Application of **Mathcad**® Software in Performing Uncertainty **Analysis Calculations to Facilitate Laboratory Instruction**

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ABSTRACT
    Uncertainty analysis is a technique for determining an estimate for the interval about a reported result within which the true result is expected to lie with a certain degree of confidence. The uncertainty analysis is an extremely useful tool for all phases of an experimental program from the initial planning (general uncertainty analysis) to detailed design, debugging, testing, and data analysis (detailed uncertainty analysis).

    The main objective of this article is to describe how Mathcad® software maybe used to facilitate uncertainty analysis calculations in undergraduate laboratory instruction. This objective is accomplished by performing general uncertainty analysis calculations for several example problems using **Mathcad**®. These examples show how quick uncertainty analysis calculations using Mathcad® during the planning phase of an experiment may assist students to select appropriate measurement equipment. Furthermore, an example of detailed uncertainty analysis is presented to investigate the contributions of the elemental bias and precision error sources, obtain estimates of the bias and precision limits for each measured variable, and illustrate how the measurement errors propagate into the experimental result.

INTRODUCTION
    Uncertainty analysis[1] is a technique for determining an estimate for the interval about a reported result within which the true result is expected to lie with a certain degree of confidence. The uncertainty analysis is an extremely useful tool for all phases of an experimental program from the initial planning (general uncertainty analysis) to detailed design, debugging, testing, and data analysis (detailed uncertainty analysis).

    The undergraduate laboratory experience usually includes experimentation with several previously designed experiments using various measurement equipment. For instance, a group of students is assigned to an experimental setup to obtain forced convection data for a heated horizontal cylinder placed in the test section of a wind tunnel. The students measure the amount of power supplied to the cylinder, surface temperature of the heated cylinder, air temperature, surrounding temperatures, air velocity, etc. using various transducers via either manual or automated data acquisition system. Upon completion of data collection, the heat transfer coefficient for the cylinder is determined and compared with the available empirical correlations. The experimental results are expected to be reported with uncertainty limits due to the measurement errors associated with each transducer or measurement equipment. The uncertainty limits for experimental results are estimated by performing detailed uncertainty analysis calculations. In some laboratory experiments, the students may perform general uncertainty analysis to select appropriate measurement equipment *a priori* to experimentation. In either case, the students are required to quantify the measurement uncertainties and

estimate the interval about the reported result within which the true result is expected to lie with a certain degree of **confidence** using uncertainty analysis techniques.

The students are faced with understanding the objectives of the laboratory experiment, details of the experimental setup (transducers, wiring, measurement equipment, etc.), experimental procedures (operation of different components) and **performing** uncertainty analysis. The experimental uncertainty analysis is an important part of experimentation. However, occasionally students become so immersed in the many details of the uncertainty analysis that the overall objective of the experiment is forgotten.

The main objective of this article is to describe how **Mathcad® software**[2] may be used to facilitate uncertainty analysis calculations in undergraduate laboratory instruction. This objective is accomplished by performing general uncertainty analysis calculations for several example problems using **Mathcad®** and presenting a comparison with the manual analysis. These examples show how quick uncertainty analysis calculations using **Mathcad®** during the planning phase of an experiment may assist students to select appropriate measurement equipment. Furthermore, an example of detailed uncertainty analysis is presented to investigate the contributions of the elemental bias error sources, obtain estimates of the bias and precision limits for each measured variable, and illustrate how the errors propagate into the experimental result.

UNCERTAINTY ANALYSIS

Laboratory instruction is one of the most important components of the engineering education. Experimentation and reporting of results are major parts of the laboratory experience. In a laboratory, students seek an experimental answer to a question. Depending upon the complexity of the question, a single or many physical variables are measured and the results are determined. This process may require application of various transducers and measurement equipment that individually or combined provide experimental data that maybe used to deduce the result. However, experimentation is not just data collection. Students are expected to learn and master a systematic approach to experimentation by dividing the experimental program (i.e., the process of obtaining the desired result) into different phases such as planning, design, **construction,** debugging, execution, data analysis, and reporting of results. There are not sharp divisions between these phases and it is expected to proceed with several phases simultaneously in an iterative **manner;** however, the concept of dividing an experimental program into different phases provides the students with a clear and logical direction. In the planning phase, the students evaluate the applicability of various approaches in finding an answer to the question being addressed (i.e., the result). The general uncertainty analysis will be most helpful in this phase of the experimental plan. The detailed uncertainty analysis procedure is very **useful** in the **design, construction,** debugging, data analysis, and reporting phases of an experiment. It is through the detailed uncertainty analysis that the students will investigate the contributions of the elemental bias (fixed) error sources due to different transducers and measurement instruments, obtain estimates of the bias and precision limits for each measured variable, and learn how these errors propagate into the experimental result.

To facilitate the uncertainty calculations and enhance understanding of this important experimentation tool, application of **Mathcad®** software as an effective instructional tool is described herein.

GENERAL UNCERTAINTY ANALYSIS

In the planning phase, general uncertainty analysis is used to determine whether a particular experiment is feasible, which measurements are more important than others, which techniques may provide more reliable results, etc. Thus, the behavior of the experimental result as a **function** of the uncertainties in the measured

variables is being investigated. At this stage, estimated uncertainties are assigned to measured variables and properties obtained from reference sources and an overall uncertainty for the result is obtained. No equipment is selected or purchased yet, but the calculations establish the feasibility of the experiment.

Consider a general case in which an experimental result, r, is a function of J variables X_i

$$r = r(X_1, X_2, x_3, \ldots\ldots, X_J) \tag{1}$$

Equation (1) is called the data reduction equation. The uncertainty in the result is given by

$$U_r = [(\frac{\partial r}{\partial X_1} U_{X_1})^2 + (\frac{\partial r}{\partial X_2} U_{X_2})^2 + \ldots\ldots + (\frac{\partial r}{\partial XJ} U_{X_J})^2]^{1/2} \tag{2}$$

where U_r is the uncertainty in the result and the U_{X_i} are the uncertainties in the measured variables X,. The "$\frac{\partial r}{\partial X_1}$" is called an "absolute sensitivity coefficient"; it represents a partial derivative of the result with respect to the measured variable X_1. In equation (2), it is assumed that the measured variables X_i are independent of each other and that the uncertainties in the measured variables are also independent of one another.

For instance, the students are asked to experimentally determine the density of air (ρ) in a pressurized tank and the conditions are such that the equation of state of an ideal gas (p = *P/R T*) is applicable. **Then,** the determination of air density (p) depends on the evaluation of the pressure (P), temperature (T), and the gas constant (R). It is not necessary to measure the gas constant since it is available with great accuracy in reference sources[3], thus, the air pressure and temperature are the quantities to be measured in the laboratory. The students know that there are uncertainties (measurement errors) associated with each measured variables, however, *the key question for them is, how do these uncertainties in the individual variables propagate through the data reduction equation (governing equation) into the result?* The following examples of general uncertainty analysis during the planning phase of the experiment show the flexibility on estimating uncertainties for measured variables and the propagation of these uncertainties into the experimental result.

Example Problem -1

A pressurized air tank is stored in a laboratory where the nominal ambient temperature is 27 "C. The air temperature is measured using a sensor with an uncertainty of 1 "C and the uncertainty in pressure measurement is 10/o. Evaluate the uncertainty in determining the air density using the equation of state for an ideal gas.

A. Solution (Manual Analysis):

The data reduction equation is

$$' = R \frac{p}{T} \tag{3}$$

The uncertainty expression becomes

$$up = [(\frac{\partial \rho}{\partial p} U_p)^2 + (\frac{\partial \rho}{\partial R} U_R)^2 + (\frac{\partial \rho}{\partial T} U_T)^2]^{1/2} \tag{4}$$

The partial derivatives are

$$\frac{\partial \rho}{\partial p} = \frac{1}{RT} \tag{5}$$

1996 ASEE Annual Conference Proceedings

$$\frac{\partial \rho}{\partial R} = -\frac{P}{R^2 T} \tag{6}$$

$$\frac{\partial \rho}{\partial T} = -\frac{P}{R T^2} \tag{7}$$

Substitution of the partial derivatives into (4) gives

$$u\rho ' [(\frac{1}{R T} U_p)^2 + (\frac{-p}{R^2 T} U_R)^2 + (\frac{-p}{R T^2} U_T)^2]^{1/2} \tag{8}$$

Dividing the uncertainty expression (8) by the data reduction equation (3) gives

$$\frac{u_\rho}{\rho} [(\frac{U_p}{p})^2 + (\frac{U_R}{R})^2 + (\frac{U_T}{T})^2]^{1/2} \tag{9}$$

Values for the variables are

$T = 27"C + 273 = 300 K$

$U_T = 1 \text{ "C } (=1 K)$

$up/p = 0.01$

$U_R = 0.0$

The uncertainty in **R**, i.e. U_R, is assumed negligible since the gas constant is available in reference sources with a great degree of certainty. Substitution of the numerical values in (9) gives the uncertainty in the air density

$U_\rho/\rho = 0.011 = 1.1\%$

The analysis presented for the above example problem shows the necessary steps in a general uncertainty analysis. The analysis for this simple problem is straightforward, however, manual evaluation of the derivatives and repeated calculations for more complex problem can become tedious. Application of **Mathcad®** for uncertainty analysis of complex problems provides the convenience and the flexibility to facilitate laboratory instruction by reducing the difficulty of calculations (i.e., taking derivatives) and increasing the speed in repeating computations.

B. Solution (Using Mathcad®):

The step-by-step solution procedure using **Mathcad®** is given in Figure 1. **Mathcad®** provides the flexibility to **perform** repeated calculations and show the results at various intermediate steps. As shown in Figure 1, the uncertainty contributions from pressure and temperature terms are presented separately. It is clear that the contribution due to the pressure is much larger than temperature. This step illustrates the utility of uncertainty analysis in **identifying** the more sensitive measurement variables with which special care must be taken. Suppose we want to reduce the uncertainty of the result (density) to 0.5%. Then, we can **vary** the estimated uncertainty in temperature measurement, pressure measurement, or both. Reducing the uncertainty in temperature measurement alone will not **satisfy** our requirement since setting it even to zero reduces the uncertainty in density to 10/o. Thus, acquiring more expensive temperature measurement device for better accuracy is unnecessary. By improving the pressure measurement and reducing its uncertainty to 0.3°/0, the uncertainty in density reduces to O. S"/o. The key point in this exercise is to show how the students may vary the uncertainties in each measured variables as they select proper transducers based on manufacturers' specifications *a priori* to purchasing the transducers. It also illustrates how quickly and conveniently the values of measured variables and associated uncertainties maybe varied in **Mathcad®** and obtain accurate results. The **Mathcad®** flexibility in allowing to enter measured values in different system of units (i.e. English or S1) as obtained from different measurement equipment is an added convenience.

Example Problem -2

A client needs to determine the behavior of the convective heat transfer from a heated circular cylinder of finite length that will be immersed in an air flow whose velocity might range from 1 to 100 m/sec. The expected convective heat transfer coefficient for these velocities range up to 1000 W/m² "C. The cylinder tube to be tested is 152 mm long with a diameter of 25.4 mm and a wall thickness of 1.27 mm. The primary case of interest is for an air temperature of about 25 °C with the cylinder at about 45 'C. The cylinder is made of aluminum, for which the specific heat has been found from a table as 903 J/kg °C at 300 K. The mass of the tube has been measured as 0.020 kg. The client wants to know the convective heat transfer coefficient (h) within 5°A and must be assured that "h" can be determined within 10°/0 before funding for the experiment will be allocated.

The two experimental techniques being considered for determining the heat transfer coefficient in this example problem are a steady-state and a transient technique, respectively. The heat transfer from the cylinder by radiation or conduction is assumed negligible in this initial analysis.

The data reduction equation for the steady-state technique is

$$h = \frac{W}{\pi D L (T - T_a)} \tag{1o}$$

For a given air velocity, the convective heat transfer coefficient h is determined by measuring the electrical power W, the cylinder diameter D and length L, and the temperatures of the cylinder surface T and the air stream T_a.

The data reduction equation for a transient technique is

$$h = \frac{M c}{\pi \tau D L} \tag{11}$$

The convective heat transfer coefficient h for a given air velocity is determined by using the cylinder mass M, specific heat c, diameter D and length L, and the time constant τ. In this approach, the cylinder is heated until it reaches a temperature TO and then the power is switched off. The time constant τ is defined as the time taken for the cylinder temperature T to change 63 .2% of the way from TO to T_a.

A. Steady State Solution (Manual Analysis):

The uncertainty expression for the steady-state technique is

$$U_h = \left[\left(\frac{\partial h}{\partial W} U_W \right)^2 + \left(\frac{\partial h}{\partial D} U_D \right)^2 + \left(\frac{\partial h}{\partial L} U_L \right)^2 + \left(\frac{\partial h}{\partial T} U_T \right)^2 + \left(\frac{\partial h}{\partial T_a} U_{T_a} \right)^2 \right]^{1/2} \tag{12}$$

For this analysis, assume that the temperature difference (T-T_a) will be measured directly. Therefore, the uncertainty expression simplifies as

$$U_h = \left[\left(\frac{\partial h}{\partial W} U_W \right)^2 + \left(\frac{\partial h}{\partial D} U_D \right)^2 + (\$ \ U_L)^2 + (\$_T `A*) 2 1"2 \tag{13}$$

Students are expected to estimate the uncertainties in W, **D, L, and** AT. **Assume the following measurement** uncertainties:

$$U_D = 0.025 \text{ mm}$$
$$U_L = 0.25 \text{ mm}$$

$$U_{\Delta T} = 0.25 \text{ "C} \text{ and } 0.5 \text{ °C}$$
$$U_W = 0.5 \text{ Wand } 1.0 \text{ W}$$

Manual evaluation of the partial derivatives, substitution into (13) and determination of the uncertainty for several h values are presented elsewhere and will not be repeated here. However, general uncertainty analysis using **Mathcad®** is presented herein.

B. Steady State Solution (using Mathcad®):

The steady-state solution using **Mathcad®** is presented in Figure 2.

A. Transient Solution (Manual Analysis):

The uncertainty expression for the transient technique is

$$U_h = [(\frac{\partial h}{\partial M} U_M)^2 + (\frac{\partial h}{\partial c} U_c)^2 + (\frac{\partial h}{\partial D} U_D)^2 + (\frac{\partial h}{\partial L} U_L)^2 + (\frac{\partial h}{\partial \tau} U_\tau)^2]^{1/2} \qquad (14)$$

The uncertainties in M, c, D, L, and τ need to be estimated. Uncertainties in D and L are the same as those used in the steady-state analysis. Let us use the following measurement uncertainties in this analysis:

$$U_D = 0.025 \text{ mm}$$
$$U_L = 0.25 \text{ mm}$$
$$U_M = 0.0001 \text{ Kg}$$
$$U_c/c = 1.0\%$$
$$U_\tau = 5, 10, \text{ and } 20 \text{ msec}$$

Manual evaluation of the partial derivatives, substitution into (14) and determination of the uncertainty are presented elsewhere and will not be repeated here. However, **Mathcad®** evaluations are presented herein.

B. Transient Solution (using Mathcad®):

The transient solution using **Mathcad®** is presented in Figure 3.

DETAILED UNCERTAINTY ANALYSIS

The detailed uncertainty analysis[4] is very useful in the design, construction, debugging, data analysis, and reporting phases of an experimental program. This approach is more complex than the general uncertainty method since the bias and precision components of uncertainty are considered separately.

The elemental bias limits from all sources for each measured variable are combined using RSS to obtain a resultant bias limit for that variable. For the ith variable influenced by biases from M significant elemental error sources, the resultant bias limit is calculated by

$$B_i = [(B_i)_1^2 + (B_i)_2^2 + \ldots\ldots + (B_i)_M^2]^{1/2} \qquad (15)$$

The precision limit, Pi, for each variable, X_i, is estimated from multiple readings of the variable over the appropriate time period or from previous experince.

Once the bias and precision limits for each variable have been estimated, the bias and precision limits for the experimental result are found from the uncertainty analysis expressions (16) and (17), respectively, which assume no correlated bias or precision uncertainties and incorporates the large sample assumption as discussed in Coleman and Steele (1995)[4].

$$B_r = [(\frac{\partial r}{\partial X_1} B_1)^2 + (\frac{\partial r}{\partial X_2} B_2)^2 + \dots + (\frac{\partial r}{\partial XJ} B_J)^2]^{1/2} \qquad (16)$$

and

$$P_r = [(\frac{\partial r}{\partial X_1} P_1)^2 + (\frac{\partial r}{\partial X_2} P_2)^2 + \dots + (\frac{\partial r}{\partial XJ} P_J)^2]^{1/2} \qquad (17)$$

The bias component (B_r) is a fixed error that may in some instances be reduced by calibration. The precision component (P_r) is a variable error that may be reduced by averaging repeated measurements taken over an appropriate time period. The uncertainty in the result is then given

$$U_r = [B_r^2 + P_r^2]^{1/2} \qquad (18)$$

The uncertainty U_r gives 95% coverage of the true value when the bias and precision contributions are estimated for a 95°A confidence level. In practice, dividing each term in expressions (16), (17) and(18) by the result may sometimes result in simplifications, and the uncertainty results are expressed in percentages as shown in Figure 4.

Example Problem -3

The changes in air velocity in an exhaust duct of a manufacturing plant is being monitored. To specify what magnitude of changes in the measured velocity should be viewed as a drift in the operating point of the system, one must know the range of velocity measurements expected if the system is operating at a "steady" set point. Analyze the system and estimate the expected range of velocity readings at a point in the duct exit plane using the measurement system described below.

The measurement system consists of a Pitot tube mounted in the exit plane of the duct, an analog differential pressure gauge with a range of O to 2 in. of water and a least scale division of 0.05 in. of water, a thermometer with a least scale division of 1 "C, and a barometer with a least scale division of 1 mm Hg. The normal operating temperature and pressure are given as 300 K and 760 mm Hg, respectively. The nominal dynamic pressure measured by the differential pressure gauge is 1.5 in. H_2O.

The Pitot tube senses both stagnation pressure and static pressure, and these are fed into the differential pressure gauge. The gauge then indicates the dynamic pressure, Ap, which is the difference between stagnation and static pressures.

The air velocity is given by

$$V = (\frac{2\Delta p}{\rho})^{1/2} \qquad (19)$$

where the air density ρ is calculated from the ideal gas relation

$$\rho = \frac{p}{R T} \qquad (20)$$

Substituting (20) into (19), gives the data reduction equation

$$V = (\frac{2R \cdot T \Delta p}{\rho})^{1/2} \qquad (21)$$

This data reduction equation is used for the detailed uncertainty analysis. The estimates for error sources are required *a priori*. The elemental uncertainties are presented and used in the detailed uncertainty analysis in Figure 4.

SUMMARY AND CONCLUSIONS

The uncertainty analysis calculations for the example problems presented in this article show that regardless of the complexity of the data reduction equation, Mathcad® software maybe used quite effectively as an instructional tool. The convenience in taking derivatives of the data reduction equation, the flexibility in allowing graphical presentation of results, the capability in checking the accuracy of units for various measured variables, and the convenience of repeating calculations are the useful features of the Mathcad® software for uncertainty analysis. These features may enhance students interest in laboratory experimentation and facilitate learning. The students are expected to benefit from reduced analysis time by devoting more time to understand the importance of uncertainty analysis in all phases of an experimental program.

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1. Define nominal values and units of variables:

$mol = 22.4 \cdot liter$ $kjoule = 1000 \cdot joule$ $T = 298 \cdot K$

$kmol := 1000 \cdot mol$ $p = 100 \cdot atm$ $M = 28.97 \cdot \dfrac{kg}{kmol}$

$RR = 8.314 \cdot \dfrac{kjoule}{kmol \cdot K}$ Universal Gas Constant

$R := \dfrac{RR}{M}$ $R = 286.987 \cdot m^2 \text{ "see}^{-2} \bullet K"]$

2. Define uncertainty values and units:

$U_T = 1 \cdot K$ $Up = 1.0 \cdot atm$

3. Perform general uncertainty analysis:

  a. State the data reduction equation in original form, and in functional form in terms of each measured variables. In this case, in terms of P and T, as shown below, for differentiation.

$$\rho := \dfrac{p}{R \cdot T} \qquad \rho p(p) = \dfrac{p}{R \cdot T} \qquad \rho T(T) = \dfrac{p}{R \cdot T}$$

  b. Instruct Mathcad to take derivatives

$$f\rho p(p) = \dfrac{d}{dp}\rho p(p) \qquad f\rho T(T) = \dfrac{d}{dT}\rho T(T)$$

  c. Substitute the derivatives, i.e., sensitivity coefficients, and uncertainty estimates into the general uncertainty analysis expression and divide each term by the result

$$Uncertainty\_of\_p\_over\_p = \left[\left(\dfrac{f\rho p(p)}{\rho p(p)} \cdot U_p \right)^2 + \left(\dfrac{f\rho T(T)}{\rho T(T)} \cdot U_T \right)^2 \right]^{\frac{1}{2}} \cdot 100$$

$Uncertainty\_of\_\rho\_over\_\rho = 1.055$ The uncertainty in determination of density is 1. 1%

4. Conclusion:

  The above uncertainty analysis shows that $U\rho/\rho = 1.1\%$ due to the uncertainties in p and T.

  It will be easy in Mathcad to list the contribution of each term within the general uncertainty analysis expression as shown below.

$$Pressure = \left(\dfrac{f\rho p(p)}{\rho p(p)} \cdot U_p \right)^2 \qquad\qquad Temperature = \left(\dfrac{f\rho T(T)}{\rho T(T)} \cdot U_T \right)^2$$

$Pressure = 1 \bullet 10\text{-}4$ $Temperature = 1. 126 \bullet 10\text{-"}5$

  By listing the contribution of each term squared, it is clear that the contribution of the pressure to the overall uncertainty of the result is higher than temperature.

Figure 1. EXAMPLE-1 (Mathcad® Solution).

1. Define nominal values and units of variables:

$$W := \begin{vmatrix} 2.4 \cdot \text{watt} \\ 4.9 \cdot \text{watt} \\ 12 \cdot \text{watt} \\ 24 \cdot \text{watt} \\ 49 \cdot \text{watt} \\ 121 \cdot \text{watt} \\ 243 \cdot \text{watt} \end{vmatrix}$$

$L := 0.152 \cdot m$

$D := 0.0254. \, m$

$AT = 20 \cdot K$

$i := 0 .. 6$

i represents the number of nominal values for power W.

The power matrix corresponds to the range of potential h values upto 1000 W/m²‘C

2. Define uncertainty values and units:

$U_{\Delta T} = 0.5 \cdot K$ $U_D = 0.025 \cdot mm$ $U_L \, 0.25 \cdot mm$

$U_W \cdot \begin{pmatrix} 0.5 \cdot \text{watt} \\ 1.0 \cdot \text{watt} \end{pmatrix}$ $j = 0 .. 1$ **j represents the number of uncertainty limits to be used for power W.**

3. General uncertainty analysis according to the data reduction equation:

a. State the data reduction equation in original form, and in functional form in terms of each measured variables.

$h \quad \dfrac{\text{----}}{\pi \cdot D \cdot L \cdot \Delta T} \qquad hAT(AT) \quad \dfrac{W_i}{\pi \cdot D \cdot L \cdot \Delta T} \qquad hD(D) \quad \dfrac{W_i}{\pi \cdot D \cdot L \cdot \Delta T} \qquad hL(L) \quad \dfrac{W_i}{\pi \cdot D \cdot L \cdot \Delta T} \qquad hW(W) \quad \dfrac{w}{\pi \cdot D \cdot L \cdot \Delta T}$

b. Instruct Mathcad to take **derivatives**

$fhW(W) = \dfrac{d}{dW} hW(W)$ $fhD(D) = \dfrac{d}{dD} hD(D)$ $fh\Delta T(\Delta T) \dfrac{d}{d\Delta T} h\Delta T(\Delta T)$ $fhL(L) := \dfrac{d}{dL} hL(L)$

c. Substitute the derivatives, i.e., sensitivity coefficients, and uncertainty estimates into the general uncertainty analysis expression

$$U_{h_{i,j}} := \left[\left(\dfrac{fhW(W_i)}{hW(W_i)} \cdot U_{W_j} \right)^2 + \left(\dfrac{fhD(D)}{hD(D)} \cdot U_D \right)^2 + \left(\dfrac{fhL(L)}{hL(L)} \cdot U_L \right)^2 + \left(\dfrac{fh\Delta T(\Delta T)}{h\Delta T(\Delta T)} \cdot U_{\Delta T} \right)^2 \right]^{\frac{1}{2}} \cdot 100$$

The results of U_h /h for $U_{\Delta T}$ = 0.50 °C are presented below. U_h vectors show the U_h\h in % for 0.5 W and 1,0 W uncertainty estimates in power measurements. The hW(W_i) vector shows the corresponding convective coefficients (h).

	5·W	1.0·W		hW(W_i)
	21	41.7		$9.9 \cdot kg \cdot sec^{3} \cdot K$ ‘
	10.5	20.6		$20.2.kg.see'' ^{3} \cdot K^{-1}$
	4.9	8.7		$49.5 \cdot kg \cdot sec^{-3} \cdot K^{-1}$
$U_h =$	3.3	4.9		$98.9 \cdot kg \cdot sec^{-3} \cdot K^{-1}$
	2.7	3.2		$202 \cdot kg \cdot sec^{3} \cdot K^{-1}$
	2.5	2.6		$498.8 \cdot kg \cdot sec^{-3} \cdot K$
	2.5	2.5		$1. 10^{3} \cdot kg \cdot sec^{3} \cdot K^{-}$

Figure 2. EXAMPLE-2 (Mathcad® Solution).

The figure below is a plot of U_h/h versus h values for both 0.5 W and 1.0 W uncertainties estimates in power measurement for $U_{\Delta T}$ = 0.5 K.
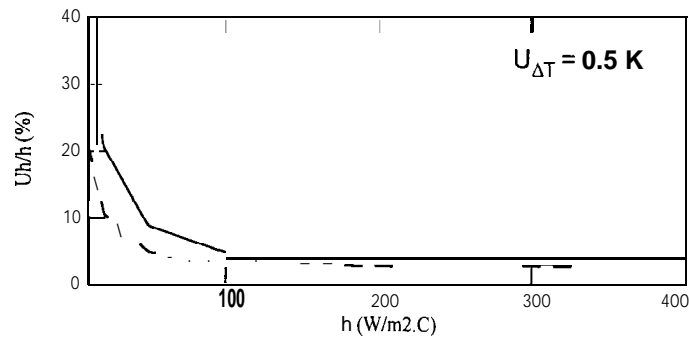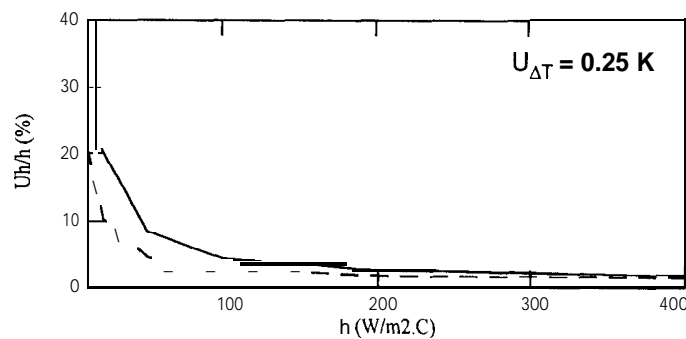
And the results of uncertainty analysis for $U_{\Delta T}$ = 0.25 °C are:

$$\mathbf{U}_{\Delta T} := 0.25 \cdot K$$

$$U_{h_{i,j}} := \left[\left(\frac{fhW(W_i)}{hW(W_i)} \cdot U_{W_j} \right)^2 + \left(\frac{fhD(D)}{hD(D)} \cdot U_D \right)^2 + \left(\frac{fhL(L)}{hL(L)} \cdot U_L \right)^2 + \left(\frac{fh\Delta T(\Delta T)}{h\Delta T(\Delta T)} \cdot U_{\Delta T} \right)^2 \right]^{\frac{1}{2}} \cdot 100$$

$$0.5 \cdot W \quad 1.0 \cdot W \qquad hW(W_i)$$

$$U_h = \begin{vmatrix} 20.9 & 41.7 \\ 10.3 & 20.4 \\ 4.4 & 8.4 \\ 2.4 & 4.4 \\ 1.6 & 2.4 \\ 1.3 & 1.5 \\ 1.3 & 1.3 \end{vmatrix}$$

$9.9 \cdot kg \cdot sec^{-3} \cdot K^{-1}$
$20.2 \cdot kkg \cdot sec^{-3} \cdot K^{-1}$
$49.5 \cdot kkg \cdot sec^{-3} \cdot K^{-1}$
$98.9 \cdot kkg \cdot sec^{-3} \cdot K^{-1}$
$202 \cdot kkg \cdot sec^{-3} \cdot K^{-1}$
$498.8 \cdot kg \cdot sec^{-3} \cdot K^{-1}$
$1 \cdot 10^3 \cdot kg \cdot sec^{-3} \cdot K^{-1}$

4. Conclusion:

The uncertainties of 0.5 watt and 1.0 watt in the power measurement are adequate at higher values of h (>100 $W/m^2.k$), but do not satisfy the requirements at low and moderate h values. The uncertainty in the power measurement is the dominant uncertainty (for the assumption made in this analysis) in the low and moderate h range, and would have to be reduced below 0.5 watt for the requirement of about 5% uncertainty in h to be met.

Figure 2. EXAMPLE-2 (Mathcad® Solution Continued).

1. Define nominal values and units of variables:

$$M := 0.02 \cdot kg \qquad c := 903 \frac{joule}{kg \cdot K} \qquad D \quad 25.4 \cdot mm \qquad L \quad 152 \cdot mm$$

$$\tau := \begin{vmatrix} 149.0 \cdot & sec \\ 74.4 \cdot sec \\ 29.8 \cdot sec \\ 14.9 \cdot sec \\ 7.4 \cdot sec \\ 3.0 \cdot sec \\ 1.5 \cdot sec \end{vmatrix} \qquad i \quad 0..6$$

The time constant matrix corresponds to the range of potential h values up to 1000 W/m^2. °C.

2. Define uncertainty values and units of variables:

$$U_M := 0.0001 \cdot kg \qquad U_c = c \cdot \frac{1}{100} \qquad u_D = 0.025 \cdot mm \qquad U_L = 0.25 \cdot mm$$

$$U_\tau - \begin{pmatrix} 0.05 \cdot sec \\ 0.10 \cdot sec \\ 0.20 \cdot sec \end{pmatrix} \qquad j \quad 0..2$$

3. General uncertainty analysis based on the data reduction equation:

   a. State the data reduction equation in original form, and in functional form in terms of each measured variables,

$$h_i := \frac{M \cdot c}{\pi \cdot D \cdot L \cdot \tau_i} \qquad hM(M) = \frac{M \cdot c}{\pi \cdot D \cdot L \cdot \tau_i} \qquad hc(c) = \frac{M \cdot c}{\pi \cdot D \cdot L \cdot \tau_i}$$

$$hD(D) := \frac{M \cdot c}{\pi \cdot D \cdot L \cdot \tau_i} \qquad hL(L) = \frac{M \cdot c}{\pi \cdot D \cdot L \cdot \tau_i} \qquad h\tau(\tau) := \frac{M \cdot c}{\pi \cdot D \cdot L \cdot \tau}$$

   b. Instruct Mathcad to take derivatives

$$fhM(M) := \frac{d}{dM} hM(M) \qquad fhc(c) = \frac{d}{dc} hc(c) \qquad fh\tau(\tau) = \frac{d}{d\tau} h\tau(\tau)$$

$$fhD(D) := \frac{d}{dD} hD(D) \qquad fhL(L) = \frac{d}{dL} hL(L)$$

   c. Substitute **the derivatives, i.e., sensitivity coefficients, and uncertainty estimates into the general uncertainty analysis expression**

$$U_{h_{i,j}} := \left[\left(\frac{fhM(M)}{hM(M)} \cdot U_M\right)^2 + \left(\frac{fhc(c)}{hc(c)} \cdot U_c\right)^2 + \left(\frac{fhD(D)}{hD(D)} \cdot U_D\right)^2 + \left(\frac{fhL(L)}{hL(L)} \cdot U_L\right)^2 + \left(\frac{fh\tau(\tau_i)}{h\tau(\tau_i)} \cdot U_{\tau_j}\right)^2 \right]^{\frac{1}{2}} \cdot 100$$
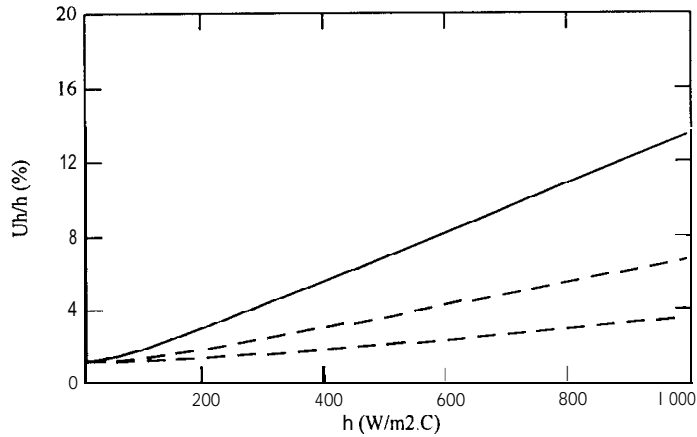
Figure 3. EXAMPLE-2 (Mathcad® Solution).

$$0.05 \cdot sec \quad 0.1 \cdot sec \quad 0.2 \cdot \ sec \qquad\qquad h\tau\left(\tau_i\right)$$

$$U_h = \begin{vmatrix} 1.1 & 1.1 & 1.1 \\ 1.1 & 1.1 & 1.2 \\ 1.1 & 1.2 & 1.3 \\ 1.2 & 1.3 & 1.8 \\ 1.3 & 1.8 & 2.9 \\ 2 & 3.5 & 6.8 \\ 3.5 & 6.8 & 13.4 \end{vmatrix}$$

$9.993 \cdot kg \cdot sec^{-3} \cdot K^{-1}$
$20.013 \cdot kg \cdot sec^{-3} \cdot K^{-1}$
$49.966 \cdot kg \cdot sec^{-3} \cdot K^{-1}$
$99.932 \cdot kg \cdot sec^{-3} \cdot K^{-1}$
$201.214 \cdot kg \cdot sec^{-3} \cdot K^{-1}$
$496.329 \cdot kg \cdot sec^{-3} \cdot K^{-1}$
$992.657 \cdot kg \cdot sec^{-3} \cdot K^{-1}$

4. **Conclusion:**
   For the conditions and estimates used in this analysis, the transient technique will meet the requirements if the time constant can be measured with an acceptable uncertainty.

Figure 3. EXAMPLE-2 (Mathcad® Solution Continued).

1. Define nominal values and units of variables:

$mol := 22.4 \cdot liter$ 　　　$M , 28.97 \cdot \dfrac{g}{mol}$ 　　　　$In\_H20 \ -112.62.43. \ 144 \cdot psi$

$RR = 8.314 \cdot \dfrac{joule}{mol \cdot K}$ 　　Universal Gas Constant 　　　$mm\_Hg \ \dfrac{1}{25.4} \cdot in\_Hg$

$Ap \quad 1.5 \cdot In\_H2o$

$R := \dfrac{RR}{M}$ 　　　　$R = 0.029 \cdot kg \cdot m \cdot K^{1}$ 　　　$P \quad 760 \cdot mm\_Hg$

$T = 300 \cdot K$

2. Define uncertainty values and units for elemental bias sources:

　　　Differential pressure gauge:

　　　$B\_\Delta p\_o = 0.05 \cdot In\_H2o$ 　　　　　zeroth order bias limit

　　　Pitot tube:

　　　$B\_\Delta p\_I = 0.015 . In\_H2o$ 　　　　　bias limit for installation

　　　Barometer:

　　　$B\_p\_o \ 2. \ 0 \cdot mm\_Hg$ 　　　　　zeroth order bias limit

　　　Thermometer:

　　　$B\_T\_o := 2.0 \cdot K$ 　　　　　zeroth order bias limit

3. Perform detailed uncertainty analysis:

　a. Determine bias limits for measurements of individual variables

　　$B\_\Delta p = \left[\left(B\_\Delta p\_o\right)^2 + \left(B\_\Delta p\_I\right)^2 \right]^{2}$

　　$B\_Ap = 0.052 \cdot In\_H2o$ 　　　　Bias limit for Ap

　b. Estimate precision limits for measurements of individual variables using previous experiences
　　with the system:

　　$P\_Ap := 0.061 . In\_H2o$ 　　　　Precision limit for Ap

　　$P\_p\_o = 0.25 . mm\_Hg$ 　　　　zeroth order precision limit

　　$P\_T\_o := 0.5 \cdot K$ 　　　　zeroth order precision limit

　c. State the data reduction equation in original form, and in functional form in terms of each
　　measured variables.

　　$" \ \cdot \left(\dfrac{2 \cdot \cdot R \cdot T \cdot \Delta p}{P}\right)^{2}$ 　　　　　$VP(P) \ \left(\dfrac{2 \cdot R \cdot T \cdot \Delta p}{P}\right)^{2}$

Figure 4. EXAMPLE-3 (Mathcad® Solution).

$$VT(T) := \left(\frac{2 \cdot R \cdot T \cdot \Delta p}{P}\right)^{\frac{1}{2}} \qquad\qquad V\Delta p(\Delta p) := \left(\frac{2 \cdot R \cdot T \cdot \Delta p}{P}\right)^{\frac{1}{2}}$$

d. Instruct Mathcad to take derivatives

$$fVP(P) := \frac{d}{dP} VP(P) \qquad\qquad fV\Delta p\ (\Delta p)\ \frac{d}{d\Delta p} V\Delta p\ (\Delta p)$$

$$fVT(T)\ \ \frac{d}{dT} VT(T)$$

e. Substitute the derivatives, i.e., sensitivity coefficients, and bias estimates into the uncertainty analysis expression and divide by the result

$$Bias\_V\_over\_V := \left[\left(\frac{fVP(P)}{VP(P)} \cdot B\_p\_o\right)^2 + \left(\frac{fVT(T)}{VT(T)} \cdot B\_T\_o\right)^2 + \left(\frac{fV\Delta p(\Delta p)}{V\Delta p(\Delta p)} \cdot B\_\Delta p\right)^2\right]^{\frac{1}{2}} \cdot 100$$

$Bias\_V\_over\_V = 1.777$ \qquad Thus, the bias limit for the velocity is 1.8%

f. Substitute the derivatives, i.e., sensitivity coefficients, and precision estimates into the uncertainty analysis expression and divide by the result

$$Precision\_V\_over\_V = \left[\left(\frac{fVP(P)}{VP(P)} \cdot P\_p\_o\right)^2 + \left(\frac{fVT(T)}{VT(T)} \cdot P\_T\_o\right)^2 + \left(\frac{fV\Delta p(\Delta p)}{V\Delta p(\Delta p)} \cdot P\_\Delta p\right)^2\right]^{\frac{1}{2}} \cdot 100$$

$Precision\_V\_over\_V = 2.035$ \qquad Thus, the precision limit for the velocity is 2%

g. Calculate the uncertainty in the result using the bias and precision limits

$$U\_V\_over\_V = \left[(Bias\_V\_over\_V)^2 + (Precision\_V\_over\_V)^2\right]^{\frac{1}{2}} \qquad U\_V\_over\_V = 2.701$$

4. Conclusion:
   The above detailed uncertainty analysis shows that Uv/v = 2.7%

Figure 4. EXAMPLE-3 (Mathcad® Solution Continued).