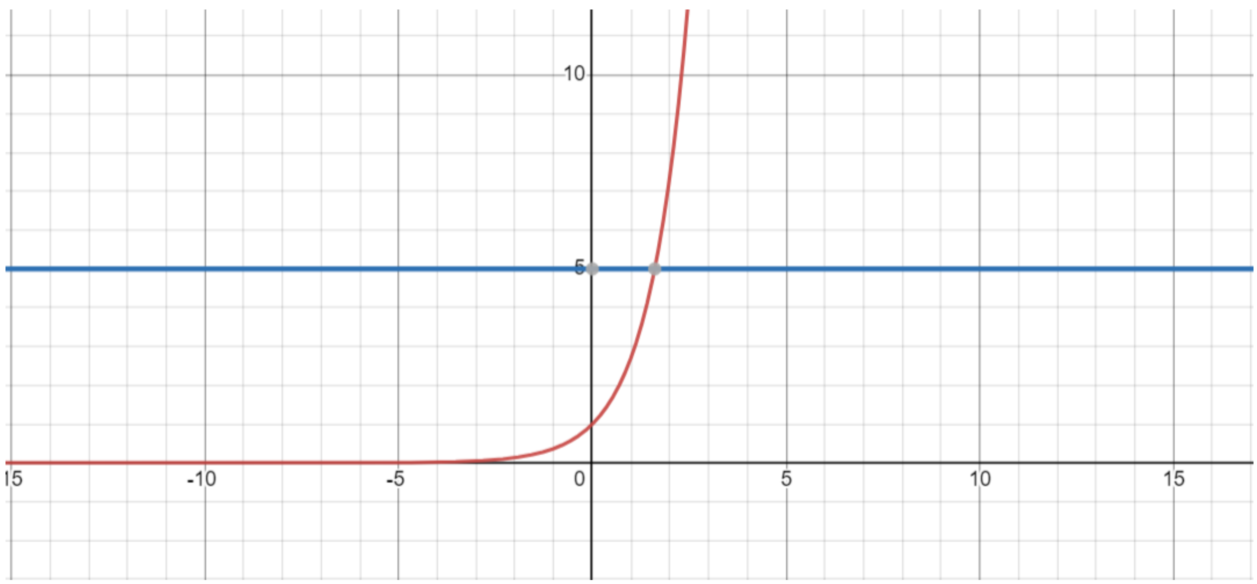


Figure 3

The blue horizontal line intersects the graph at only one point. When a function passes the horizontal line test it is classified as a one-to-one function, and this implies that an inverse function exists. The inverse function takes in the y-values from the exponential function and returns the exponent x for a given specified base b.

Note in both Case 1 and 2 the base b is a **constant** and the exponent is a variable. The domain for the exponential function is all real numbers $(-\infty, \infty)$. The Range for the exponential function is $(0, \infty)$.

In summary for the Exponential Function:

$$y = f(x) = b^x$$

Domain: $(-\infty, \infty)$ The allowable x-values

Range: $(0, \infty)$ The resulting y-values

<u>The Inverse Function of the Exponential Function</u>

The exponential function is classified as a one-to-one function which means that it has an inverse function. The role of the inverse function is to simply switch an ordered pair such as (x, y) to (y, x). Some one-to-one functions have an inverse function that can be algebraically determined as in the following example.

$$y = f(x) = 2x + 8$$

This is a linear equation. All linear lines are one-to-one functions, except for vertical and horizontal lines. To determine the inverse function f(y), we solve the above equation for x.

$$The\ Inverse\ function = x = f(y) = \frac{y - 8}{2}$$

Note the Inverse function is designated as $x = f(y)$.

To show the role of the inverse function let's first determine the y-value from

$y = f(x) = 2x + 8$, when x is set equal to 3.

$$y = f(3) = 2(3) + 8 = 14$$

Therefore, the point (3,14) lies on the graph of f(x). The inverse function f(y) now takes in the calculated y-value of 14 and returns the x-value that was used.

$$x = f(y) = \frac{y - 8}{2}$$

Using y = 14

$$x = f(14) = \frac{14 - 8}{2} = \frac{6}{2} = 3$$

Therefore, the point (14,3) lies on the graph of the inverse function. This is all the inverse function does, it takes in the y-values of a one-to-one function f(x) and returns the x-values that were previously used.

However, for the exponential function it is not algebraically possible to solve the equation for x. For example, if we started with the exponential function $y = f(x) = e^x$ we cannot algebraically solve the equation for x. However, mathematicians have developed what are called transcendental functions that are in the form of a Power Serries Expansion and for $y = f(x) = e^x$ the inverse function can be written as the following.

$$\text{x} = \text{f(y)} = \log_e(y) = \ln(y) = 2A + \frac{2}{3}A^3 + \frac{2}{5}A^5 + \frac{2}{7}A^7 + \frac{2}{9}A^9 + \ldots$$

$$Where\ A = \frac{y - 1}{y + 1}\ and\ y > 0$$

The above formula has an infinite number of terms. As you calculate the terms far from the equal sign, they get closer and closer to zero. The above formula is programmed into a calculator. It would be impossible to calculate an infinite number of terms. For practical purposes we can get a good approximation by using only the first three terms as in the equation below. Your calculator does the same thing, except it uses many more terms.

$$\text{x} = \text{f(y)} = \ln(y) \approx 2A + \frac{2}{3}A^3 + \frac{2}{5}A^5$$

The **General Form** of the inverse logarithmic functions is defined as the following.

$$x = f(y) = \log_b(y)$$

Note that the f in $f(y)$ is being replaced by log_b. Two special notations exist for logarithms of base e and base 10.

$$\log_e(y) = \ln(y)$$

$$\log_{10}(y) = \log(y)$$

The logarithmic formula changes depending on the base b. Each exponential function with a specific base b has its own unique inverse function f(y). Similarly, the inverse function for the exponential function $y = f(x) = 10^x$ is given below.

$$x = \log\,(y) = (0.4342945)(2A + \tfrac{2}{3}A^3 + \tfrac{2}{5}A^5 + \tfrac{2}{7}A^7 + \tfrac{2}{9}A^9 + \,.\,.\,.)$$

$$Where\ A = \frac{y\ -\ 1}{y + 1}\ and\ y > 0$$

Note the new coefficient in front of the power series of 0.4342945. For practical purposes we can get a good approximation by using only the first three terms as in the equation below. Your calculator does the same thing, except it uses many more terms.

$$x = \log\,(y) \approx (0.4342945)(2A + \frac{2}{3}A^3 + \frac{2}{5}A^5)$$

Example 1)

Determine $\ln(2)$ and the ln (0.01) using the power series expansion formula and then with a calculator. We are also using the calculator to see how well our approximation of three terms works.

Using the Power Series Expansion formula for the ln(2)

$$\mathrm{x} = \ln\,(y) \approx 2A + \frac{2}{3}A^3 + \frac{2}{5}A^5$$

$$Where\ A = \frac{y\ -\ 1}{y + 1}\ and\ y > 0$$

First calculate A. The value of y is 2.

$$A = \frac{2 -\ 1}{2 + 1} = \frac{1}{3}$$

$$x = \ln(2) \approx 2\left({}^{1}/_{3}\right) + \frac{2}{3}\left({}^{1}/_{3}\right)^{3} + \frac{2}{5}\left({}^{1}/_{3}\right)^{5}$$

$$x = \ln(2) \approx 0.693004115$$

Using the calculator for the ln(2)

x = ln (2) = 0.693147181 using a Texas Instrument Calculator

Note, our power series expansion formula value is very close to the value determined when the "ln" calculator key was used.

Using the Power Series Expansion formula for the ln(0.01)

$$x = \ln(0.01) \approx -2.950171567$$

Using the calculator for the ln(0.01)

x = ln (0.01) = -4.605170186 using a Texas Instrument Calculator

Note, our calculated value which we approximated with only the first three terms is **not** that close to the value determined when the "ln" calculator key was used which uses many more than the first 3 terms from the formula. To get a better fit we would need to use more than the first 3 terms.

Hopefully you can now see that the "ln" key on a calculator represents an actual function.

Example 2) Verify the roll of the inverse function for base b =10.

For the following Exponential Function $y = f(x) = 10^{x}$,

confirm the following y-values from the given x-values.

Given x-values	Calculated y-values
0.5	$\sqrt{10}$
1	10
1.1	12.58925412

Using the following exponential function $y = f(x) = 10^{x}$.

$$y = f(0.5) = 10^{0.5} = \sqrt{10} \approx 3.16227766$$

$$y = f(1) = 10^1 = 10$$

$$y = f(1.1) = 10^{1.1} = 12.58925412$$

The Inverse Function takes in the y-values just calculated and returns the previously given x-values. The Inverse function for base 10 has the following general form:

$$x = f(y) = \log(y) \approx (0.4342945)(2A + \frac{2}{3}A^3 + \frac{2}{5}A^5)$$

$$Where\ A = \frac{y\ -\ 1}{y + 1}\ and\ y > 0$$

$$For\ y = \sqrt{10}, determine\ A$$

$$A = \frac{\sqrt{10} -\ 1}{\sqrt{10} + 1} = 0.5194939$$

$$x \approx (0.4342945)[2(0.5194939) + \frac{2}{3}(0.5194939)^3 + \frac{2}{5}(0.5194939)^5]$$

$$x \approx 0.498390812$$

Note the approximated value is very close to actual value used of 0.5. Similarly for the remaining two y-values we have the following.

For y = 10

$$x = \log(10) \approx 0.93293439$$

This value is close to 1

For y = 12.58925412

$$x = f(12.58925412) = \log(12.58925412) \approx 0.998709277$$

This value is close to 1.1

Hopefully you can once again see that there is an actual formula for inverse logarithmic function, but it is much easier to use the “ln” and “log” keys on a calculator.

Solving Exponential Equations

When solving an exponential or logarithmic equation the following standard is applied.

$$The\ General\ Exponential\ Equation\ y = f(x) = b^x$$

The variable x will always represent the exponent in the exponential equation.

The variable y is always the y-value obtained from the exponential equation.

The variable b will always represent the base.

Type 1 - Easy Exponential Equations

These problems will have just 1 exponential equation with the general form consisting of an unknown exponent x, and a given numerical y-value and base b.

Easy exponential equations follow the following rule.

$$if\ b^{x1} = b^{x2}, then\ x1 = x2$$

Example 1)

Solve the following exponential equation for x.

$$32 = 2^{x+4}$$

Knowns:

32 = The y-value

2 = The base b

$$32 = 2^5 = 2^{x+4}$$

$$2^5 = 2^{x+4}$$

Since the bases are the same on both sides of the equation, we can set the two exponents equal. Note how 32 was written as 2 to the fifth power.

$$5 = x + 4$$

$$1 = x$$

We can now check the solution:

$$32 = 2^{1+4} = 2^5 = 32$$

Type 2 - Easy Exponential Equations

These problems will involve two different exponential equations. We set the equations equal. The general form consists of an unknown exponent x, and a given base b. The base b for each of the two exponential equations must be the same number.

Example 2)

Solve for the unknown x value given in the 2 exponential equations below.

$y_1 = 6^{2x}$ and $y_2 = 6^{1-3x}$

We want to determine an x-value that makes $y_1 = y_2$. The y-values are the same for both functions. This is equivalent to finding the point of intersection when the 2 functions are plotted on the same graph.

$$y_1 = y_2$$

$$6^{2x} = 6^{1-3x}$$

Since the bases are the same on both sides of the equation, we can set the two exponents equal.

$$2x = 1 - 3x$$

$$x = {}^{1}/_{5}$$

Example 3)

Solve for the unknown x value given in the 2 exponential equations below.

$y_1 = 64^{4x}$ and $y_2 = 16^{x-4}$

Note the two bases are different. We must find a common base.

$$64 = 2^6 \; and \; 16 = 2^4$$

$$y_1 = (2^6)^{4x} \text{ and } y_2 = (2^4)^{x-4}$$

$$y_1 = 2^{24x} \text{ and } y_2 = 2^{4x-16}$$

$$y_1 = y_2$$

$$2^{24x} = 2^{4x-16}$$

$$24x = 4x - 16$$

$$x = -\frac{4}{5}$$

Solving Logarithmic Equations

Example 4)

Solve for the unknown base b in the following logarithmic equation.

$$3 = \log_b({}^{8}/_{27})$$

Note that the exponent x = 3 and the y-value = 8/27. To solve a logarithmic equation, we go back to the associated exponential equation. For this example, the equation would be the following.

$${}^{8}/_{27} = b^3$$

This is an algebraic equation. To solve for base b, we must take the cube root of both sides of the equation.

$$\sqrt[3]{{}^{8}/_{27}} = \sqrt[3]{b^3}$$

$${}^{2}/_{3} = b$$

Example 5)

Solve for the unknown y-value in the following logarithmic equation.

$${}^{5}/_{2} = \log_4(y)$$

We go back to the associated exponential equation. For this example, the equation would be the following.

$$y = 4^{5/2} = \left(\sqrt{4}\right)^5 = 2^5 = 32$$

Example 6)

Solve for the unknown exponent x in the following logarithmic equation.

$$x = \log_{49}\left(\sqrt[3]{7}\right)$$

We go back to the associated exponential equation. For this example, the equation would be the following.

$$\sqrt[3]{7} = 49^{x}$$

This is a Type 1 - Easy Exponential Equation. We must rewrite the equation so that each side has the same base b.

$$7^{1/3} = (7^2)^x = 7^{2x}$$

$$1/3 = 2x \text{ or } x = 1/6$$

Example 7)

Solve for the unknown base b in the following logarithmic equation.

$$2 = \log_b(9)$$

We go back to the associated exponential equation. For this example, the equation would be the following.

$$9 = b^2$$

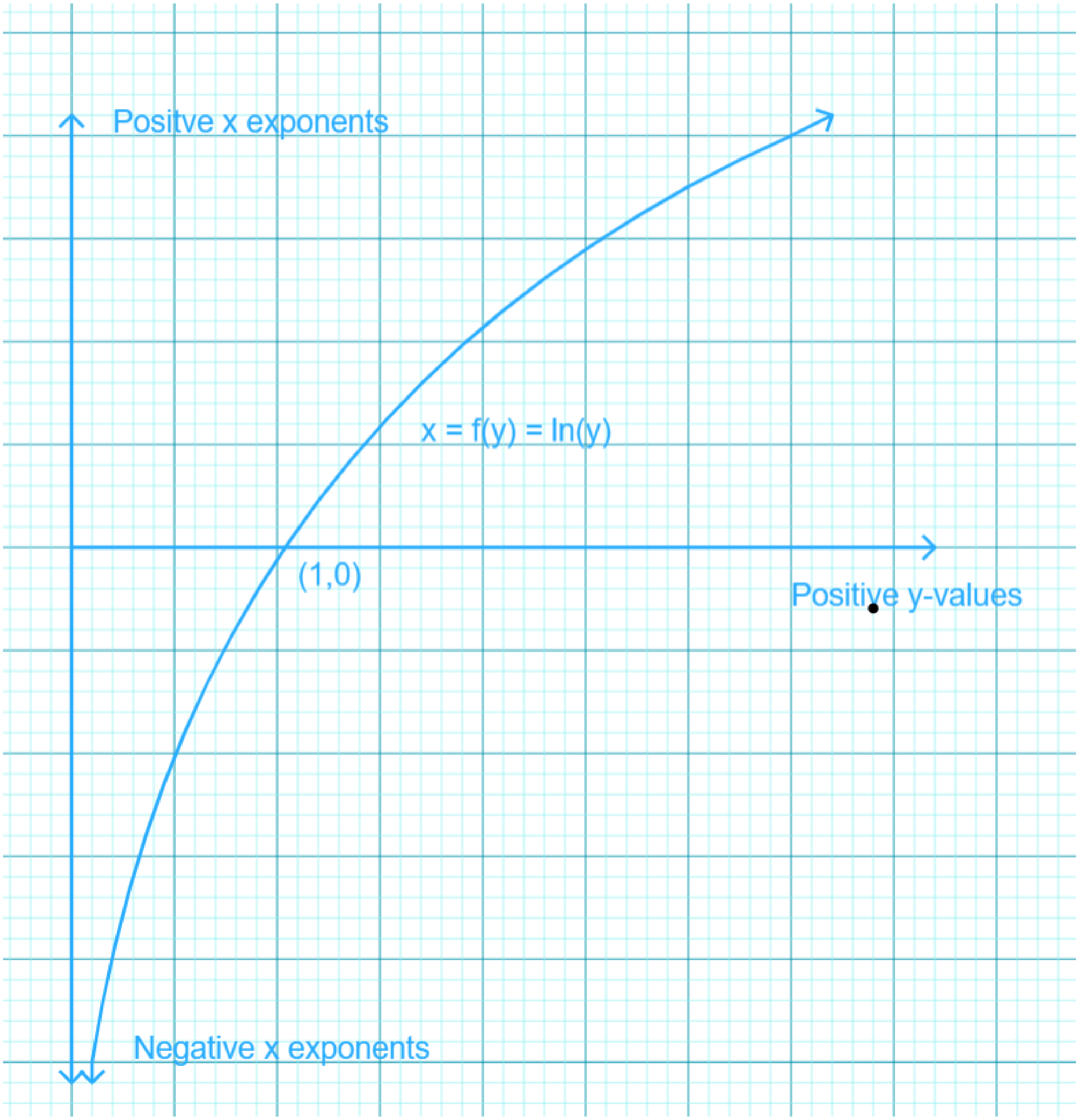
This is an algebraic equation. To solve for base b, we must take the square root of both sides of the equation.

$$\sqrt{9} = \sqrt{b^2}$$

$$\pm 3 = b$$

Note that we solved a quadratic equation which always gives 2 roots. We must choose the correct root. The exponential equation must have a base b greater than zero or a positive number. Therefore, we use b = 3.

Lastly the graph of the inverse logarithmic function $x = f(y) = \ln(y)$ is plotted below. Note the vertical axis is labeled with the exponent x and the horizontal axis is label as the y-values.



Note that the domain of the function above is $(0, \infty)$ which was the range of the exponential function. The range of the function above is $(-\infty, \infty)$ which was the domain of the exponential function. This is how every function - inverse function pair works.

Discussions and Conclusions

In conclusion, this paper hopes to explain the connection between the logarithmic power series function and the "ln" and "log" keys on a calculator. This paper is for the mathematically under prepared student. Students will hopefully understand that the given functions are somewhat tedious to work with and that these formulas are programmed into their calculators. They will also be reinforced to understand that the inverse function performs the role of accepting the y-values from a given one-to one function (the exponential function) and returns the x-values used in the exponential function. It has been my experience over the last several years that students have responded very favorably to seeing the connection between the calculator logarithmic keys and the actual power series formula. After a few calculations using the formula they are happy to use the Ln calculator key. They have also appreciated the fact that once the variables x, y, and b have been defined they remain constant when solving equations.