AC 2007-11: ANALYSIS OF STATICALLY INDETERMINATE REACTIONS AND DEFLECTIONS OF BEAMS USING MODEL FORMULAS: A NEW APPROACH

Ing-Chang Jong, University of Arkansas

Ing-Chang Jong serves as Professor of Mechanical Engineering at the University of Arkansas. He received a BSCE in 1961 from the National Taiwan University, an MSCE in 1963 from South Dakota School of Mines and Technology, and a Ph.D. in Theoretical and Applied Mechanics in 1965 from Northwestern University. He was Chair of the Mechanics Division, ASEE, in 1996-97. His research interests are in mechanics and engineering education.

Joseph Rencis, University of Arkansas

Joseph J. Rencis is Professor and Head of the Department of Mechanical Engineering at the University of Arkansas in Fayetteville. From 1985 to 2004 he was in the Mechanical Engineering Department at the Worcester Polytechnic Institute. His research focuses on the development of boundary and finite element methods for analyzing solid, heat transfer and fluid mechanics problems with a focus on multi-scale modeling. He serves on the editorial board of Engineering Analysis with Boundary Elements and is associate editor of the International Series on Advances in Boundary Elements. He is currently the Secretary/Treasurer of the ASEE Mechanical Engineering Division and Vice Chair of the ASME Mechanical Engineering Department Heads Committee. He was Chair of the ASEE Mechanics Division, 1999-2000, received award as the 2002 ASEE New England Section Teacher of the Year, and is an ASME fellow. In 2004 he received the ASEE New England Section Outstanding Leader Award and in 2006 the ASEE Mechanics Division James L. Meriam Service Award. He received his B.S. from the Milwaukee School of Engineering in 1980, an M.S. from Northwestern University in 1982, and a Ph.D. from Case Western Reserve University in 1985. V-mail: 479-575-4153; E-mail: jjrencis@uark.edu.

Analysis of Statically Indeterminate Reactions and Deflections of Beams Using Model Formulas: A New Approach

Abstract

This paper is intended to share with educators and practitioners in mechanics a new approach that employs a set of four model formulas in analyzing statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. The model formulas, in algebraic form, are derived using singularity functions. They are expressed in terms of *(a)* flexural rigidity of the beam; *(b)* slopes and deflections, as well as shear forces and bending moments, at both ends of the beam; and *(c)* applied loads on the beam. The types of applied loads include: *(i)* concentrated force and moment; *(ii)* uniformly distributed moment; and *(iii)* linearly varying distributed force. Thus, these model formulas are applicable to most problems encountered in the teaching and learning of mechanics of materials, as well as in practice. As a salient feature, this new approach allows one to treat reactions at supports, even not at the ends of a beam, simply as concentrated forces or moments, where corresponding boundary conditions at the points of supports are imposed using also the model formulas. This feature allows one to readily determine statically indeterminate reactions at supports, as well as slopes and deflections at any positions, of beams. A beam needs to be divided into segments for analysis only when it has discontinuity in slope or in flexural rigidity. Several examples are provided to illustrate this new approach.

I. Introduction

There are different well-known methods for determining deflections of beams in mechanics of materials. These methods may include the following:[1–10]

(a) method of double integration (*with* or *without* the use of singularity functions),
(b) method of superposition,
(c) method using moment-area theorems,
(d) method using Castigliano's theorem, and
(e) conjugate beam method.

This paper extends an earlier study on method of segments[11] by using singularity functions and model formulas. As a result, the proposed new approach allows a considerable reduction in the number of segments required in the study. This new approach makes available an effective method for mechanics educators and practitioners when it comes to determining reactions and deflections of beams. It is aimed at contributing to the enrichment of one's learning experience and to provide a means for independent checking on solutions obtained by other methods.

The paper goes over the description of sign conventions and derives four model formulas for the slope and deflection of a beam segment having a constant flexural rigidity and carrying a variety of commonly applied loads. These formulas, derived using singularity functions, form the basis for a new approach to solving problems involving reactions and deflections of beams. In contrast to the method of segments,[11] the proposed new approach does not have to divide a beam into multiple segments even if the beam has multiple concentrated loads or simple supports not at its

ends. Application of these model formulas is direct and requires no further integration or writing of continuity equations. The model formulas can readily be extended to the analysis of beams having discontinuity in slope (e.g., at hinge connections) or in flexural rigidity (e.g., in stepped segments). It can solve both statically determinate and statically indeterminate beam problems.

II. Sign Conventions for Beams

In the analysis of beams, it is important to adhere to the generally agreed positive and negative signs for loads, shear forces, bending moments, slopes, and deflections. The free-body diagram for a beam *ab* having a constant flexural rigidity *EI* and carrying loads is shown in Fig. 1. The positive directions of shear forces $\mathbf{V}_a$ and $\mathbf{V}_b$, moments $\mathbf{M}_a$ and $\mathbf{M}_b$, at ends *a* and *b* of the beam, the concentrated force **P** and concentrated moment **K**, as well as the distributed loads, are illustrated in this figure.

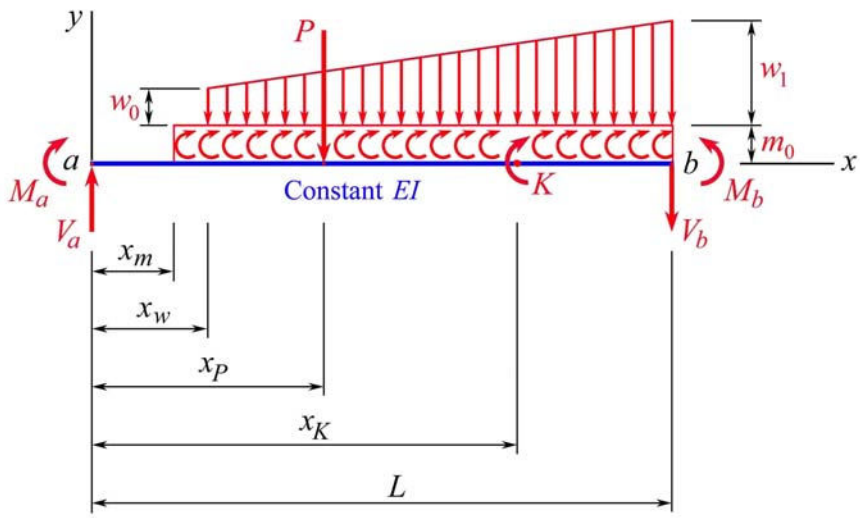

Figure 1. Positive directions of shear forces, moments, and applied loads

In general, we have the following sign conventions for shear forces, moments, and applied loads:

- A *shear force* is *positive* if it acts upward on the left (or downward on the right) face of the beam element (e.g., $\mathbf{V}_a$ at the left end *a*, and $\mathbf{V}_b$ at the right end *b* in Fig. 1).
- At ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam (e.g., $\mathbf{M}_a$ at the left end *a*, and $\mathbf{M}_b$ at the right end *b* in Fig. 1).
- Not at ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam just to the right of the position where it acts (e.g., the concentrated moment **K** and the uniformly distributed moment with intensity m_0 in Fig. 1).
- A *concentrated force* or a *distributed force* applied to the beam is *positive* if it is directed downward (e.g., the concentrated force **P**, the linearly varying distributed force with intensity w_0 on the left side and w_1 on the right side in Fig. 1, where the distribution becomes uniform if $w_1 = w_0$).

Furthermore, we adopt the following sign conventions for deflection and slope of a beam:

- A *positive deflection* is an upward displacement.
- A *positive slope* is a counterclockwise angular displacement.

III. Derivation of Model Formulas

Any beam element of differential width dx at any position x may be perceived to have a *left face* and a *right face*. Using singularity functions,[8-10] we may write, for the beam ab in Fig. 1, the loading function q, shear force V, and bending moment M acting on the **left face** of the beam element at any position x for this beam as follows:

$$q = V_a < x >^{-1} + M_a < x >^{-2} - P < x - x_P >^{-1} + K < x - x_K >^{-2} - w_0 < x - x_w >^{0} - \frac{w_1 - w_0}{L - x_w} < x - x_w >^{1} + m_0 < x - x_m >^{-1} \tag{1}$$

$$V = V_a < x >^{0} + M_a < x >^{-1} - P < x - x_P >^{0} + K < x - x_K >^{-1} - w_0 < x - x_w >^{1} - \frac{w_1 - w_0}{2(L - x_w)} < x - x_w >^{2} + m_0 < x - x_m >^{0} \tag{2}$$

$$M = V_a < x >^{1} + M_a < x >^{0} - P < x - x_P >^{1} + K < x - x_K >^{0} - \frac{w_0}{2} < x - x_w >^{2} - \frac{w_1 - w_0}{6(L - x_w)} < x - x_w >^{3} + m_0 < x - x_m >^{1} \tag{3}$$

It is important to note that Eqs. (1) through (3) are written for the beam in the range $0 \le x < L$, and we have $x - L < 0$ even though $x \to L$ at the right end of beam. By the definition of singularity functions, the value of $< x - L >^{n}$ is **zero** whenever $x - L < 0$, regardless of the value of n. Therefore, values of the terms $-V_b < x - L >^{-1} - M_b < x - L >^{-2}$, as well as the integrals of these terms, are always equal to zero for the beam. This is the reason why these two terms are trivial and may simply be omitted in the expression for the loading function q in Eq. (1).

Noting that EI is the flexural rigidity, y is the deflection, y' is the slope, y'' is the second derivative of y with respect to x for any section of the beam ab, and $EIy'' = M$, we write[4]

$$EIy'' = V_a < x >^{1} + M_a < x >^{0} - P < x - x_P >^{1} + K < x - x_K >^{0} - \frac{w_0}{2} < x - x_w >^{2} - \frac{w_1 - w_0}{6(L - x_w)} < x - x_w >^{3} + m_0 < x - x_m >^{1} \tag{4}$$

$$EIy' = \frac{1}{2} V_a < x >^{2} + M_a < x >^{1} - \frac{1}{2} P < x - x_P >^{2} + K < x - x_K >^{1} - \frac{w_0}{6} < x - x_w >^{3} - \frac{w_1 - w_0}{24(L - x_w)} < x - x_w >^{4} + \frac{1}{2} m_0 < x - x_m >^{2} + C_1 \tag{5}$$

$$EIy = \frac{1}{6} V_a < x >^{3} + \frac{1}{2} M_a < x >^{2} - \frac{1}{6} P < x - x_P >^{3} + \frac{1}{2} K < x - x_K >^{2} - \frac{w_0}{24} < x - x_w >^{4} - \frac{w_1 - w_0}{120(L - x_w)} < x - x_w >^{5} + \frac{1}{6} m_0 < x - x_m >^{3} + C_1 x + C_2 \tag{6}$$

The slope and deflection of the beam in Fig. 1 at its left end a (i.e., at $x = 0$) are θ_a and y_a, respectively. Imposition of these two boundary conditions on Eqs. (5) and (6) yields

$$C_1 = EI\theta_a \tag{7}$$

$$C_2 = EIy_a \tag{8}$$

Substituting Eqs. (7) and (8) into Eqs. (5) and (6), we obtain the **model formulas** for the slope y' and deflection y, at any position x, of the beam ab in Fig. 1 as follows:

$$y' = \theta_a + \frac{V_a}{2EI}x^2 + \frac{M_a}{EI}x - \frac{P}{2EI}<x-x_P>^2 + \frac{K}{EI}<x-x_K>^1 - \frac{w_0}{6EI}<x-x_w>^3 - \frac{w_1-w_0}{24EI(L-x_w)}<x-x_w>^4 + \frac{m_0}{2EI}<x-x_m>^2 \tag{9}$$

$$y = y_a + \theta_a x + \frac{V_a}{6EI}x^3 + \frac{M_a}{2EI}x^2 - \frac{P}{6EI}<x-x_P>^3 + \frac{K}{2EI}<x-x_K>^2 - \frac{w_0}{24EI}<x-x_w>^4 - \frac{w_1-w_0}{120EI(L-x_w)}<x-x_w>^5 + \frac{m_0}{6EI}<x-x_m>^3 \tag{10}$$

By letting $x = L$ in Eqs. (9) and (10), we obtain the **model formulas** for the slope θ_b and deflection y_b at the right end b of the beam ab as follows:

$$\theta_b = \theta_a + \frac{V_a L^2}{2EI} + \frac{M_a L}{EI} - \frac{P}{2EI}(L-x_P)^2 + \frac{K}{EI}(L-x_K) - \frac{3w_0+w_1}{24EI}(L-x_w)^3 + \frac{m_0}{2EI}(L-x_m)^2 \tag{11}$$

$$y_b = y_a + \theta_a L + \frac{V_a L^3}{6EI} + \frac{M_a L^2}{2EI} - \frac{P}{6EI}(L-x_P)^3 + \frac{K}{2EI}(L-x_K)^2 - \frac{4w_0+w_1}{120EI}(L-x_w)^4 + \frac{m_0}{6EI}(L-x_m)^3 \tag{12}$$

IV. Applications of Model Formulas

The set of four **model formulas** given by Eqs. (9) through (12) may be used as the basis upon which to formulate a new approach to analyzing statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. The beams may carry a variety of applied loads, as illustrated in Fig. 1.

Note that L in the model formulas is a *parameter* representing the total length of the beam segment, to which the model formulas are to be applied. These formulas have already taken into account the boundary conditions of the beam at its ends. Furthermore, this approach allows one to treat reactions at interior supports (those *not* at the ends of the beam) as applied concentrated forces or moments. All one has to do is to simply impose the additional corresponding boundary conditions at the interior supports for the beam segment. Thus, the new approach allows one to readily determine statically indeterminate reactions as well as slopes and deflections of beams.

A beam needs to be divided into separate segments for analysis only if *(a)* it is a combined beam (e.g., *Gerber beam*) having discontinuity in slope at hinge connections between segments, and *(b)* it contains segments of different flexural rigidities. The new approach proposed in this paper can best be understood with illustrations. Therefore, simple as well as more challenging problems are included in the following examples.

Example 1. A cantilever beam with a constant flexural rigidity EI and a length L is acted on by three concentrated forces of magnitudes P, $2P$, and $3P$ as shown in Fig. 2. For this beam, determine the slope θ_A and deflection y_A at its free end A.

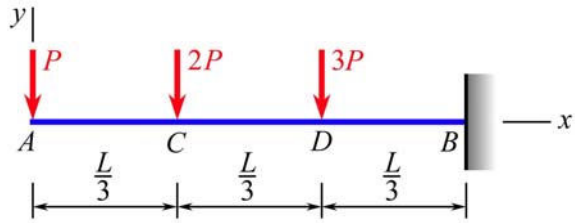

Figure 2. Cantilever beam carrying three concentrated forces

Solution. There is no need to divide the beam into segments for study. At end A, the moment M_A is zero and the shear force V_A is $-P$. At end B, the slope θ_B and deflection y_B are both zero. Since we have multiple concentrated forces acting on the beam, we need to apply the concentrated force term (the term containing P) *multiple times* in all model formulas. Applying the model formulas in Eqs. (11) and (12), successively, to this beam, we write

$$0 = \theta_A + \frac{-PL^2}{2EI} + 0 - \frac{2P}{2EI}\left(L - \frac{L}{3}\right)^2 - \frac{3P}{2EI}\left(L - \frac{2L}{3}\right)^2 + 0 - 0 + 0$$

$$0 = y_A + \theta_A L + \frac{-PL^3}{6EI} + 0 - \frac{2P}{6EI}\left(L - \frac{L}{3}\right)^3 - \frac{3P}{6EI}\left(L - \frac{2L}{3}\right)^3 + 0 - 0 + 0$$

The above *two* simultaneous equations, containing the *two* unknowns θ_A and y_A, yield

$$\theta_A = \frac{10PL^2}{9EI} \qquad\qquad y_A = -\frac{67PL^3}{81EI}$$

Consistent with the defined sign conventions, we report that

$$\theta_A = \frac{10PL^2}{9EI} \circlearrowleft \qquad\qquad y_A = \frac{67PL^3}{81EI} \downarrow$$

Example 2. The right end B of a fix-ended beam AB, which has a constant flexural rigidity EI and a length L, is shifted upward by an amount Δ as shown in Fig. 3. For such a relative vertical shifting of supports, determine *(a)* the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ developed at the left end A, *(b)* the deflection y of the beam at any position x.

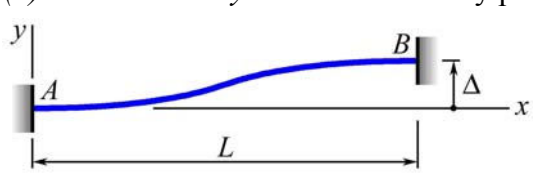

Figure 3. Relative vertical shifting of supports in a fix-ended beam

Solution. This beam is statically indeterminate to the *second* degree. At the fixed support A, the deflection y_A and slope θ_A are zero. At the fixed support B, the deflection y_B is Δ, but the slope θ_B is zero. Applying the model formulas in Eqs. (11) and (12), successively, to this beam, we write

$$0 = 0 + \frac{A_y L^2}{2EI} + \frac{M_A L}{EI} - 0 + 0 - 0 + 0$$

$$\Delta = 0 + 0 + \frac{A_y L^3}{6EI} + \frac{M_A L^2}{2EI} - 0 + 0 - 0 + 0$$

The above *two* simultaneous equations, containing the *two* unknowns A_y and M_A, yield

$$A_y = -\frac{12\,EI\Delta}{L^3} \qquad\qquad M_A = \frac{6\,EI\Delta}{L^2}$$

Consistent with the defined sign conventions, we report that

$$\mathbf{A}_y = \frac{12\,EI\Delta}{L^3}\ \downarrow \qquad\qquad \mathbf{M}_A = \frac{6\,EI\Delta}{L^2}\ \circlearrowright$$

Substituting the obtained values of A_y and M_A into the model formula in Eq. (10), we write

$$y = 0 + 0 + \frac{A_y}{6EI}x^3 + \frac{M_A}{2EI}x^2 - 0 + 0 - 0 - 0 + 0 = -\frac{2\Delta}{L^3}x^3 + \frac{3\Delta}{L^2}x^2$$

$$y = \Delta\left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right)$$

Example 3. A cantilever beam *AB* with a constant flexural rigidity *EI* and carrying a uniformly distributed moment of intensity m_0 over its entire length *L* is shown in Fig. 4. Determine *(a)* the slope θ_B and deflection y_B at the free end *B*, *(b)* the deflection *y* of the beam at any position *x*.

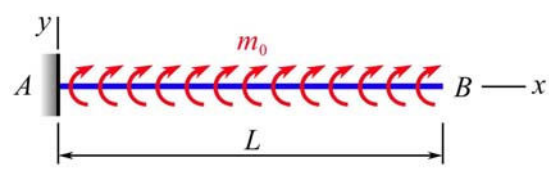

Figure 4. Cantilever beam carrying a uniformly distributed moment

Solution. The free-body diagram of the cantilever beam *AB*, which is in equilibrium as shown in Fig. 5, indicates that the beam has only a counterclockwise reaction moment of magnitude $m_0 L$ at its end *A* besides a uniformly distributed moment of intensity m_0 over its entire length *L*.

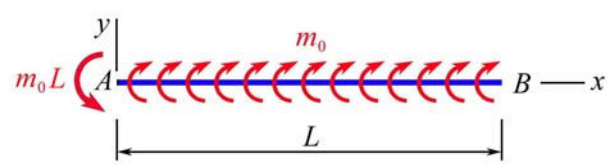

Figure 5. Free-body diagram of the cantilever beam *AB*

We note that the deflection y_A, slope θ_A, and the shear force A_y at end *A* in Fig. 5 are all zero. At end *B*, the moment M_B and shear force B_y are both zero. Applying the model formulas in Eqs. (11) and (12), successively, to this beam and noting that $x_m = 0$, we write

$$\theta_B = 0 + 0 + \frac{(-m_0 L)L}{EI} - 0 + 0 - 0 + \frac{m_0}{2\,EI}(L-0)^2$$

$$y_B = 0 + 0 + 0 + \frac{(-m_0 L)L^2}{2\,EI} - 0 + 0 - 0 + \frac{m_0}{6\,EI}(L-0)^3$$

The above *two* simultaneous equations, containing the *two* unknowns θ_B and y_B, yield

$$\theta_B = -\frac{m_0 L^2}{2EI} \qquad y_B = -\frac{m_0 L^3}{3EI}$$

Consistent with the defined sign conventions, we report that

$$\theta_B = \frac{m_0 L^2}{2EI} \circlearrowright \qquad y_B = \frac{m_0 L^3}{3EI} \downarrow$$

Substituting the obtained values of θ_B and y_B into the model formula in Eq. (10), we write

$$y = 0 + 0 + 0 + \frac{-m_0 L}{2EI}x^2 - 0 + 0 - 0 - 0 + \frac{m_0}{6EI}(x-0)^3 = -\frac{m_0 x^2(3L-x)}{6EI}$$

$$y = -\frac{m_0 x^2(3L-x)}{6EI}$$

Example 4. A cantilever beam *AB* with a constant flexural rigidity *EI* and a length 2*L* is propped by a linear tension-compression spring of modulus *k*, and it carries a concentrated moment **K** at its midpoint *C* as shown in Fig. 6. Determine the slope θ_A and deflection y_A at its left end *A*.

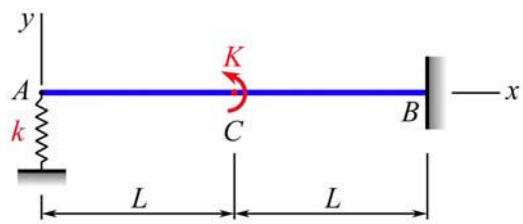

Figure 6. Cantilever beam propped by a linear spring and carrying a concentrated moment

Solution. At end *A* of this beam, the moment M_A is zero and shear force $V_A = -ky_A$, which is based on the initial assumption that y_A is upward and the linear spring force of ky_A acts downward at end *A*. At end *B*, the slope θ_B and deflection y_B are both zero. Note that we need to set the parameter *L* in the model formulas in Eqs. (11) and (12) equal to 2*L* for this beam *AB*. Letting $x_K = L$ and applying Eqs. (11) and (12), successively, to this beam, we write

$$0 = \theta_A + \frac{(-ky_A)(2L)^2}{2EI} + 0 - 0 + \frac{-K}{EI}(2L-L) - 0 + 0$$

$$0 = y_A + \theta_A(2L) + \frac{(-ky_A)(2L)^3}{6EI} + 0 - 0 + \frac{-K}{2EI}(2L-L)^2 - 0 + 0$$

The above *two* simultaneous equations, containing the *two* unknowns θ_A and y_A, yield

$$\theta_A = \frac{KL(3EI-kL^3)}{EI(3EI+8kL^3)} \qquad y_A = -\frac{9KL^2}{2(3EI+8kL^3)}$$

Consistent with the defined sign conventions, we report that

$$\theta_A = \frac{KL(3EI-kL^3)}{EI(3EI+8kL^3)} \circlearrowleft \qquad y_A = \frac{9KL^2}{2(3EI+8kL^3)} \downarrow$$

Example 5. A beam AB, which has a constant flexural rigidity EI, a roller support at A, a roller support at C, a fixed support at B, and a length of $2L$, carries a linearly distributed load with maximum intensity w_1 at B as shown in Fig. 7. Determine *(a)* the vertical reaction force $\mathbf{A}_y$ and the slope θ_A at A, *(b)* the vertical reaction force $\mathbf{C}_y$ and the slope θ_C at C.

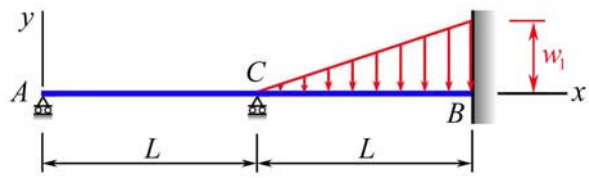

Figure 7. Beam supported by two rollers and a fixed support

Solution. This beam is statically indeterminate to the *second* degree. There is no need to divide the beam AB into two segments for analysis in the solution by the proposed new approach. We can simply treat the vertical reaction force $\mathbf{C}_y$ at C as an unknown applied concentrated force, directed upward, and regard the beam AB as one that has a total length of $2L$, which is to be used as the value for the parameter L in the model formulas in Eqs. (9) through (12). The boundary conditions of this beam reveal that the moment M_A and the deflection y_A at A are both zero, the slope θ_B and deflection y_B at B are also both zero, and the deflection y_C at C is zero. The shear force at the left end A is the vertical reaction force $\mathbf{A}_y$ at A, which may be assumed to be acting upward. Applying Eqs. (11) and (12) to the entire beam and Eq. (10) for imposing that $y_C = 0$ at C, in that *order*, we write

$$0 = \theta_A + \frac{A_y(2L)^2}{2EI} + 0 - \frac{-C_y}{2EI}(2L-L)^2 + 0 - \frac{0+w_1}{24EI}(2L-L)^3 + 0$$

$$0 = 0 + \theta_A(2L) + \frac{A_y(2L)^3}{6EI} + 0 - \frac{-C_y}{6EI}(2L-L)^3 + 0 - \frac{0+w_1}{120EI}(2L-L)^4 + 0$$

$$0 = 0 + \theta_A L + \frac{A_y}{6EI}L^3 + 0 - 0 + 0 - 0 - 0 + 0$$

The above *three* simultaneous equations, containing the *three* unknowns A_y, θ_A, and C_y, yield

$$A_y = -\frac{w_1 L}{70} \qquad \theta_A = \frac{w_1 L^3}{420EI} \qquad C_y = \frac{19\,w_1 L}{140}$$

Consistent with the defined sign conventions, we report that

$$\mathbf{A}_y = \frac{w_1 L}{70} \downarrow \qquad \theta_A = \frac{w_1 L^3}{420EI} \circlearrowleft \qquad \mathbf{C}_y = \frac{19\,w_1 L}{140} \uparrow$$

The slope θ_C is simply y' evaluated at C, which is located at $x = L$. Applying the model formula in Eq. (9) and utilizing the preceding solutions for θ_A and A_y, we write

$$\theta_C = \theta_A + \frac{A_y}{2EI}L^2 + 0 - 0 + 0 - 0 - 0 - 0 = \frac{w_1 L^3}{420EI} - \frac{w_1 L}{70}\left(\frac{L^2}{2EI}\right) = -\frac{w_1 L^3}{210EI}$$

$$\theta_C = \frac{w_1 L^3}{210EI} \circlearrowright$$

Example 6. A combined beam (*Gerber beam*), with a constant flexural rigidity *EI*, fixed supports at its ends *A* and *D*, a hinge connection at *B*, and carrying a concentrated moment **K** at *C*, is shown in Fig. 8. Determine *(a)* the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at *A*, *(b)* the deflection y_B of the hinge at *B*, *(c)* the slopes θ_{BL} and θ_{BR} just to the *left* and just to the *right* of the hinge at *B*, respectively, and *(d)* the slope θ_C and the deflection y_C at *C*.

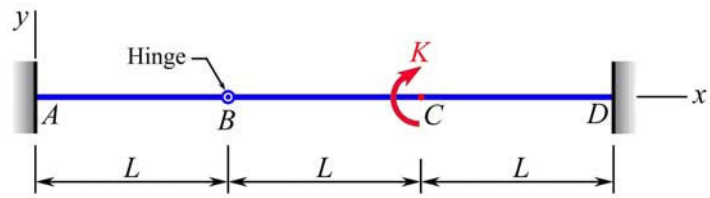

Figure 8. Fix-ended beam with a hinge connector

Solution. This beam is statically indeterminate to the *first* degree. Because of the discontinuity in slope at the hinge connection *B*, this beam needs to be divided into two segments *AB* and *BD* for analysis in the solution, as shown in Figs. 9 and 10, where the deflected shapes are shown to highlight the discontinuity in slope at *B*; i.e., $\theta_{BL} \neq \theta_{BR}$. The boundary conditions of this beam reveal that slope and deflection at *A* and *D* are all equal to zero.

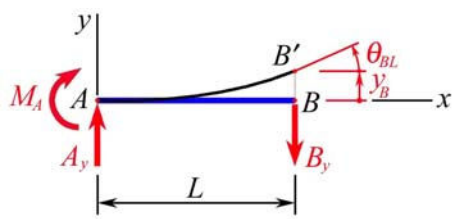

Figure 9. Free-body diagram for segment *AB* and its deflection

Applying the model formulas in Eqs. (11) and (12), successively, to segment *AB*, as shown in Fig. 9, we write

$$\theta_{BL} = 0 + \frac{A_y L^2}{2EI} + \frac{M_A L}{EI} - 0 + 0 - 0 + 0 \qquad (a)$$

$$y_B = 0 + 0 + \frac{A_y L^3}{6EI} + \frac{M_A L^2}{2EI} - 0 + 0 - 0 + 0 \qquad (b)$$

For equilibrium of segment *AB* in Fig. 9, we write

$$+\uparrow \Sigma F_y = 0: \qquad A_y - B_y = 0 \qquad (c)$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -M_A - L A_y = 0 \qquad (d)$$

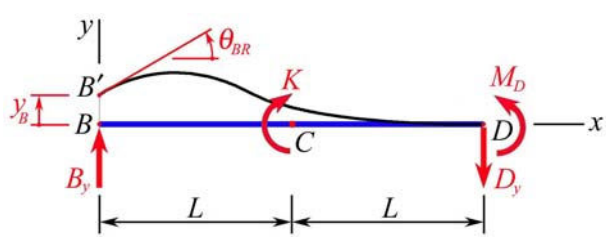

Figure 10. Free-body diagram for segment *BD* and its deflection

Applying the model formulas in Eqs. (11) and (12), successively, to segment *BD*, as shown in Fig. 10, we write

$$0 = \theta_{BR} + \frac{B_y(2L)^2}{2EI} + 0 - 0 + \frac{K}{EI}(2L - L) - 0 + 0 \qquad (e)$$

$$0 = y_B + \theta_{BR}(2L) + \frac{B_y(2L)^3}{6EI} + 0 - 0 + \frac{K}{2EI}(2L - L)^2 - 0 + 0 \qquad (f)$$

For equilibrium of segment *BD* in Fig. 10, we write

$$+\uparrow \Sigma F_y = 0: \qquad B_y - D_y = 0 \qquad (g)$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -K - 2LD_y + M_D = 0 \qquad (h)$$

The above *eight* simultaneous equations, in *eight* unknowns, yield

$$A_y = -\frac{K}{2L} \qquad B_y = -\frac{K}{2L} \qquad M_A = \frac{K}{2} \qquad \theta_{BL} = \frac{KL}{4EI}$$

$$\theta_{BR} = 0 \qquad D_y = -\frac{K}{2L} \qquad M_D = 0 \qquad y_B = \frac{KL^2}{6EI}$$

Consistent with the defined sign conventions, we report that

$$\mathbf{A}_y = \frac{K}{2L} \downarrow \qquad \mathbf{M}_A = \frac{K}{2} \circlearrowright$$

$$y_B = \frac{KL^2}{6EI} \uparrow \qquad \theta_{BL} = \frac{KL}{4EI} \circlearrowleft \qquad \theta_{BR} = 0$$

The position *C* is located at $x = L$, as shown in Fig. 10. Applying the model formulas in Eqs. (9) and (10), successively, to the segment *BD* in this figure and utilizing the preceding solutions for θ_{BR} and B_y, we write

$$\theta_C = \theta_{BR} + \frac{B_y}{2EI}(L^2) + 0 - 0 + 0 - 0 - 0 + 0 = -\frac{KL}{4EI} \qquad \theta_C = \frac{KL}{4EI} \circlearrowright$$

$$y_C = y_B + \theta_{BR}L + \frac{B_y}{6EI}(L^3) + 0 - 0 + 0 - 0 - 0 + 0 = \frac{KL^2}{12EI} \qquad y_C = \frac{KL^2}{12EI} \uparrow$$

Based on the preceding solutions, the deflections of the combined beam *AD* may be illustrated as shown in Fig. 11.

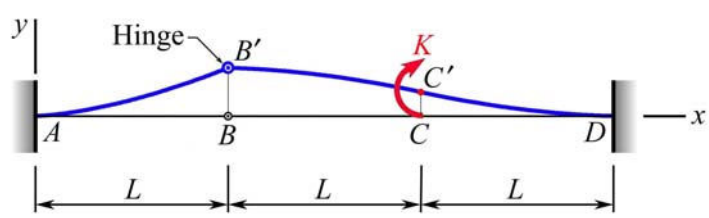

Figure 11. Deflection of the combined beam

Example 7. A stepped beam ABC carries a uniformly distributed load w_0 as shown in Fig. 12, where the segments AB and BC have flexural rigidities EI_1 and EI_2, respectively. Determine *(a)* the deflection y_B at B, *(b)* the slopes θ_A, θ_B, and θ_C at A, B, and C, respectively.

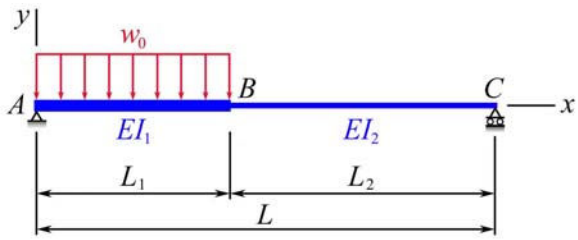

Figure 12. Stepped beam carrying a uniformly distributed load

Solution. Because of the *discontinuity in flexural rigidities* at B, this beam needs to be divided into two segments AB and BC for analysis in the solution. The boundary conditions of this beam reveal that the deflections at A and C are zero.

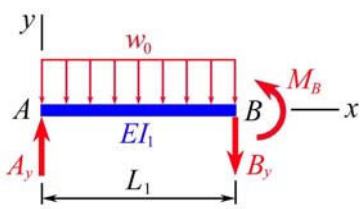

Figure 13. Free-body diagram for segment AB

Applying the model formulas in Eqs. (11) and (12), successively, to segment AB and noting that $w_1 = w_0$, as shown in Fig. 13, we write

$$\theta_B = \theta_A + \frac{A_y L_1^2}{2EI_1} + 0 - 0 + 0 - \frac{3w_0 + w_0}{24EI_1} + 0 \qquad (a)$$

$$y_B = 0 + \theta_A L_1 + \frac{A_y L_1^3}{6EI_1} + 0 - 0 + 0 - \frac{4w_0 + w_0}{120EI_1}(L_1 - 0)^4 + 0 \qquad (b)$$

For equilibrium of segment AB in Fig. 13, we write

$$+\uparrow \Sigma F_y = 0: \qquad A_y - B_y - w_0 L_1 = 0 \qquad (c)$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -A_y L_1 + \frac{w_0 L_1^2}{2} + M_B = 0 \qquad (d)$$

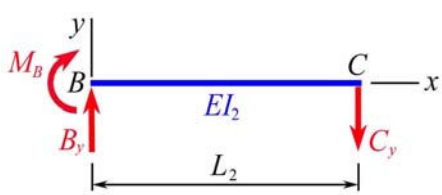

Figure 14. Free-body diagram for segment BC

Applying the model formulas in Eqs. (11) and (12), successively, to segment BC, as shown in Fig. 14, we write

$$\theta_C = \theta_B + \frac{B_y L_2^2}{2EI_2} + \frac{M_B L_2}{EI_2} - 0 + 0 - 0 + 0 \qquad (e)$$

$$0 = y_B + \theta_B L_2 + \frac{B_y L_2^3}{6EI_2} + \frac{M_B L_2^2}{2EI_2} - 0 + 0 - 0 + 0 \qquad (f)$$

For equilibrium of segment *BC* in Fig. 14, we write

$$+\uparrow \Sigma F_y = 0: \qquad B_y - C_y = 0 \qquad (g)$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -M_B - B_y L_2 = 0 \qquad (h)$$

The above *eight* simultaneous equations, in *eight* unknowns, yield

$$A_y = \frac{w_0 L_1 (L_1 + 2L_2)}{2(L_1 + L_2)} \qquad C_y = -\frac{w_0 L_1^2}{2(L_1 + L_2)} \qquad B_y = -\frac{w_0 L_1^2}{2(L_1 + L_2)} \qquad M_B = \frac{w_0 L_1^2 L_2}{2(L_1 + L_2)}$$

$$y_B = -\frac{w_0 L_1^3 [4L_2^3 I_1 + L_1 L_2 (L_1 + 5L_2) I_2]}{24(L_1 + L_2)^2 E I_1 I_2}$$

$$\theta_A = -\frac{w_0 L_1^2 [4L_2^3 I_1 + L_1 (L_1^2 + 5L_1 L_2 + 8L_2^2) I_2]}{24(L_1 + L_2)^2 E I_1 I_2}$$

$$\theta_B = -\frac{w_0 L_1^2 [4L_2^3 I_1 - L_1^2 (L_1 + 5L_2) I_2]}{24(L_1 + L_2)^2 E I_1 I_2}$$

$$\theta_C = \frac{w_0 L_1^2 [2L_2^2 (3L_1 + L_2) I_1 + L_1^2 (L_1 + 5L_2) I_2]}{24(L_1 + L_2)^2 E I_1 I_2}$$

The above solutions have been *assessed* and numerically *verified* to be in agreement with the answers that were independently obtained for a problem involving the same beam but being solved using a different method – the method of segments.[11]

V. Advantages of Model Formulas Approach over Method of Segments

The proposed approach using model formulas represents a significant extension of the method of segments.[11] In comparison with the method of segments, this approach embraces several major advantages by allowing the following:

- *Multiple concentrated and distributed loads*. A single beam segment is allowed to carry simultaneously any number of concentrated forces, concentrated moments, distributed forces, and distributed moments.
- *Multiple supports anywhere of the beam.* A single beam segment is allowed to have multiple supports that are (*a*) rigid (e.g., roller or hinge), not at the ends; (*b*) non-rigid (e.g., tension-

compression spring or torsional spring), not at the ends; and (*c*) either rigid (e.g., roller, hinge, or fixed) or non-rigid, at the ends.

- *Prescribed linear or angular displacements of the beam at any of its supports*. A single beam segment is allowed to have a prescribed translational or rotational displacement at any of its supports, regardless of the support being at the end or not at the end of the beam.
- *Significant reduction in required number of segments into which a beam must be divided.* Most beam problems can be solved by the model formulas approach using only one segment. Division of a beam into two or more segments is required only when the beam has discontinuity in slope (e.g., at hinge connections) or in flexural rigidity (e.g., in stepped segments).
- *Significant reduction in number of equations and unknowns generated in the solution.* The number of equations and unknowns in a beam problem increases as the required number of segments is increased. Since the model formulas approach allows most beam problems to be modeled with a single segment (except in the case of a beam having discontinuity in slope or in flexural rigidity), the number of equations and unknowns is significantly reduced.

VI. Conclusion

This paper is presented to share with educators and practitioners in mechanics a new approach that employs a set of four model formulas in solving problems involving statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. These formulas, derived using singularity functions, provide the material equations, besides the equations of static equilibrium, for the solution of the problem. They are expressed in terms of *(a)* flexural rigidity of the beam; *(b)* slopes and deflections, as well as shear forces and bending moments, at both ends of the beam; and *(c)* applied loads on the beam. Typical applied loads are illustrated in Fig. 1, which include *(i)* concentrated force and concentrated moment; *(ii)* uniformly, as well as linearly varying, distributed force, and *(iii)* uniformly distributed moment. The case of multiple concentrated forces acting on a beam is illustrated in Example 1, where the term containing P in the model formulas has been applied multiple times to account for the multiple concentrated forces. For other types of multiple loads on the beam, one may similarly apply multiple times the appropriate terms in the model formulas.

As a salient feature, the proposed new approach allows one to treat unknown reactions at supports not at the ends of a beam simply as concentrated forces or moments. The boundary conditions at such supports are readily imposed using also the model formulas. A beam needs to be divided into separate segments for analysis only when it has discontinuity in slope or in flexural rigidity. Finally, one should remember that the parameter L in the model formulas represents the total length of the beam segment, to which the formulas are to be applied.

VII. References

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