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An Alternative Form of Euler's Equation for the Rotational Dynamics of a Rigid Body Confined to Planar (2-D) Motion

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Introduction

Instructors of engineering mechanics who have previously taught statics and dynamics courses over a sustained period of time are likely familiar with the practices listed below, which address the conventional evaluation of the appropriate moments-of-forces/couples equation that governs the rotational behavior of a rigid body:

- <u>Statics</u>: Moments may be evaluated about axes through any selected point in space, which is typically on, in, or nearby the rigid body of interest.
- <u>Dynamics</u>: Moments should be evaluated about either (a) axes through the mass center of the body, or (b) a fixed axis about which the body is constrained to rotate (if applicable).

This article presents another option for evaluation of the moments-of-forces/couples equation for the targeted case of dynamics. The scope of application of the method proposed herein will be restricted to planar (2-D) motion of rigid bodies, though it is possible to extend this method to spatial (3-D) motion as well. However, it is generally more involved in this context, and it might be less suited for (and of less interest to) engineering students at the undergraduate level.

In this method, the moments-of-forces/couples equation may be evaluated at any point on or in the rigid body, but it must be an embedded point at which the kinematics of the body motion is either already provided or readily assessed. However, as will be discussed and demonstrated in this article, the equation associated with this method lends itself especially well to problems that involve a *composite rigid body* (i.e., a set of rigid elements which are rigidly joined together).

Two illustrative examples are considered in this article to both introduce and apply the method advocated. These examples will reveal the advantage of moment evaluation about a point that is different from the mass center of the body. When the method is properly applied, the associated effort is typically less involved than is experienced in the traditional practices because locating the position and assessing the motion of the mass center is often challenging in many problems.

Some alternative forms of the moments-of-forces/couples equation for the rotational dynamics of a rigid body, which is frequently called *Euler's equation* in the literature, may be found in [1–3]. The author has examined standard textbooks and other technical references, and it appears that the specific form of the equation presented in this article is novel and useful.

Simple/Single Rigid-Body Case

From the analysis of a general system of particles subjected to both external and mutual-internal forces (for which a modest particle-interaction restriction is assumed), the equations that govern the translational and rotational dynamics of the system are most commonly expressed [4–6] as

$$\mathbf{F} = m\,\mathbf{a}_{\mathrm{G}} \tag{1}$$

$$\mathbf{M}_{\mathrm{G}} = \dot{\mathbf{H}}_{\mathrm{G}} \tag{2}$$

where the subscript G identifies a quantity associated with the mass center of the system, while $\mathbf{F}$, $\mathbf{M}_{\mathrm{G}}$, and $\mathbf{H}_{\mathrm{G}}$ respectively denote the total external force acting on the system, total moment about G of the external forces acting on the system, and total angular momentum of the system about G at any instant t in time. As customarily adopted in engineering and physics, each dot above a quantity denotes a derivative operation with respect to t applied to that quantity. Also, m denotes the total mass of the system, which is presumed to be fixed since (in most situations of practical interest) the system consists of a well-defined aggregate of particles whose collective behavior will be assessed as the system state dynamically evolves over time.

Consider a general point P. From basic principles of mechanics, it can be easily shown that

$$\mathbf{M}_{\mathrm{P}} = \mathbf{M}_{\mathrm{G}} + \mathbf{r}_{\mathrm{G/P}} \times \mathbf{F} \tag{3}$$

where $\mathbf{M}_{\mathrm{P}}$ denotes the total moment about P of the external forces acting on the system, while $\mathbf{r}_{\mathrm{G/P}}$ denotes the relative position vector of G with respect to P, for which $\mathbf{r}_{\mathrm{G/P}} \equiv \mathbf{r}_{\mathrm{G}} - \mathbf{r}_{\mathrm{P}}$. Then, after performing substitutions with Eqs. (1) and (2), Eq. (3) becomes

$$\begin{aligned} \mathbf{M}_{\mathrm{P}} &= \dot{\mathbf{H}}_{\mathrm{G}} + \mathbf{r}_{\mathrm{G/P}} \times (m\,\mathbf{a}_{\mathrm{G}}) \\ &= \dot{\mathbf{H}}_{\mathrm{G}} + m(\mathbf{r}_{\mathrm{G/P}} \times \mathbf{a}_{\mathrm{G}}) \end{aligned} \tag{4}$$

Next, suppose that the system of particles corresponds to a *rigid body* confined to planar (2-D) motion, and that point P is embedded on/in this body. When such a body is also geometrically symmetric about the reference plane in which the motion occurs, it is well established [7] that

$$\dot{\mathbf{H}}_{\mathrm{G}} = I_{\mathrm{G}}\,\boldsymbol{\alpha} \tag{5}$$

Furthermore, general rigid-body kinematics [8] enables $\mathbf{a}_{\mathrm{G}}$ to be alternatively expressed as

$$\begin{aligned} \mathbf{a}_{\mathrm{G}} &= \mathbf{a}_{\mathrm{P}} + \boldsymbol{\alpha} \times \mathbf{r}_{\mathrm{G/P}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{\mathrm{G/P}}) \\ &= \mathbf{a}_{\mathrm{P}} + \boldsymbol{\alpha} \times \mathbf{r}_{\mathrm{G/P}} + (\boldsymbol{\omega} \cdot \mathbf{r}_{\mathrm{G/P}})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{r}_{\mathrm{G/P}} \\ &= \mathbf{a}_{\mathrm{P}} + \boldsymbol{\alpha} \times \mathbf{r}_{\mathrm{G/P}} - \omega^2\,\mathbf{r}_{\mathrm{G/P}} \end{aligned} \tag{6}$$

since $\boldsymbol{\omega} \cdot \boldsymbol{\omega} = \omega^2$ and $\boldsymbol{\omega} \cdot \mathbf{r}_{\mathrm{G/P}} = 0$ because $\boldsymbol{\omega} = \omega\,\mathbf{k}$ and $\mathbf{r}_{\mathrm{G/P}} = x_{\mathrm{G/P}}\,\mathbf{i} + y_{\mathrm{G/P}}\,\mathbf{j}$ in the case of planar motion. Of course, $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ denote the angular acceleration and angular velocity of the body; I_{G} denotes the mass-moment of inertia of the body about G. The useful vector identity [9]

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{7}$$

has been utilized to simplify the triple vector product $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{G/P})$ in Eq. (6). With the result of Eq. (6) and the identity in Eq. (7), it is observed that

$$\begin{aligned} m(\mathbf{r}_{G/P}\times\mathbf{a}_G) &= m(\mathbf{r}_{G/P}\times\mathbf{a}_P)+m\,[\mathbf{r}_{G/P}\times(\boldsymbol{\alpha}\times\mathbf{r}_{G/P})]-\omega^2\,m(\mathbf{r}_{G/P}\times\mathbf{r}_{G/P}) \\ &= m(\mathbf{r}_{G/P}\times\mathbf{a}_P)+m\,[(\mathbf{r}_{G/P}\cdot\mathbf{r}_{G/P})\boldsymbol{\alpha}-(\mathbf{r}_{G/P}\cdot\boldsymbol{\alpha})\mathbf{r}_{G/P}] \\ &= m(\mathbf{r}_{G/P}\times\mathbf{a}_P)+m\,r_{G/P}^2\,\boldsymbol{\alpha} \end{aligned} \tag{8}$$

since $\mathbf{r}_{G/P}\times\mathbf{r}_{G/P}=\mathbf{0}$, $\mathbf{r}_{G/P}\cdot\mathbf{r}_{G/P}=r_{G/P}^2$, and $\mathbf{r}_{G/P}\cdot\boldsymbol{\alpha}=0$ because $\boldsymbol{\alpha}=\alpha\,\mathbf{k}$ and $\mathbf{r}_{G/P}=x_{G/P}\,\mathbf{i}+y_{G/P}\,\mathbf{j}$ in the case of planar motion. Consequently, based upon Eqs. (5) and (8), Eq. (4) becomes

$$\mathbf{M}_P = I_P\,\boldsymbol{\alpha}+\mathbf{r}_{G/P}\times(m\,\mathbf{a}_P) \tag{9}$$

since $I_P = I_G + m\,r_{G/P}^2$ via the *parallel-axis theorem* [10], where $r_{G/P}=\|\mathbf{r}_{G/P}\|$. It is beneficial to recast Eq. (9) into a slightly different form, which has an intriguing interpretation:

$$\mathbf{M}_P-\mathbf{r}_{G/P}\times(m\,\mathbf{a}_P)=I_P\,\boldsymbol{\alpha} \tag{10}$$

This result suggests that it is possible to evaluate the moments-of-forces/couples equation about an arbitrary (but embedded) point P if an extra term 'produced' by an *inertial force* $(-m\,\mathbf{a}_P)$ is included. This apparent force is 'applied' at the mass center G of the body (as indicated by the relative position vector $\mathbf{r}_{G/P}$), and it may be interpreted as a reactive (i.e., virtual) contribution to the moment about P already produced by the external (i.e., actual) forces acting on the body. It is vital to understand that this apparent force is not included in the evaluation of Eq. (1), which only accounts for forces that are actually exerted on the body.

Finally, a closer inspection of Eq. (10) reveals that it reduces to the familiar forms expected for the moments-of-forces/couples equation when P is (a) the mass center of the body ($\mathbf{r}_{G/P}=\mathbf{0}$), or (b) a fixed point ($\mathbf{a}_P=\mathbf{0}$). Hence, Eq. (10) may be regarded as a *generalization* of the equation that governs the rotational dynamics of a rigid body confined to planar motion. Of course, this equation must be supplemented with Eq. (1) in order to completely characterize the dynamics of the rigid body of interest.

Composite Rigid-Body Case

Consider a rigid body composed of n rigid elements which are rigidly joined together. Suppose that an arbitrary point P is embedded on/in one of the elements of this body. Then the rotational dynamics of this composite rigid body can be described by a modified form of Eq. (10) which is adapted for this kind of body. Based upon the development of Eq. (10), it can be concluded that an equation governing the rotational motion of the entire set of elements is given by

$$\mathbf{M}_{\mathrm{P}} - \left[\sum_{k=1}^{n} m_k \left(\overline{\mathbf{r}}_k - \mathbf{r}_{\mathrm{P}} \right) \right] \times \mathbf{a}_{\mathrm{P}} = \left[\sum_{k=1}^{n} \left(I_{\mathrm{P}} \right)_k \right] \boldsymbol{\alpha} \qquad (11)$$

since Eq. (10) may be separately applied to each rigid element, and then the equations obtained are summed together to yield Eq. (11). $\overline{\mathbf{r}}_k$ identifies the position of the mass center of the k-th rigid element with mass m_k (see Figure 1), which is typically located by inspection, whereas $\overline{I}_k$ denotes the mass-moment of inertia of the k-th element about this same location, from which

$$\left(I_{\mathrm{P}} \right)_k = \overline{I}_k + m_k \, d_k^2 \quad , \quad d_k = \| \overline{\mathbf{r}}_k - \mathbf{r}_{\mathrm{P}} \| \qquad (12)$$

The selection of point P is based upon convenience, as guided by insight and experience, but the particular point selected should provide a definite advantage (for the analysis) over other points which could have been chosen. As in the former case considered, the composite rigid body must be confined to planar motion, and it must be symmetric with respect to the reference plane if the external forces that contribute to $\mathbf{M}_{\mathrm{P}}$ act only parallel to this plane [11].

For this kind of body, it is typically easier to apply Eq. (11) than Eq. (4) since the need to locate the mass center for the overall body is obviated, and it is presumed that the evaluation of $\mathbf{a}_\mathrm{P}$ is much simpler (or much more convenient) than the evaluation of $\mathbf{a}_\mathrm{G}$. Furthermore, based upon the definition of the mass center for the overall body, two successive time-differentiations yield

$$m\,\mathbf{r}_\mathrm{G} \equiv \sum_{k=1}^{n} m_k\,\overline{\mathbf{r}}_k \quad \Rightarrow \quad m\,\mathbf{a}_\mathrm{G} = \sum_{k=1}^{n} m_k\,\overline{\mathbf{a}}_k \tag{13}$$

in which case Eq. (1) becomes

$$\mathbf{F} = \sum_{k=1}^{n} m_k\,\overline{\mathbf{a}}_k \tag{14}$$

In some situations, the $\overline{\mathbf{a}}_k$ can be directly assessed due to constraints on the motion of the body. In others, it is easier to evaluate the $\overline{\mathbf{a}}_k$ for Eq. (14) in terms of $\mathbf{a}_\mathrm{P}$ via the kinematic relation

$$\overline{\mathbf{a}}_k = \mathbf{a}_\mathrm{P} + \boldsymbol{\alpha}\times(\overline{\mathbf{r}}_k - \mathbf{r}_\mathrm{P}) - \omega^2\,(\overline{\mathbf{r}}_k - \mathbf{r}_\mathrm{P}) \tag{15}$$

In summary, Eqs. (11) through (15) are generally preferred for the analysis of a composite rigid body, and it is essential to acknowledge that $\mathbf{F}$ and $\mathbf{M}_\mathrm{P}$ respectively denote the total (resultant) force and total moment about P produced by the external forces acting on <u>any</u> of the elements of this kind of body.

Some objects are fabricated from a single piece of homogeneous stock material but actually can be viewed/treated as a composite rigid body due to their complicated shape. In these situations, it is natural to virtually partition such an object into a set of continuous rigid elements, which are geometrically evident and rigidly joined together, for the purposes of design and/or analysis. In this scenario, the presence of composite rigid bodies might be expected in many applications.

Examples of the Method Proposed

Each of the examples considered below involve rigid bodies for which the path of motion of the mass center is non-trivial, so it is preferable to select an alternate point at which the equations of motion for the rigid body under consideration may be evaluated. A simple/single rigid body will be initially examined, and a composite rigid body will be subsequently examined.

EXAMPLE PROBLEM 1 – A Simple/Single Rigid Body

A slender uniform bar of mass m_{b} and length l_{b} is constrained so that its ends must move along support surfaces (or guide rails) without incurring any frictional effects, as depicted in Figure 2. The angle θ uniquely identifies the configuration of the bar. The end sliders are assumed to be sufficiently small so that both their size and mass may be neglected. The bar is released from a *state of rest*, with $\theta = 30^\circ$ at $t = 0$. The case in which gravity alone drives the motion of the bar is investigated. Evaluate the reaction forces $\{R_{\mathrm{A}}, R_{\mathrm{B}}\}$ and angular acceleration α at the instant in time when the bar is released. The data considered for this problem consists of $m_{\mathrm{b}} = 50\ \mathrm{lbm}$, $l_{\mathrm{b}} = 4\ \mathrm{ft}$, $\beta = 45^\circ$, and $\omega = 0$. Accordingly, $m_{\mathrm{b}} = 50/32.174 = 1.554\ (\mathrm{lbf} \cdot \mathrm{s}^2)/\mathrm{ft}$.

Constraints and Relations

As a direct result of the geometric constraints on the bar motion, simple vector relationships and two successive time-differentiations yield

$$\mathbf{r}_{\mathrm{G}} = \tfrac{1}{2}(\mathbf{r}_{\mathrm{A}} + \mathbf{r}_{\mathrm{B}}) \quad \Rightarrow \quad \mathbf{a}_{\mathrm{G}} = \tfrac{1}{2}(\mathbf{a}_{\mathrm{A}} + \mathbf{a}_{\mathrm{B}}) \tag{16}$$

Other kinematic constraints for the two-dimensional motion of the falling/sliding bar include

$$\begin{aligned} \mathbf{a}_{\mathrm{A}} &= a_{\mathrm{A}}\,\mathbf{i} \\ \mathbf{a}_{\mathrm{B}} &= a_{\mathrm{B}}\left[(\cos\beta)\,\mathbf{i} - (\sin\beta)\,\mathbf{j}\right] \end{aligned} \quad , \quad \begin{aligned} \mathbf{a}_{\mathrm{G}} &= a_x\,\mathbf{i} + a_y\,\mathbf{j} \\ \boldsymbol{\alpha} &= \alpha\,\mathbf{k} \end{aligned} \tag{17}$$

When selected relations from Eqs. (16) and (17) are combined, it can be shown that

$$a_x = \tfrac{1}{2}\left[a_{\mathrm{A}} + a_{\mathrm{B}}\cos\beta\right] \quad , \quad a_y = -\tfrac{1}{2}a_{\mathrm{B}}\sin\beta \tag{18}$$

However, as revealed by the kinematic constraint below, a_{A}, a_{B}, and α are related:

$$\mathbf{a}_{\mathrm{B}} = \mathbf{a}_{\mathrm{A}} + \boldsymbol{\alpha}\times\mathbf{r}_{\mathrm{B/A}} - \omega^2\,\mathbf{r}_{\mathrm{B/A}} \quad ; \quad \mathbf{r}_{\mathrm{B/A}} = l_{\mathrm{b}}\left[(-\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}\right] \tag{19}$$

Taken together, the relations in Eqs. (17), (18), and (19), along with $\omega = 0$, yield

$$\begin{aligned} a_{\mathrm{A}} &= \left[\tfrac{\cos\theta}{\tan\beta} + \sin\theta\right] l_{\mathrm{b}}\,\alpha \\ a_{\mathrm{B}} &= \tfrac{\cos\theta}{\sin\beta}\, l_{\mathrm{b}}\,\alpha \end{aligned} \quad \Rightarrow \quad \begin{aligned} a_x &= \left[\tfrac{\cos\theta}{\tan\beta} + \tfrac{1}{2}\sin\theta\right] l_{\mathrm{b}}\,\alpha \\ a_y &= -\tfrac{1}{2}(\cos\theta)\, l_{\mathrm{b}}\,\alpha \end{aligned} \tag{20}$$

From the parallel-axis theorem, the mass-moment of inertia of the bar about end A is given by

$$\begin{aligned} I_{\mathrm{A}} &= I_{\mathrm{G}} + m_{\mathrm{b}}\, r_{\mathrm{G/A}}^2 \\ &= \tfrac{1}{12} m_{\mathrm{b}}\, l_{\mathrm{b}}^2 + m_{\mathrm{b}}\left(\tfrac{1}{2} l_{\mathrm{b}}\right)^2 \\ &= \tfrac{1}{3} m_{\mathrm{b}}\, l_{\mathrm{b}}^2 \end{aligned} \tag{21}$$

For the reference frame adopted (see Figure 2), the relevant external forces acting on the bar are identified from the free-body diagram and expressed as

$$\mathbf{W}_{\mathrm{b}} = -(m_{\mathrm{b}}\,g)\,\mathbf{j} \quad , \quad \begin{aligned} \mathbf{R}_{\mathrm{A}} &= R_{\mathrm{A}}\,\mathbf{j} \\ \mathbf{R}_{\mathrm{B}} &= R_{\mathrm{B}}\left[(\sin\beta)\,\mathbf{i} + (\cos\beta)\,\mathbf{j}\right] \end{aligned} \tag{22}$$

<u>Newtonian/Eulerian Equations of Motion</u>

From the free-body diagram for the bar, with Eqs. (17), (20), and (22), Eq. (1) becomes

$$(\sin\beta)\,R_{\mathrm{B}}\,\mathbf{i}+(\cos\beta)\,R_{\mathrm{B}}\,\mathbf{j}+R_{\mathrm{A}}\,\mathbf{j}-(m_{\mathrm{b}}\,g)\,\mathbf{j} = m_{\mathrm{b}}\left[\tfrac{\cos\theta}{\tan\beta}+\tfrac{1}{2}\sin\theta\right]l_{\mathrm{b}}\,\alpha\,\mathbf{i}-m_{\mathrm{b}}\,\tfrac{1}{2}(\cos\theta)\,l_{\mathrm{b}}\,\alpha\,\mathbf{j} \tag{23}$$

which, by equating corresponding components on each side, yields the scalar equations

$$(\sin\beta)\,R_{\mathrm{B}}-m_{\mathrm{b}}\left[\tfrac{\cos\theta}{\tan\beta}+\tfrac{1}{2}\sin\theta\right]l_{\mathrm{b}}\,\alpha=0 \tag{24}$$

$$R_{\mathrm{A}}+(\cos\beta)\,R_{\mathrm{B}}+\tfrac{1}{2}m_{\mathrm{b}}\,(\cos\theta)\,l_{\mathrm{b}}\,\alpha=m_{\mathrm{b}}\,g \tag{25}$$

The advantage of the method advocated herein is revealed in the evaluation of the special form of the moments equation, which is Eq. (10) for this case. In this problem, end A of the bar will serve as the reference point P for this equation. If moments of forces indicated in the free-body diagram (as well as the inertial force depicted) are taken about end A, then R_{A} is <u>not</u> involved, which enables Eq. (10) to be more easily evaluated as

$$\mathbf{r}_{\mathrm{G/A}}\times\mathbf{W}_{\mathrm{b}}+\mathbf{r}_{\mathrm{B/A}}\times\mathbf{R}_{\mathrm{B}}-\mathbf{r}_{\mathrm{G/A}}\times(m_{\mathrm{b}}\,\mathbf{a}_{\mathrm{A}})=I_{\mathrm{A}}\,\boldsymbol{\alpha} \tag{26}$$

where $\mathbf{r}_{\mathrm{G/A}}=\tfrac{1}{2}\,\mathbf{r}_{\mathrm{B/A}}$. When results from Eqs. (17) through (22) are combined with Eq. (26), and the indicated operations are performed, the equation obtained involves only terms based upon the unit vector $\mathbf{k}$, so this equation can be simply expressed in the scalar form

$$m_{\mathrm{b}}\,g\left(\tfrac{1}{2}l_{\mathrm{b}}\cos\theta\right)-(R_{\mathrm{B}}\cos\beta)(l_{\mathrm{b}}\cos\theta)-(R_{\mathrm{B}}\sin\beta)(l_{\mathrm{b}}\sin\theta) + m_{\mathrm{b}}\left[\tfrac{\cos\theta}{\tan\beta}+\sin\theta\right]l_{\mathrm{b}}\,\alpha\left(\tfrac{1}{2}l_{\mathrm{b}}\sin\theta\right)=\tfrac{1}{3}m_{\mathrm{b}}\,l_{\mathrm{b}}^{2}\,\alpha \tag{27}$$

or, equivalently,

$$\cos(\beta-\theta)\,R_{\mathrm{B}}+\left\{\tfrac{1}{3}-\tfrac{1}{2}(\sin\theta)\left[\tfrac{\cos\theta}{\tan\beta}+\sin\theta\right]\right\}m_{\mathrm{b}}\,l_{\mathrm{b}}\,\alpha=\tfrac{1}{2}(\cos\theta)\,m_{\mathrm{b}}\,g \tag{28}$$

where the identity $\cos(\beta-\theta)=\cos\beta\cos\theta+\sin\beta\sin\theta$ has been utilized. Equations (24), (25), and (28) form a set of relations for the unknown variables $\{R_{\mathrm{A}}, R_{\mathrm{B}}\}$ and α. At this juncture, it is prudent to enter the data given for the parameters within these relations, which become (when reordered) the following system of equations:

$$\begin{aligned} 0.9659\,R_{\mathrm{B}}-0.05080\,\alpha&=21.65\\ 0.7071\,R_{\mathrm{B}}-6.937\,\alpha&=0\\ R_{\mathrm{A}}+0.7071\,R_{\mathrm{B}}+2.692\,\alpha&=50 \end{aligned} \tag{29}$$

Because the first two equations involve only the variables R_{B} and α (which is made possible by the special form of the moments equation utilized above), they can be jointly solved to yield

$$\begin{aligned} \alpha&=2.297\ \text{rad/s}\\ R_{\mathrm{B}}&=22.53\ \text{lbf} \end{aligned} \quad\Rightarrow\quad R_{\mathrm{A}}=27.88\ \text{lbf} \tag{30}$$

where the value of R_{A} is an immediate consequence of the last equation in Eqs. (29). ■

Remark: ↑ Example Problem 1 ↑ is best suited for an introductory-level dynamics course.

EXAMPLE PROBLEM 2 – A Composite Rigid Body

Consider a composite rigid body consisting of a slender uniform bar which is rigidly joined to a disk (of uniform thickness) that rolls without slipping on a horizontal planar support surface, as depicted in Figure 3 (where basic dimensions are indicated). The disk and bar are homogeneous in material composition with masses m_{d} and m_{b} (respectively). The disk-bar system is confined to planar (2-D) motion, and its configuration is uniquely identified by the angle θ.

In addition to the effect of gravity, the system is subjected to a specified force of magnitude F_a, which may be time-varying, but it always acts in the horizontal direction. Develop the standard equations of motion for this system. Also, in the special case with $F_a = 0$, determine the natural (undamped) frequency ω_n for small-amplitude oscillations of the system due to gravity alone.

Constraints and Relations

Based upon the rolling-without-slipping condition for the disk on the support surface, it is found that the kinematics and relative positions of various points on the system are governed by

$$x_P = r_d\,\theta \quad \Rightarrow \quad v_P = r_d\,\dot{\theta} \quad \Rightarrow \quad a_P = r_d\,\ddot{\theta} \tag{31}$$

$$\mathbf{r}_{Q/P} = -\tfrac{1}{2} l_b \left[\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right] \quad , \quad \mathbf{r}_{C/P} = -r_d\,\mathbf{j} \tag{32}$$

where points P and Q identify the respective mass centers of the disk and bar, and C identifies the point of contact (always directly below P) between the disk and the support surface. Other kinematic constraints for the two-dimensional motion of the disk-bar system include

$$\begin{array}{l} \mathbf{v}_P = v_P\,\mathbf{i} \\ \mathbf{a}_P = a_P\,\mathbf{i} \end{array} \quad , \quad \begin{array}{l} \boldsymbol{\omega} = -\dot{\theta}\,\mathbf{k} \\ \boldsymbol{\alpha} = -\ddot{\theta}\,\mathbf{k} \end{array} \tag{33}$$

The (separate) mass-moments of inertia for the disk and bar are given below. These results then may be combined via the parallel-axis theorem to obtain the total mass-moment of inertia for the disk-bar system (composite rigid body) about point P as

$$(I_P)_{\text{disk}} \triangleq \bar{I}_d = \tfrac{1}{2} m_d\, r_d^2 \quad , \quad (I_Q)_{\text{bar}} \triangleq \bar{I}_b = \tfrac{1}{12} m_b\, l_b^2 \tag{34}$$

$$
\begin{aligned}
(I_\text{P})_\text{body} &= (I_\text{P})_\text{disk} + (I_\text{P})_\text{bar} \\
&= \overline{I}_\text{d} + (\overline{I}_\text{b} + m_\text{b}\, r_\text{Q/P}^2) \\
&= \tfrac{1}{2} m_\text{d}\, r_\text{d}^2 + \tfrac{1}{3} m_\text{b}\, l_\text{b}^2
\end{aligned}
\tag{35}
$$

For the reference frame adopted (see Figure 3), the relevant external forces acting on the overall body are identified from the free-body diagram and expressed as

$$
\begin{aligned}
\mathbf{W}_\text{d} &= -(m_\text{d}\, g)\,\mathbf{j} \\
\mathbf{W}_\text{b} &= -(m_\text{b}\, g)\,\mathbf{j}
\end{aligned}
\quad , \quad \mathbf{F}_\text{a} = F_\text{a}\,\mathbf{i} \quad , \quad
\begin{aligned}
\mathbf{F}_\text{f} &= F_\text{f}\,\mathbf{i} \\
\mathbf{F}_\text{n} &= F_\text{n}\,\mathbf{j}
\end{aligned}
\tag{36}
$$

Newtonian/Eulerian Equations of Motion

As previously introduced, Eq. (14) was specifically developed to facilitate an assessment of the *translational motion* of a composite rigid body. In this instance, Eq. (14) with $n = 2$ becomes

$$
\mathbf{F} = m_\text{d}\,\mathbf{a}_\text{P} + m_\text{b}\,\mathbf{a}_\text{Q}
\tag{37}
$$

where $\mathbf{a}_\text{P}$ is evaluated as indicated in Eqs. (31) and (33), while $\mathbf{a}_\text{Q}$ is obtained via Eq. (6) as

$$
\begin{aligned}
\mathbf{a}_\text{Q} &= \mathbf{a}_\text{P} + \boldsymbol{\alpha}\times\mathbf{r}_\text{Q/P} - \omega^2\,\mathbf{r}_\text{Q/P} \\
&= \mathbf{a}_\text{P} + \tfrac{1}{2} l_\text{b}\big[(\sin\theta)\,\mathbf{j} - (\cos\theta)\,\mathbf{i}\big]\,\ddot{\theta} + \tfrac{1}{2} l_\text{b}\big[(\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}\big]\,\dot{\theta}^2 \\
&= \mathbf{a}_\text{P} + \tfrac{1}{2} l_\text{b}\big[(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta)\big]\,\mathbf{i} + \tfrac{1}{2} l_\text{b}\big[(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta)\big]\,\mathbf{j}
\end{aligned}
\tag{38}
$$

Based upon the free-body diagram provided in Figure 3, the expressions for the external forces given in Eqs. (36), and the expressions for $\mathbf{a}_\text{P}$ and $\mathbf{a}_\text{Q}$ obtained above, the $\mathbf{i}$ and $\mathbf{j}$ components on each side of Eq. (37) yield the scalar equations

$$
F_\text{f} + F_\text{a} = (m_\text{d} + m_\text{b})\, r_\text{d}\,\ddot{\theta} + \tfrac{1}{2} m_\text{b}\, l_\text{b}\,(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta)
\tag{39}
$$

$$F_{\mathrm{n}} = (m_{\mathrm{d}} + m_{\mathrm{b}})\, g + \tfrac{1}{2} m_{\mathrm{b}}\, l_{\mathrm{b}} \,(\dot{\theta}^2 \cos\theta + \ddot{\theta} \sin\theta) \tag{40}$$

As previously introduced, Eq. (11) was specifically developed to facilitate an assessment of the *rotational motion* of a composite rigid body. In this instance, Eq. (11) with $n = 2$ becomes

$$\mathbf{r}_{\mathrm{Q/P}} \times \mathbf{W}_{\mathrm{b}} + \mathbf{r}_{\mathrm{C/P}} \times \mathbf{F}_{\mathrm{f}} - (m_{\mathrm{d}}\, \mathbf{r}_{\mathrm{P/P}} + m_{\mathrm{b}}\, \mathbf{r}_{\mathrm{Q/P}}) \times \mathbf{a}_{\mathrm{P}} = \big[(I_{\mathrm{P}})_{\mathrm{disk}} + (I_{\mathrm{P}})_{\mathrm{bar}}\big]\, \boldsymbol{\alpha} \tag{41}$$

where $\mathbf{r}_{\mathrm{P/P}} = \mathbf{0}$, a consequence of the particular choice of the reference point P, which happens to coincide with the mass center of one of the rigid elements of the composite rigid body. Based upon Eqs. (31) through (36) and the free-body diagram, Eq. (41) [just as in the case of Eq. (26), only vector components that involve $\mathbf{k}$ survive] yields the single scalar equation

$$\tfrac{1}{2} m_{\mathrm{b}}\, g\, l_{\mathrm{b}} \sin\theta + F_{\mathrm{f}}\, r_{\mathrm{d}} - \tfrac{1}{2} m_{\mathrm{b}}\, r_{\mathrm{d}}\, l_{\mathrm{b}}\, \ddot{\theta} \cos\theta = -\big(\tfrac{1}{2} m_{\mathrm{d}}\, r_{\mathrm{d}}^2 + \tfrac{1}{3} m_{\mathrm{b}}\, l_{\mathrm{b}}^2\big)\, \ddot{\theta} \tag{42}$$

This equation is already greatly simplified because three of the five external forces acting on the system do not produce moments about point P (because their lines of action all pass through P), which is the primary reason for the selection of that particular point as the reference point.

Equations (39) and (42) can be combined to easily eliminate F_{f}, and thereby obtain

$$\begin{aligned} \tfrac{1}{2} m_{\mathrm{b}}\, g\, l_{\mathrm{b}} \sin\theta + \big[(m_{\mathrm{d}} + m_{\mathrm{b}})\, r_{\mathrm{d}}\, \ddot{\theta} + \tfrac{1}{2} m_{\mathrm{b}}\, l_{\mathrm{b}}\,(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) + F_{\mathrm{a}}\big]\, r_{\mathrm{d}} \\ = \big[\tfrac{1}{2} m_{\mathrm{b}}\, r_{\mathrm{d}}\, l_{\mathrm{b}} \cos\theta - \big(\tfrac{1}{2} m_{\mathrm{d}}\, r_{\mathrm{d}}^2 + \tfrac{1}{3} m_{\mathrm{b}}\, l_{\mathrm{b}}^2\big)\big]\, \ddot{\theta} \end{aligned} \tag{43}$$

or, equivalently,

$$\begin{aligned} \big[\big(\tfrac{3}{2} m_{\mathrm{d}} + m_{\mathrm{b}}\big)\, r_{\mathrm{d}}^2 + \tfrac{1}{3} m_{\mathrm{b}}\, l_{\mathrm{b}}^2 - m_{\mathrm{b}}\, r_{\mathrm{d}}\, l_{\mathrm{b}} \cos\theta\big]\, \ddot{\theta} + \big(\tfrac{1}{2} m_{\mathrm{b}}\, r_{\mathrm{d}}\, l_{\mathrm{b}} \sin\theta\big)\, \dot{\theta}^2 \\ + \tfrac{1}{2} m_{\mathrm{b}}\, g\, l_{\mathrm{b}} \sin\theta = -F_{\mathrm{a}}\, r_{\mathrm{d}} \end{aligned} \tag{44}$$

Equations (39), (40), and (44) are the standard equations of motion for the disk-bar system when F_a is specified. At this juncture, it must be emphasized that the development of Eq. (44) in the manner presented above is far superior to the conventional approach, which proceeds by starting with Eq. (2) in the form which is strictly valid for planar (2-D) motion: $\mathbf{M}_G = I_G\,\boldsymbol{\alpha}$. The author has taken this approach and confirmed that identical results are indeed obtained, but it involves a somewhat arduous and protracted effort (as may be verified by interested readers).

Next, consider the case in which (a) $F_a = 0$, and (b) the system is subjected to initial conditions $(\theta, \dot{\theta}) = (\theta_o, 0)$ at $t = 0$, with $0 < \theta_o \leq \frac{\pi}{6}$. In this case, the system will manifest small-amplitude angular oscillations for which $|\theta| \leq \theta_o$. Also, under these conditions, it is well known [12] that

$$\begin{matrix} \sin\theta \approx \theta \\ \cos\theta \approx 1 \end{matrix} \quad ; \quad (\sin\theta)\,\dot{\theta}^2 \approx 0 \tag{45}$$

As a result, Eq. (44) then approximately becomes

$$\ddot{\theta} + \omega_n^2\,\theta = 0 \quad ; \quad \omega_n = \sqrt{\frac{\frac{1}{2} m_b\, g\, l_b}{\left(\frac{3}{2} m_d + m_b\right) r_d^2 + m_b\, l_b \left(\frac{1}{3} l_b - r_d\right)}} \tag{46}$$

This ordinary differential equation corresponds to *simple harmonic motion* [13]. Therefore, ω_n is an estimate, since this equation is only an approximation, of the natural (undamped) frequency for small-amplitude oscillations exhibited by the disk-bar system when it behaves as a vibratory system driven by gravity alone ($F_a = 0$).

It also should be mentioned that Eqs. (39) and (40) may be utilized to enforce $|F_f| \leq \mu F_n$, which establishes a necessary condition upon μ in order to achieve the rolling-without-slipping motion presumed to occur, apart from which the entire analysis conducted above is invalid. This inquiry has been pursued by the author, and it reveals some intriguing results for the frictional quantities F_{f} and μ, but it is beyond the intended scope of this article. ■

Remark: ↑ Example Problem 2 ↑ is best suited for an intermediate-level dynamics course.

Pedagogical Effectiveness and Learning Enhancement

At his academic institution, the author teaches two distinct undergraduate dynamics courses:

- Dynamics I – the traditional course in an introductory engineering mechanics sequence
- Dynamics II – an intermediate-level course; covers further theory, topics, and methods

The method proposed herein is briefly introduced in Dynamics I, when the consideration of rigid bodies is undertaken and after several examples (involving simple/single rigid bodies) have been solved via the traditional method of analysis for such bodies: Eqs. (1) and (2). This exposure is intended to emphasize to students that greater care must be exercised when applying a moments equation in dynamics as compared to the relative ease of application experienced in statics. But students in Dynamics I are not expected (in terms of learning objectives) to demonstrate mastery of the method advocated in this article; it is merely offered as an enhancement to basic methods.

When the method is presented in this course, most students are generally favorable toward it, and some students even express that they would prefer to learn the generalization of Euler's equation at the beginning of the material on rigid-body kinetics because of its flexibility for evaluation. A survey instrument was developed to measure these sentiments, and it was administered during a recent academic term in order to acquire definitive student feedback concerning opinions on both the method and the course for assessment purposes. The survey was voluntarily completed, and responses were anonymously submitted. The survey instrument appears in the Appendix of this article; the questions ('Inquiry Items') are reproduced below for the convenience of the reader.

Inquiry Items:

1. I understand the difference between the alternative form of Euler's equation and the other two standard forms of this equation (for the *mass center* and a *fixed point*, respectively) covered in previous class lectures.

2. I recognize that the alternative form of Euler's equation could offer an advantage over the other two standard forms of this equation for the solution of some dynamics problems.

3. With some further examples, guidance, and practice, I believe that I could effectively apply the alternative form of Euler's equation to solve certain dynamics problems.

4. Although ENGR 212 has been a challenging course in my engineering degree program, I have enjoyed learning the basic concepts and principles of introductory dynamics.

5. I am interested in taking a higher-level dynamics course, as an elective engineering course, after I have completed ENGR 212.

Table 1. Survey Results for Opinions on an Alternative Form of Euler's Equation

Inquiry Item	Definitely Agree	Partially Agree	Neutral	Partially Disagree	Definitely Disagree
#1	50.0%	27.3%	13.6%	9.1%	0.0%
#2	59.2%	31.8%	0.0%	4.5%	4.5%
#3	63.7%	22.7%	9.1%	0.0%	4.5%
#4	54.6%	31.8%	9.1%	4.5%	0.0%
#5	22.7%	27.3%	27.3%	22.7%	0.0%

Note: 22 respondents completed the survey from a class membership of 24 students.

From these results, it is evident that over 90% of respondents recognize the potential usefulness of the alternative form of Euler's equation, and upwards of 85% of this same group believe they could effectively apply this alternative form to solve dynamics problems. It seems to be the case that the generalized form of Euler's equation has definite merit and could be deserving of closer attention (and adoption) by course instructors and textbook authors of engineering mechanics.

In contrast, in Dynamics II, the method proposed herein facilitates the study of various forms of Euler's equation for the rotational dynamics of a rigid body which are needed to treat problems involving 3-D motion. When first presented in the context of problems involving 2-D motion, it offers the students a simpler application scenario which can be later extended and generalized to more sophisticated situations. Also, it is not until Dynamics II that students are exposed to more complicated systems such as composite rigid bodies, where the method advocated in this article has been shown to be very advantageous (in the last example considered).

Because the generalization of Euler's equation is a required topic in Dynamics II, the opinions of the students (enrolled in this course) regarding this topic have not been formally assessed, but it can be reported that these students are able to easily assimilate and apply the method involved.

Final Remarks

It is the sincere hope of the author that the alternative form of Euler's equation presented in this article, including the versions for the simple/single and composite rigid-body cases, will become widely known through this venue and potentially be advocated in popular engineering mechanics textbooks. The greater flexibility it affords for the evaluation of moments produced by forces is practicable and convenient for dynamics problems involving various kinds of rigid bodies.

Even though it is possible to utilize the conventional approach in lieu of the proposed approach, it is worthwhile to present this option in engineering mechanics courses. With over two decades of experience as an engineering, mathematics, and physics educator, the author is convinced that there is an enduring benefit in exposing students to multiple analytical approaches, especially for students who desire a deeper understanding of, and a greater proficiency in, dynamics.

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[13] J. C. Hayen, "A Rocking/Rolling Half-Disk Vibratory System", Paper ID #18222, 2017 ASEE Annual Conference and Exposition, June 2017, Columbus, Ohio, pg. 9.

A Simplified Model of a Piston-Crankshaft Connecting Rod

n : Number of Rigid Elements that Comprise the Composite Rigid Body

G_k : Point that Identifies the Mass Center of the k-th Element $(1 \leq k \leq n)$

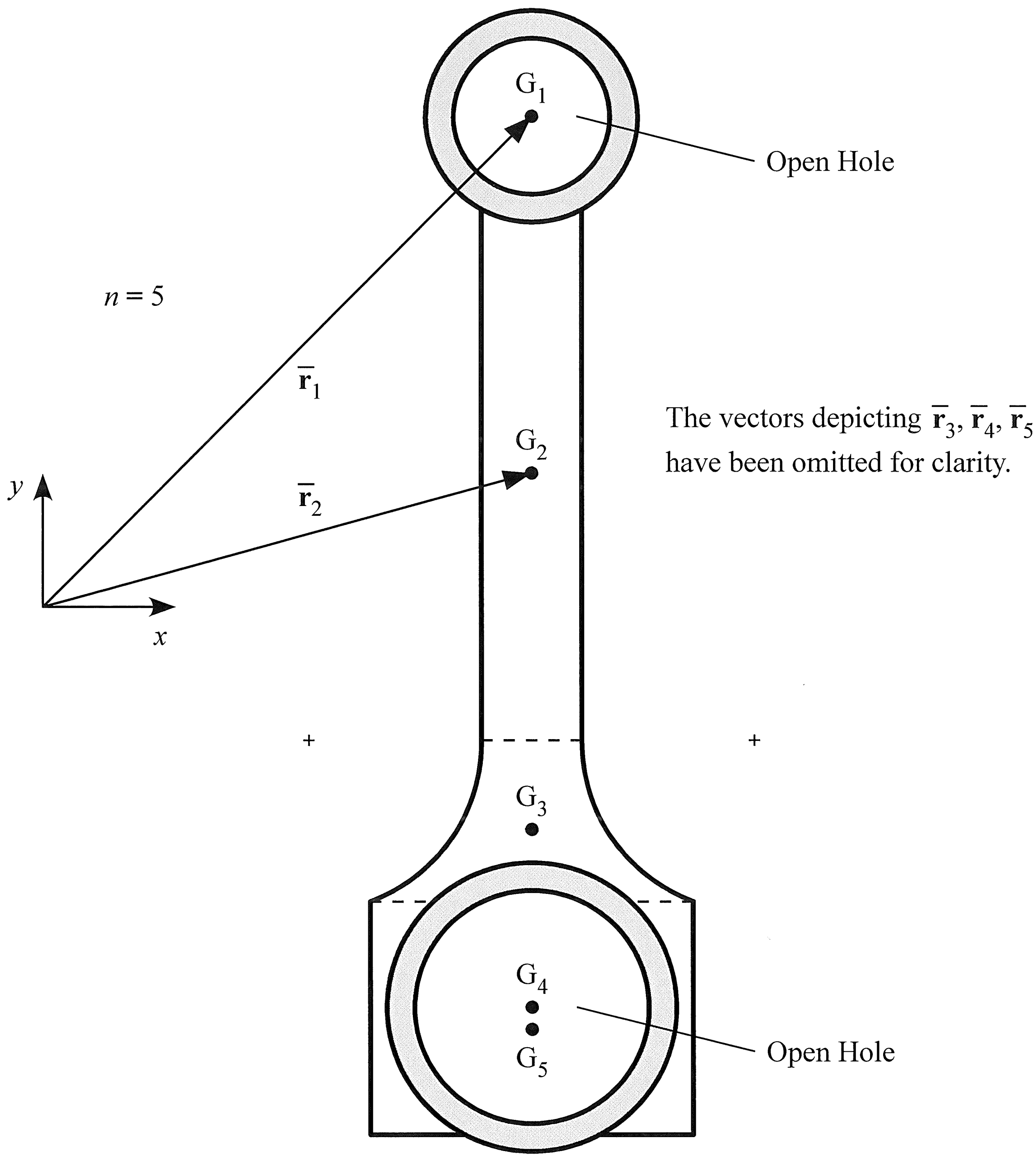
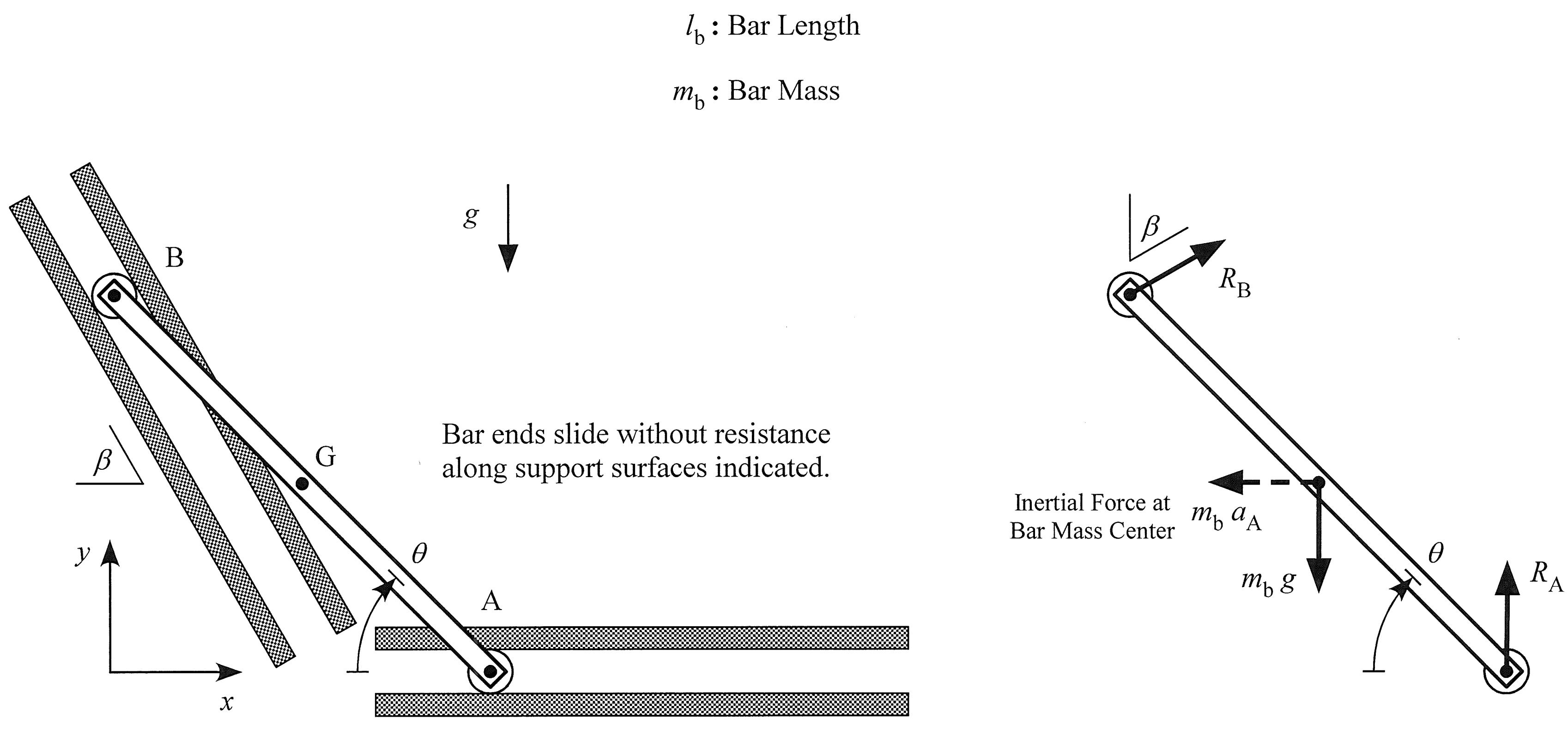


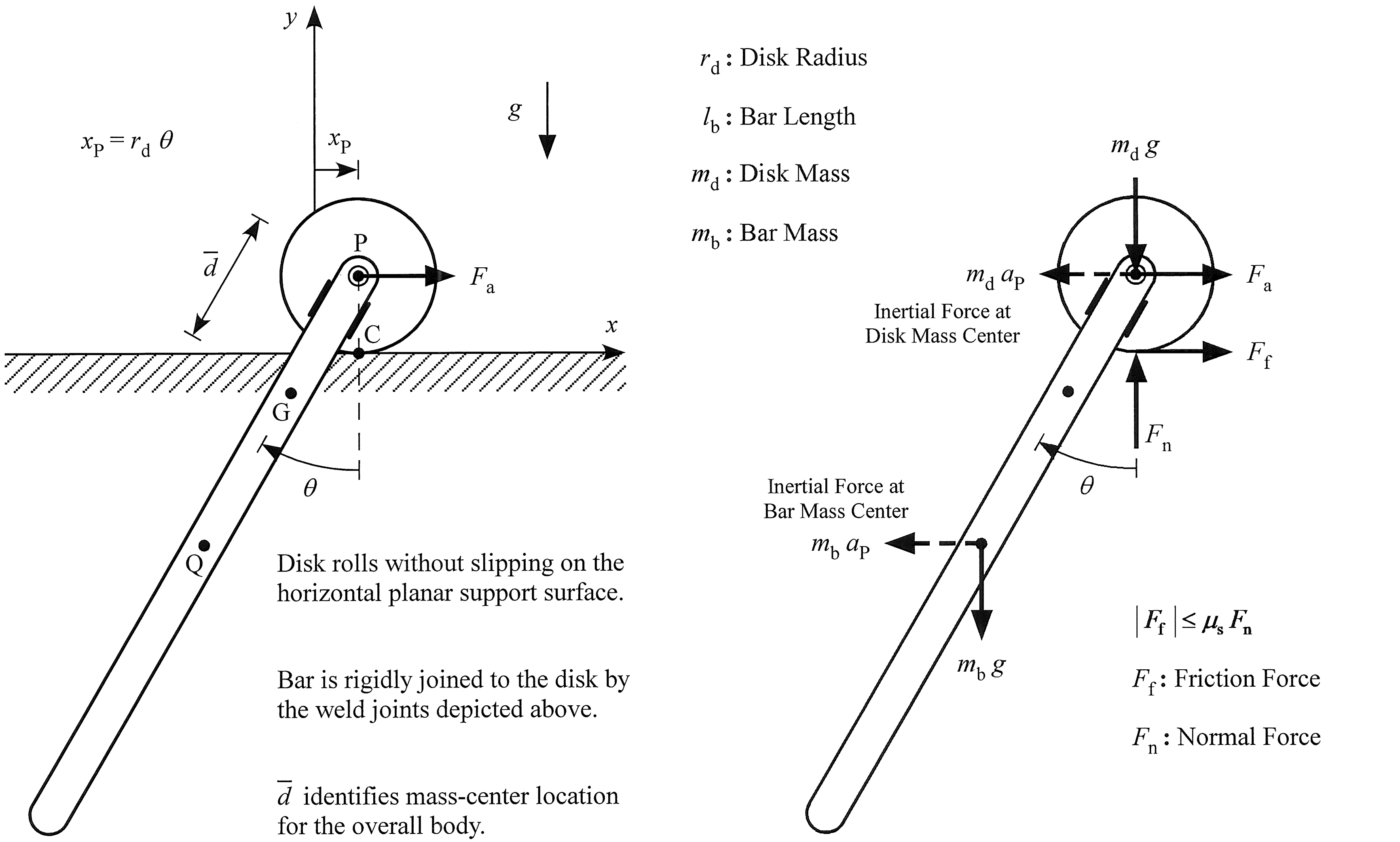
Figure 1 – Example Representation of a Composite Rigid Body



(a) Configuration Diagram

(b) Free-Body Diagram

Figure 2 – Configuration and Free-Body Diagrams for the Falling/Sliding Bar



(a) Configuration Diagram

(b) Free-Body Diagram

Figure 3 – Configuration and Free-Body Diagrams for the Disk-Bar System

Dynamics I (ENGR 212)
Survey for Presentation on / Example of an Alternative Form of Euler's Equation

Please respond to the following inquiry items concerning your understanding of and opinions on the supplemental topic presented in our class meeting today. Please select the response that best represents your understanding/opinions from the options available for each item. Your efforts in thoughtfully and carefully completing this survey are greatly appreciated.

1. I understand the difference between the alternative form of Euler's equation and the other two standard forms of this equation (for the *mass center* and a *fixed point*, respectively) covered in previous class lectures.

☐ Definitely Agree ☐ Partially Agree ☐ Neutral ☐ Partially Disagree ☐ Definitely Disagree

2. I recognize that the alternative form of Euler's equation could offer an advantage over the other two standard forms of this equation for the solution of some dynamics problems.

☐ Definitely Agree ☐ Partially Agree ☐ Neutral ☐ Partially Disagree ☐ Definitely Disagree

3. With some further examples, guidance, and practice, I believe that I could effectively apply the alternative form of Euler's equation to solve certain dynamics problems.

☐ Definitely Agree ☐ Partially Agree ☐ Neutral ☐ Partially Disagree ☐ Definitely Disagree

4. Although ENGR 212 has been a challenging course in my engineering degree program, I have enjoyed learning the basic concepts and principles of introductory dynamics.

☐ Definitely Agree ☐ Partially Agree ☐ Neutral ☐ Partially Disagree ☐ Definitely Disagree

5. I am interested in taking a higher-level dynamics course, as an elective engineering course, after I have completed ENGR 212.

☐ Definitely Agree ☐ Partially Agree ☐ Neutral ☐ Partially Disagree ☐ Definitely Disagree

Thank you for your participation in this survey. Your responses will remain anonymous.