An Algorithm for Computing **Q**uotient and **R**emainder Polynomials

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ABSTRACT

The task of dividing one polynomial by another is encountered in continuous fraction expansion (CFE) and other engineering and systems science computations. This note presents an efficient algorithm for performing the division. A method for constructing synthetic division tableaus (SDT) for polynomials over any coefficient field is formulated and the relative ease in extracting the solution from the tableau is demonstrated. The beauty of the method lies in its simplicity even for manual calculations; and above its efficiency, a minimal memory space is needed for program execution. While other programs and algorithms exist for performing this task, the algorithm introduced in this correspondence promises high efficiency and simplicity in formulation than many of the existing methods. To demonstrate its effectiveness and efficiency, the new method is compared to other existing methods.

I. Introduction

Polynomial long division (PLD) is often encountered in system science. It is used for computing the greatest common divisor of two polynomials. A description of the operations of polynomial long division can be found in many texts on algebraic computing. Most of these descriptions are simply extensions or direct application of Euclid's algorithm. Also described in the literature is synthetic division algorithm for polynomials applicable only in the case of a first order denominator (devisor) polynomial. PLD operations have been implemented in several different algebraic programs, over the years, with varying efficiencies and computer memory requirements. Examples of softwares (old and new) with PLD operations include Altran, Derive, Macsyma, MathCad, Mathematica, Maple, Reduce, SAC, and SMP.

The purpose of this correspondence is to introduce a novel algorithm for polynomial long division which is comparable to the Euclidean in efficiency, if not superior. It is applicable to polynomials of any degrees and over any coefficient field. This new algorithm also requires minimal memory space and thus will be as useful as existing operations for computer program implementation. Perhaps the most attractive features of the proposed algorithm are its conceptual simplicity and convenience for calculations.

This paper constructs a PLD tableau based on a pattern of relationships between the operands in a long division process. An algorithm for constructing the tableau is described. An algorithm for constructing the tableau, the termination criterion and interpretation of results are described. An example is used to illustrate the application of the proposed algorithm.

II. Formulation of Algorithm

Consider two polynomials in *s*, *N(s)* and *D(s)* over a field, given by:

$$N(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0, \quad \textit{and} \qquad (1)$$

$$D(s) = b_d s^d + b_{d-1}s^{d-1} + b_{d-2}s^{d-2} + \ldots + d_0$$

Where d > or = n.

$$D(s) = Q(s)N(s) + R(s). \qquad (2)$$

It can be shown that the quotient polynomial Q(s) is of the form:

$$Q(s) = b_d s^{d-n}/a_n + \{c_1 s^{d-n-1} + c_2 s^{d-n-2} + \ldots + c_{d-n}\} \qquad (3)$$

and the remainder polynomial *R(s)* is given by:

$$R(s) = r_1 s^{n-1} + r_2 s^{n-2} + \ldots + r_n \qquad (4)$$

A tableau can be constructed from which the coefficients c_t of *Q(s)* and r_j of *R(s)* are obtained. The first two rows of the tableau are formed by the coefficients, $a_{n-j,}$ of *N(s),* the divisor - first row, and the coefficients $b_{d-j,}$ of *D(s),* the dividend - second row;

j = 0,1,2,...d. Elements of subsequent rows of the tableau are derived from the determinants of 2x2 matrices formed from an array of the first and last rows. In addition to the rows formed with the coefficients of the divisor, N(s) - the Pivot row, and the coefficients of the dividend, D(s) defined as *'row zero'*, d-n+1 rows are generated. Thus the synthetic division tableau so formed is a (d-n+2) x (d+1) matrix. This format is illustrated in Table 1. below.

Table 1. Format for Synthetic Division Tableau

Description	j=0	j=1	j=2	...	d
PIVOT ROW: a_{n-j}	a_n				0
ROW ZERO: b_{d-j} t=0	$c_{0,0} = b_d$				b_0
t=1	$c_{1,0}$	$c_{1,1}$	$c_{1,2}$		
t=2	$c_{2,0}$	$c_{2,1}$			
.					
.					
t=d-n	$c_{d-n,0}$				
REMAINDER COEFFICIENT ROW: t=d-n+1					

Based on the above row definitions, equation (3) simply becomes

$$Q(s) = 1/a_n\{c_{0,0}s^{d-n-0} + c_{1,0}s^{d-n-1} + c_{2,0}s^{d-n-2} + \ldots + c_{d-n,0}\} \qquad (5)$$

Where $c_{t,0}$ (t = 0,1,2, . . ., d-n) represents the 'row t, column zero' element (entry) on the tableau. Also, the entry $c_{t,j}$ on the tableau is given by

$$c_{t,j} = \{ a_n c_{t-1,j+1} - a_{n-1-j} c_{t-1,0}\}/ a_n ; \quad \text{for } t = 1,2,\ldots,d-n+1; \quad j= 0,1,2,\ldots,d-1 \qquad (6)$$

i.e. $c_{t,j}$ are calculated using the pivot row and the last row.

The following postulates are useful for determining the termination of the algorithm and for obtaining the results from the tableau:

(i) *Q(s)* has d-n+1 possible terms
(ii) *R(s)* has n terms
(iii) The coefficient of the s^{d-n-j} term in *Q(s)* is $c_{t,0}$; for t = j = 0,1,2,…d-n
(iv) The coefficient of the s^{n-j-1} term in *R(s) is* $c_{d-n+1,j}$; for j = 0,1,2…n-1.

Thus we observe that once the tableau is constructed the quotient polynomial can easily be assembled from the first row, j=1, and the coefficients for the remainder polynomial are simply the elements of the last row, t=d-n+1, and in descending power of the variable.

III. Illustrative Example

To illustrate the proposed algorithm, we consider the division D(s)/N(s) of the following pair of Polynomials given by:

$N(s) = 2 s^2 + 5s + 2$, and
$D(s) = 2 s^4 + 11 s^3 + 27 s^2 + 32s + 12$

d = 4, n = 2, the order of Q(s) = d-n = 2, the order of R(s) = n-1 = 1, and the number of steps = d-n+1 = 3; t = 0,1,2,3.

a_{n-j}:	2	5	2	0	0
t_0:	2	11	27	32	12
t_1:	6	25	32	12	0
t_2:	10	26	12	0	0
t_3:	1	2	0	0	

Hence, $Q(s) = s^2 + 3s + 5$, and $R(s) = s + 2$.

IV. Concluding Remarks

A nontrivial algorithm for finding the quotient and remainder polynomials is derived. It will be noted that the algorithm requires an explicit division only by a_n the leading coefficient of N(s) and it should also be observed that there are only d–n+1 steps required in the computation of the coefficients of Q(s) and R(s). In these regards this algorithm is similar to the Euclidean and since there are no loops or recursions in this algorithm it might be more efficient than most long division process described in algebraic texts.

Bibliography
[1] Kalu, A.; "An Engineering Approach to Model Order Reduction and its Application to Controller Design." Ph.D. dissertation, Louisiana State University, Baton Rouge, Louisiana, July 1985.

[2] Gourishankar, V. and Donald H. Kelly; "Electromechanical Energy Conversion, 2nd Edition; Intent Educational Publishers, New York.

[3] Knth, Donald E.; Seminumerical Algorithms vol-2 , The Art of Computer Programming." Addison-Wesley Publishing Company, Reading, Mass.

[4] Johnson, Richard E.; "University Algebra." Prentice-Hall, Inc.; Englewwod Cliffs, New Jersey

[5] Bajpai, Mustoc and Walker; "Engineering Mathematics." John Wiley and Sons, London, England.

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