AC 2007-11: ANALYSIS OF STATICALLY INDETERMINATE REACTIONS AND DEFLECTIONS OF BEAMS USING MODEL FORMULAS: A NEW APPROACH

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Analysis of Statically Indeterminate Reactions and Deflections of Beams Using Model Formulas: A New Approach

Abstract

This paper is intended to share with educators and practitioners in mechanics a new approach that employs a set of four model formulas in analyzing statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. The model formulas, in algebraic form, are derived using singularity functions. They are expressed in terms of (a) flexural rigidity of the beam; (b) slopes and deflections, as well as shear forces and bending moments, at both ends of the beam; and (c) applied loads on the beam. The types of applied loads include: (i) concentrated force and moment; (ii) uniformly distributed moment; and (iii) linearly varying distributed force. Thus, these model formulas are applicable to most problems encountered in the teaching and learning of mechanics of materials, as well as in practice. As a salient feature, this new approach allows one to treat reactions at supports, even not at the ends of a beam, simply as concentrated forces or moments, where corresponding boundary conditions at the points of supports are imposed using also the model formulas. This feature allows one to readily determine statically indeterminate reactions at supports, as well as slopes and deflections at any positions, of beams. A beam needs to be divided into segments for analysis only when it has discontinuity in slope or in flexural rigidity. Several examples are provided to illustrate this new approach.

I. Introduction

There are different well-known methods for determining deflections of beams in mechanics of materials. These methods may include the following:¹–¹⁰

(a) method of double integration (with or without the use of singularity functions),
(b) method of superposition,
(c) method using moment-area theorems,
(d) method using Castigliano’s theorem, and
(e) conjugate beam method.

This paper extends an earlier study on method of segments¹¹ by using singularity functions and model formulas. As a result, the proposed new approach allows a considerable reduction in the number of segments required in the study. This new approach makes available an effective method for mechanics educators and practitioners when it comes to determining reactions and deflections of beams. It is aimed at contributing to the enrichment of one’s learning experience and to provide a means for independent checking on solutions obtained by other methods.

The paper goes over the description of sign conventions and derives four model formulas for the slope and deflection of a beam segment having a constant flexural rigidity and carrying a variety of commonly applied loads. These formulas, derived using singularity functions, form the basis for a new approach to solving problems involving reactions and deflections of beams. In contrast to the method of segments,¹¹ the proposed new approach does not have to divide a beam into multiple segments even if the beam has multiple concentrated loads or simple supports not at its...
ends. Application of these model formulas is direct and requires no further integration or writing of continuity equations. The model formulas can readily be extended to the analysis of beams having discontinuity in slope (e.g., at hinge connections) or in flexural rigidity (e.g., in stepped segments). It can solve both statically determinate and statically indeterminate beam problems.

**II. Sign Conventions for Beams**

In the analysis of beams, it is important to adhere to the generally agreed positive and negative signs for loads, shear forces, bending moments, slopes, and deflections. The free-body diagram for a beam $ab$ having a constant flexural rigidity $EI$ and carrying loads is shown in Fig. 1. The positive directions of shear forces $V_a$ and $V_b$, moments $M_a$ and $M_b$, at ends $a$ and $b$ of the beam, the concentrated force $P$ and concentrated moment $K$, as well as the distributed loads, are illustrated in this figure.

![Figure 1. Positive directions of shear forces, moments, and applied loads](image)

In general, we have the following sign conventions for shear forces, moments, and applied loads:

- A *shear force* is positive if it acts upward on the left (or downward on the right) face of the beam element (e.g., $V_a$ at the left end $a$, and $V_b$ at the right end $b$ in Fig. 1).
- At ends of the beam, a *moment* is positive if it tends to cause compression in the top fiber of the beam (e.g., $M_a$ at the left end $a$, and $M_b$ at the right end $b$ in Fig. 1).
- Not at ends of the beam, a *moment* is positive if it tends to cause compression in the top fiber of the beam just to the right of the position where it acts (e.g., the concentrated moment $K$ and the uniformly distributed moment with intensity $m_0$ in Fig. 1).
- A *concentrated force* or a *distributed force* applied to the beam is positive if it is directed downward (e.g., the concentrated force $P$, the linearly varying distributed force with intensity $w_0$ on the left side and $w_l$ on the right side in Fig. 1, where the distribution becomes uniform if $w_l = w_0$).

Furthermore, we adopt the following sign conventions for deflection and slope of a beam:

- A *positive deflection* is an upward displacement.
- A *positive slope* is a counterclockwise angular displacement.
III. Derivation of Model Formulas

Any beam element of differential width $dx$ at any position $x$ may be perceived to have a left face and a right face. Using singularity functions, we may write, for the beam $ab$ in Fig. 1, the loading function $q$, shear force $V$, and bending moment $M$ acting on the left face of the beam element at any position $x$ for this beam as follows:

$$q = V_a <x>^{-1} + M_a <x>^{-2} - P<x-x_p>^{-1} + K<x-x_K>^{-2} - w_0 <x-x_w>^0$$
$$- \left[ \frac{w_1 - w_0}{L-x_w} <x-x_m>^1 + m_0 <x-x_m>^1 \right]$$  \hspace{1cm} (1)

$$V = V_a <x>^0 + M_a <x>^{-1} - P<x-x_p>^0 + K<x-x_K>^1 - w_0 <x-x_w>^1$$
$$- \frac{w_1 - w_0}{2(L-x_w)} <x-x_m>^2 + m_0 <x-x_m>^0$$  \hspace{1cm} (2)

$$M = V_a <x>^1 + M_a <x>^0 - P<x-x_p>^1 + K<x-x_K>^0 - \frac{w_0}{2} <x-x_w>^2$$
$$- \frac{w_1 - w_0}{6(L-x_w)} <x-x_m>^3 + m_0 <x-x_m>^1$$  \hspace{1cm} (3)

It is important to note that Eqs. (1) through (3) are written for the beam in the range $0 \leq x < L$, and we have $x-L < 0$ even though $x \to L$ at the right end of beam. By the definition of singularity functions, the value of $<x-L>^n = \text{zero}$ whenever $x-L < 0$, regardless of the value of $n$. Therefore, values of the terms $-V_b <x-L>^1 - M_b <x-L>^2$, as well as the integrals of these terms, are always equal to zero for the beam. This is the reason why these two terms are trivial and may simply be omitted in the expression for the loading function $q$ in Eq. (1).

Noting that $EI$ is the flexural rigidity, $y$ is the deflection, $y'$ is the slope, $y''$ is the second derivative of $y$ with respect to $x$ for any section of the beam $ab$, and $EIy'' = M$, we write

$$EIy'' = V_a <x>^1 + M_a <x>^0 - P<x-x_p>^1 + K<x-x_K>^0 - \frac{w_0}{2} <x-x_w>^2$$
$$- \frac{w_1 - w_0}{6(L-x_w)} <x-x_m>^3 + m_0 <x-x_m>^1$$  \hspace{1cm} (4)

$$EI y' = \frac{1}{2} V_a <x>^2 + M_a <x>^1 - \frac{1}{2} P<x-x_p>^2 + K<x-x_K>^1 - \frac{w_0}{6} <x-x_w>^3$$
$$- \frac{w_1 - w_0}{24(L-x_w)} <x-x_m>^4 + \frac{1}{2} m_0 <x-x_m>^2 + C_1$$  \hspace{1cm} (5)

$$EIy = \frac{1}{6} V_a <x>^3 + \frac{1}{2} M_a <x>^2 - \frac{1}{6} P<x-x_p>^3 + \frac{1}{2} K<x-x_K>^2 - \frac{w_0}{24} <x-x_w>^4$$
$$- \frac{w_1 - w_0}{120(L-x_w)} <x-x_m>^5 + \frac{1}{6} m_0 <x-x_m>^3 + C_1 x + C_2$$  \hspace{1cm} (6)

The slope and deflection of the beam in Fig. 1 at its left end $a$ (i.e., at $x = 0$) are $\theta_a$ and $y_a$, respectively. Imposition of these two boundary conditions on Eqs. (5) and (6) yields

$$C_1 = EI \theta_a$$  \hspace{1cm} (7)

$$C_2 = EI y_a$$  \hspace{1cm} (8)
Substituting Eqs. (7) and (8) into Eqs. (5) and (6), we obtain the model formulas for the slope \( y' \) and deflection \( y \), at any position \( x \), of the beam \( ab \) in Fig. 1 as follows:

\[
y' = \theta_a + \frac{V_a}{2EI} x^2 + \frac{M_a}{EI} x - \frac{P}{2EI} (x-x_p)^2 + \frac{K}{EI} (x-x_K)^3 - \frac{w_0}{6EI} (x-x_w)^3
\]

\[
y - y_c - \frac{1}{24EI(L-x_w)} (x-x_w)^4 + \frac{m_0}{2EI} (x-x_m)^2
\]

By letting \( x = L \) in Eqs. (9) and (10), we obtain the model formulas for the slope \( \theta_b \) and deflection \( y_b \) at the right end \( b \) of the beam \( ab \) as follows:

\[
\theta_b = \theta_a + \frac{V_a L^2}{2EI} + \frac{M_a L}{EI} - \frac{P}{2EI} (L-x_p)^2 + \frac{K}{EI} (L-x_K)
- \frac{3w_0 + w_i}{24EI} (L-x_w)^3 + \frac{m_0}{2EI} (L-x_m)^2
\]

\[
y_b = y_a + \theta_a L + \frac{V_a L^2}{6EI} + \frac{M_a L^2}{2EI} - \frac{P}{6EI} (L-x_p)^3 + \frac{K}{2EI} (L-x_K^2)
- \frac{4w_0 + w_i}{120EI} (L-x_w)^4 + \frac{m_0}{6EI} (L-x_m)^3
\]

IV. Applications of Model Formulas

The set of four model formulas given by Eqs. (9) through (12) may be used as the basis upon which to formulate a new approach to analyzing statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. The beams may carry a variety of applied loads, as illustrated in Fig. 1.

Note that \( L \) in the model formulas is a parameter representing the total length of the beam segment, to which the model formulas are to be applied. These formulas have already taken into account the boundary conditions of the beam at its ends. Furthermore, this approach allows one to treat reactions at interior supports (those not at the ends of the beam) as applied concentrated forces or moments. All one has to do is to simply impose the additional corresponding boundary conditions at the interior supports for the beam segment. Thus, the new approach allows one to readily determine statically indeterminate reactions as well as slopes and deflections of beams.

A beam needs to be divided into separate segments for analysis only if (a) it is a combined beam (e.g., Gerber beam) having discontinuity in slope at hinge connections between segments, and (b) it contains segments of different flexural rigidities. The new approach proposed in this paper can best be understood with illustrations. Therefore, simple as well as more challenging problems are included in the following examples.
**Example 1.** A cantilever beam with a constant flexural rigidity $EI$ and a length $L$ is acted on by three concentrated forces of magnitudes $P$, $2P$, and $3P$ as shown in Fig. 2. For this beam, determine the slope $\theta_A$ and deflection $y_A$ at its free end $A$.

![Figure 2. Cantilever beam carrying three concentrated forces](image)

**Solution.** There is no need to divide the beam into segments for study. At end $A$, the moment $M_A$ is zero and the shear force $V_A$ is $-P$. At end $B$, the slope $\theta_B$ and deflection $y_B$ are both zero. Since we have multiple concentrated forces acting on the beam, we need to apply the concentrated force term (the term containing $P$) *multiple times* in all model formulas. Applying the model formulas in Eqs. (11) and (12), successively, to this beam, we write:

$$0 = \theta_A + \frac{-PL^2}{2EI} + 0 - \frac{2P}{2EI} \left( L - \frac{L}{3} \right)^2 - \frac{3P}{2EI} \left( L - \frac{2L}{3} \right)^2 + 0 - 0 + 0$$

$$0 = y_A + \frac{\theta_A L}{2EI} + 0 - \frac{PL^2}{6EI} \left( L - \frac{L}{3} \right)^3 - \frac{2P}{6EI} \left( L - \frac{L}{3} \right)^3 - \frac{3P}{6EI} \left( L - \frac{2L}{3} \right)^3 + 0 - 0 + 0$$

The above *two* simultaneous equations, containing the *two* unknowns $\theta_A$ and $y_A$, yield:

$$\theta_A = \frac{10PL^2}{9EI} \quad y_A = \frac{-67PL^3}{81EI}$$

Consistent with the defined sign conventions, we report that:

$$\theta_A = \frac{10PL^2}{9EI} \quad y_A = \frac{67PL^3}{81EI}$$

**Example 2.** The right end $B$ of a fix-ended beam $AB$, which has a constant flexural rigidity $EI$ and a length $L$, is shifted upward by an amount $\Delta$ as shown in Fig. 3. For such a relative vertical shifting of supports, determine (a) the vertical reaction force $A_y$ and the reaction moment $M_A$ developed at the left end $A$, (b) the deflection $y$ of the beam at any position $x$.

![Figure 3. Relative vertical shifting of supports in a fix-ended beam](image)

**Solution.** This beam is statically indeterminate to the *second* degree. At the fixed support $A$, the deflection $y_A$ and slope $\theta_A$ are zero. At the fixed support $B$, the deflection $y_B$ is $\Delta$, but the slope $\theta_B$ is zero. Applying the model formulas in Eqs. (11) and (12), successively, to this beam, we write
\[0 = 0 + \frac{A_y L^2}{2EI} + \frac{M_A L}{EI} - 0 + 0 - 0 + 0\]
\[\Delta = 0 + 0 + \frac{A_y L^3}{6EI} + \frac{M_A L^2}{2EI} - 0 + 0 - 0 + 0\]

The above two simultaneous equations, containing the two unknowns \(A_y\) and \(M_A\), yield
\[A_y = -\frac{12 EI \Delta}{L^3} \quad \text{and} \quad M_A = \frac{6 EI \Delta}{L^2}\]

Consistent with the defined sign conventions, we report that
\[A_y = \frac{12 EI \Delta}{L^3} \quad \text{and} \quad M_A = \frac{6 EI \Delta}{L^2}\]

Substituting the obtained values of \(A_y\) and \(M_A\) into the model formula in Eq. (10), we write
\[y = 0 + 0 + \frac{A_y}{6EI} x^3 + \frac{M_A}{2EI} x^2 - 0 + 0 - 0 + 0 = -\frac{2 \Delta}{L^3} x^3 + \frac{3 \Delta}{L^2} x^2\]

\[y = \Delta \left( \frac{3 x^2}{L^2} - \frac{2 x^3}{L^3} \right)\]

**Example 3.** A cantilever beam \(AB\) with a constant flexural rigidity \(EI\) and carrying a uniformly distributed moment of intensity \(m_0\) over its entire length \(L\) is shown in Fig. 4. Determine (a) the slope \(\theta_B\) and deflection \(y_B\) at the free end \(B\), (b) the deflection \(y\) of the beam at any position \(x\).

**Solution.** The free-body diagram of the cantilever beam \(AB\), which is in equilibrium as shown in Fig. 5, indicates that the beam has only a counterclockwise reaction moment of magnitude \(m_0 L\) at its end \(A\) besides a uniformly distributed moment of intensity \(m_0\) over its entire length \(L\).

We note that the deflection \(y_A\), slope \(\theta_A\), and the shear force \(A_y\) at end \(A\) in Fig. 5 are all zero. At end \(B\), the moment \(M_B\) and shear force \(B_y\) are both zero. Applying the model formulas in Eqs. (11) and (12), successively, to this beam and noting that \(x_n = 0\), we write
\[\theta_B = 0 + 0 + \frac{(-m_0 L)L}{EI} - 0 + 0 - 0 + \frac{m_0}{2 EI} (L - 0)^2\]
\[y_B = 0 + 0 + \frac{(-m_0 L)L^2}{2 EI} - 0 + 0 - 0 + \frac{m_0}{6 EI} (L - 0)^3\]
The above two simultaneous equations, containing the two unknowns $\theta_B$ and $y_B$, yield

$$\theta_B = -\frac{m_0 L^2}{2EI} \quad y_B = -\frac{m_0 L^3}{3EI}$$

Consistent with the defined sign conventions, we report that

$$\theta_B = \frac{m_0 L^2}{2EI} \quad y_B = \frac{m_0 L^3}{3EI}$$

Substituting the obtained values of $\theta_B$ and $y_B$ into the model formula in Eq. (10), we write

$$y = 0 + 0 + 0 + \frac{-m_0 L}{2EI} x^2 - 0 + 0 + 0 - 0 + \frac{m_0}{6EI} (x-0)^3 = -\frac{m_0 x^2 (3L-x)}{6EI}$$

Example 4. A cantilever beam $AB$ with a constant flexural rigidity $EI$ and a length $2L$ is propped by a linear tension-compression spring of modulus $k$, and it carries a concentrated moment $K$ at its midpoint $C$ as shown in Fig. 6. Determine the slope $\theta_A$ and deflection $y_A$ at its left end $A$.

Figure 6. Cantilever beam propped by a linear spring and carrying a concentrated moment

Solution. At end $A$ of this beam, the moment $M_A$ is zero and shear force $V_A = -ky_A$, which is based on the initial assumption that $y_A$ is upward and the linear spring force of $ky_A$ acts downward at end $A$. At end $B$, the slope $\theta_B$ and deflection $y_B$ are both zero. Note that we need to set the parameter $L$ in the model formulas in Eqs. (11) and (12) equal to $2L$ for this beam $AB$. Letting $x_k = L$ and applying Eqs. (11) and (12), successively, to this beam, we write

$$0 = \theta_A + \frac{(-ky_A)(2L)^2}{2EI} + 0 - 0 - \frac{K}{EI} (2L-L) - 0 + 0$$

$$0 = y_A + \theta_A (2L) + \frac{(-ky_A)(2L)^3}{6EI} + 0 - 0 + \frac{K}{2EI} (2L-L)^2 - 0 + 0$$

The above two simultaneous equations, containing the two unknowns $\theta_A$ and $y_A$, yield

$$\theta_A = \frac{KL(3EI-kL^3)}{EI(3EI+8kL^3)} \quad y_A = -\frac{9KL^2}{2(3EI+8kL^3)}$$

Consistent with the defined sign conventions, we report that

$$\theta_A = \frac{KL(3EI-kL^3)}{EI(3EI+8kL^3)} \quad y_A = -\frac{9KL^2}{2(3EI+8kL^3)}$$
**Example 5.** A beam $AB$, which has a constant flexural rigidity $EI$, a roller support at $A$, a roller support at $C$, a fixed support at $B$, and a length of $2L$, carries a linearly distributed load with maximum intensity $w_1$ at $B$ as shown in Fig. 7. Determine (a) the vertical reaction force $A_y$ and the slope $\theta_A$ at $A$, (b) the vertical reaction force $C_y$ and the slope $\theta_C$ at $C$.

![Figure 7. Beam supported by two rollers and a fixed support](image)

**Solution.** This beam is statically indeterminate to the second degree. There is no need to divide the beam $AB$ into two segments for analysis in the solution by the proposed new approach. We can simply treat the vertical reaction force $C_y$ at $C$ as an unknown applied concentrated force, directed upward, and regard the beam $AB$ as one that has a total length of $2L$, which is to be used as the value for the parameter $L$ in the model formulas in Eqs. (9) through (12). The boundary conditions of this beam reveal that the moment $M_A$ and the deflection $y_A$ at $A$ are both zero, the slope $\theta_B$ and deflection $y_B$ at $B$ are also both zero, and the deflection $y_C$ at $C$ is zero. The shear force at the left end $A$ is the vertical reaction force $A_y$ at $A$, which may be assumed to be acting upward. Applying Eqs. (11) and (12) to the entire beam and Eq. (10) for imposing that $y_c=0$ at $C$, in that order, we write

$$0 = \theta_A + \frac{A_y(2L)^2}{2EI} + 0 - \frac{-C_y}{2EI} (2L-L)^2 + 0 - \frac{0+w_1}{24EI} (2L-L)^3 + 0$$

$$0 = 0 + \theta_A(2L) + \frac{A_y(2L)^3}{6EI} + 0 - \frac{-C_y}{6EI} (2L-L)^3 + 0 - \frac{0+w_1}{120EI} (2L-L)^4 + 0$$

$$0 = 0 + \theta_A L + \frac{A_y}{6EI} L^3 + 0 - 0 - 0 - 0 - 0$$

The above three simultaneous equations, containing the three unknowns $A_y$, $\theta_A$, and $C_y$, yield

$$A_y = \frac{w_1 L}{70} \quad \theta_A = \frac{w_1 L^3}{420EI} \quad C_y = \frac{19 w_1 L}{140}$$

Consistent with the defined sign conventions, we report that

$A_y = \frac{w_1 L}{70}$ \quad $\theta_A = \frac{w_1 L^3}{420EI}$ \quad $C_y = \frac{19 w_1 L}{140}$

The slope $\theta_C$ is simply $y'$ evaluated at $C$, which is located at $x=L$. Applying the model formula in Eq. (9) and utilizing the preceding solutions for $\theta_A$ and $A_y$, we write

$$\theta_C = \theta_A + \frac{A_y}{2EI} L^2 + 0 - 0 - 0 - 0 - 0 \quad = \frac{w_1 L^3}{420EI} - \frac{w_1 L}{70} \left( \frac{L^2}{2EI} \right) = - \frac{w_1 L^3}{210EI}$$

$$\theta_C = \frac{w_1 L^3}{210EI}$$
Example 6. A combined beam (Gerber beam), with a constant flexural rigidity $EI$, fixed supports at its ends $A$ and $D$, a hinge connection at $B$, and carrying a concentrated moment $K$ at $C$, is shown in Fig. 8. Determine (a) the vertical reaction force $A_y$ and the reaction moment $M_A$ at $A$, (b) the deflection $y_B$ of the hinge at $B$, (c) the slopes $\theta_{BL}$ and $\theta_{BR}$ just to the left and just to the right of the hinge at $B$, respectively, and (d) the slope $\theta_C$ and the deflection $y_C$ at $C$.

Solution. This beam is statically indeterminate to the first degree. Because of the discontinuity in slope at the hinge connection $B$, this beam needs to be divided into two segments $AB$ and $BD$ for analysis in the solution, as shown in Figs. 9 and 10, where the deflected shapes are shown to highlight the discontinuity in slope at $B$; i.e., $\theta_{BL} \neq \theta_{BR}$. The boundary conditions of this beam reveal that slope and deflection at $A$ and $D$ are all equal to zero.

Applying the model formulas in Eqs. (11) and (12), successively, to segment $AB$, as shown in Fig. 9, we write

\[
\theta_{BL} = 0 + \frac{A_y L^2}{2EI} + \frac{M_A L}{EI} - 0 + 0 - 0 - 0 \quad (a)
\]

\[
y_B = 0 + 0 + \frac{A_y L^3}{6EI} + \frac{M_A L^2}{2EI} - 0 + 0 - 0 + 0 \quad (b)
\]

For equilibrium of segment $AB$ in Fig. 9, we write

\[+\sum F_y = 0:\quad A_y - B_y = 0 \quad (c)\]

\[+\sum M_B = 0:\quad -M_A - L A_y = 0 \quad (d)\]
Applying the model formulas in Eqs. (11) and (12), successively, to segment \(BD\), as shown in Fig. 10, we write

\[
0 = \theta_{br} + \frac{B_y(2L)^2}{2EI} + 0 - 0 + \frac{K}{EI}(2L - L) - 0 + 0
\]  

\(e\)

\[
0 = y_b + \theta_{br}(2L) + \frac{B_y(2L)^3}{6EI} + 0 - 0 + \frac{K}{2EI}(2L - L)^2 - 0 + 0
\]  

\(f\)

For equilibrium of segment \(BD\) in Fig. 10, we write

\[+ \uparrow \sum F_y = 0: \quad B_y - D_y = 0\]  

\(g\)

\[+ \bigcirc \sum M_B = 0: \quad -K - 2LD_y + M_D = 0\]  

\(h\)

The above \textit{eight} simultaneous equations, in \textit{eight} unknowns, yield

\[
A_y = -\frac{K}{2L} \quad B_y = -\frac{K}{2L} \quad M_A = \frac{K}{2} \quad \theta_{bl} = \frac{KL}{4EI}
\]

\[
\theta_{br} = 0 \quad D_y = -\frac{K}{2L} \quad M_D = 0 \quad y_b = \frac{KL^2}{6EI}
\]

Consistent with the defined sign conventions, we report that

\[
A_y = \frac{K}{2L} \quad M_A = \frac{K}{2}
\]

\[
y_b = \frac{KL^2}{6EI} \quad \theta_{bl} = \frac{KL}{4EI} \quad \theta_{br} = 0
\]

The position \(C\) is located at \(x = L\), as shown in Fig. 10. Applying the model formulas in Eqs. (9) and (10), successively, to the segment \(BD\) in this figure and utilizing the preceding solutions for \(\theta_{br}\) and \(B_y\), we write

\[
\theta_c = \theta_{br} + \frac{B_y}{2EI}(L^2) + 0 - 0 + 0 - 0 + 0 = -\frac{KL}{4EI}
\]  

\[
\theta_c = \frac{KL}{4EI}
\]

\[
y_c = y_b + \theta_{br}L + \frac{B_y}{6EI}(L^3) + 0 - 0 + 0 - 0 + 0 = \frac{KL^2}{12EI}
\]  

\[
y_c = \frac{KL^2}{12EI}
\]

Based on the preceding solutions, the deflections of the combined beam \(AD\) may be illustrated as shown in Fig. 11.

![Figure 11. Deflection of the combined beam](image-url)
Example 7. A stepped beam $ABC$ carries a uniformly distributed load $w_0$ as shown in Fig. 12, where the segments $AB$ and $BC$ have flexural rigidities $EI_1$ and $EI_2$, respectively. Determine (a) the deflection $y_B$ at $B$, (b) the slopes $\theta_A$, $\theta_B$, and $\theta_C$ at $A$, $B$, and $C$, respectively.

![Figure 12. Stepped beam carrying a uniformly distributed load](image)

Solution. Because of the discontinuity in flexural rigidities at $B$, this beam needs to be divided into two segments $AB$ and $BC$ for analysis in the solution. The boundary conditions of this beam reveal that the deflections at $A$ and $C$ are zero.

![Figure 13. Free-body diagram for segment $AB$](image)

Applying the model formulas in Eqs. (11) and (12), successively, to segment $AB$ and noting that $w_1 = w_0$, as shown in Fig. 13, we write

\[ \theta_B = \theta_A + \frac{A_1 L_2^2}{2 EI_1} + 0 - 0 - \frac{3 w_0 + w_0}{24 EI_1} + 0 \]  

\[ y_B = 0 + \theta_A L_1 + \frac{A_1 L_1^3}{6 EI_1} + 0 - 0 - \frac{4 w_0 + w_0 (L_1 - 0)^4}{120 EI_1} + 0 \]  

For equilibrium of segment $AB$ in Fig. 13, we write

\[ +\Sigma F_y = 0: \quad A_y - B_y - w_0 L_1 = 0 \]  

\[ +\Sigma M_B = 0: \quad - A_y L_1 + \frac{w_0 L_1^2}{2} + M_B = 0 \]  

![Figure 14. Free-body diagram for segment $BC$](image)

Applying the model formulas in Eqs. (11) and (12), successively, to segment $BC$, as shown in Fig. 14, we write
\[ \theta_c = \theta_b + \frac{B_1L_2^3}{2EI_2} + \frac{M_2L_2^2}{EI_2} - 0 + 0 - 0 + 0 \]  
\[ 0 = y_b + \theta_bL_2 + \frac{B_1L_2^3}{6EI_2} + \frac{M_2L_2^2}{2EI_2} - 0 + 0 - 0 + 0 \]

For equilibrium of segment BC in Fig. 14, we write

\[ + \sum F_y = 0 : \quad B_y - C_y = 0 \]  
\[ + \sum M_y = 0 : \quad -M_b - B_yL_2 = 0 \]

The above eight simultaneous equations, in eight unknowns, yield

\[ A_y = \frac{w_0L_1(L_1 + 2L_2)}{2(L_1 + L_2)} \quad C_y = -\frac{w_0L_1^2}{2(L_1 + L_2)} \quad B_y = -\frac{w_0L_1^2}{2(L_1 + L_2)} \quad M_b = \frac{w_0L_1^2L_2}{2(L_1 + L_2)} \]

\[ y_b = -\frac{w_0L_1^3}{24(L_1 + L_2)^2EI_1I_2} \left[ 4L_2^3I_1 + L_1L_2(L_1 + 5L_2)I_2 \right] \]

\[ \theta_l = -\frac{w_0L_1^3}{24(L_1 + L_2)^2EI_1I_2} \left[ 4L_2^3I_1 + L_1(L_1^2 + 5L_2^2)I_2 \right] \]

\[ \theta_b = -\frac{w_0L_1^3}{24(L_1 + L_2)^2EI_1I_2} \left[ 4L_2^3I_1 - L_1^2(L_1 + 5L_2)I_2 \right] \]

\[ \theta_c = \frac{w_0L_1^3}{24(L_1 + L_2)^2EI_1I_2} \left[ 2L_1^2(3L_1 + L_2)I_1 + L_1^2(L_1 + 5L_2)I_2 \right] \]

The above solutions have been assessed and numerically verified to be in agreement with the answers that were independently obtained for a problem involving the same beam but being solved using a different method – the method of segments.\(^{11}\)

V. Advantages of Model Formulas Approach over Method of Segments

The proposed approach using model formulas represents a significant extension of the method of segments.\(^{11}\) In comparison with the method of segments, this approach embraces several major advantages by allowing the following:

- **Multiple concentrated and distributed loads.** A single beam segment is allowed to carry simultaneously any number of concentrated forces, concentrated moments, distributed forces, and distributed moments.

- **Multiple supports anywhere of the beam.** A single beam segment is allowed to have multiple supports that are (a) rigid (e.g., roller or hinge), not at the ends; (b) non-rigid (e.g., tension-
compression spring or torsional spring), not at the ends; and (c) either rigid (e.g., roller, hinge, or fixed) or non-rigid, at the ends.

- **Prescribed linear or angular displacements of the beam at any of its supports.** A single beam segment is allowed to have a prescribed translational or rotational displacement at any of its supports, regardless of the support being at the end or not at the end of the beam.

- **Significant reduction in required number of segments into which a beam must be divided.** Most beam problems can be solved by the model formulas approach using only one segment. Division of a beam into two or more segments is required only when the beam has discontinuity in slope (e.g., at hinge connections) or in flexural rigidity (e.g., in stepped segments).

- **Significant reduction in number of equations and unknowns generated in the solution.** The number of equations and unknowns in a beam problem increases as the required number of segments is increased. Since the model formulas approach allows most beam problems to be modeled with a single segment (except in the case of a beam having discontinuity in slope or in flexural rigidity), the number of equations and unknowns is significantly reduced.

### VI. Conclusion

This paper is presented to share with educators and practitioners in mechanics a new approach that employs a set of four model formulas in solving problems involving statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. These formulas, derived using singularity functions, provide the material equations, besides the equations of static equilibrium, for the solution of the problem. They are expressed in terms of (a) flexural rigidity of the beam; (b) slopes and deflections, as well as shear forces and bending moments, at both ends of the beam; and (c) applied loads on the beam. Typical applied loads are illustrated in Fig. 1, which include (i) concentrated force and concentrated moment; (ii) uniformly, as well as linearly varying, distributed force, and (iii) uniformly distributed moment. The case of multiple concentrated forces acting on a beam is illustrated in Example 1, where the term containing \( P \) in the model formulas has been applied multiple times to account for the multiple concentrated forces. For other types of multiple loads on the beam, one may similarly apply multiple times the appropriate terms in the model formulas.

As a salient feature, the proposed new approach allows one to treat unknown reactions at supports not at the ends of a beam simply as concentrated forces or moments. The boundary conditions at such supports are readily imposed using also the model formulas. A beam needs to be divided into separate segments for analysis only when it has discontinuity in slope or in flexural rigidity. Finally, one should remember that the parameter \( L \) in the model formulas represents the total length of the beam segment, to which the formulas are to be applied.

### VII. References


