Application of Micro Computer in Optimal Linearization of Nonlinear Systems

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Abstract

This paper presents a computer-assisted method to generate accurate linear models of nonlinear systems with reduced biasing errors. The technique, which is based on finite difference methods, approximates partial derivatives of a Taylor series expansion of nonlinear state equations about a nominal operating point or trajectories. The matrices of the linear state-space representation of the nonlinear system can be determined using personal-computer software. It can be shown that positive and negative perturbations in the system inputs can result in more accurate linear model. The advantages of this approach are illustrated and discussed. The proposed techniques will be useful in motivating students to pursue a graduate degree in institutions where the limited budget will not allow purchasing costly modeling/simulation packages and software.

Introduction

Most standard control design techniques have been developed for linear systems. More importantly, any derived technique for a nonlinear system may not be applicable to other nonlinear systems due to their complicated dynamics. Thus, a good linear representation of these physical systems must be derived. In practice, it is found that some behavior of nonlinear systems only occurs if they are driven into certain operation regions. For these systems, the linear model may give relatively accurate results over a wide range of operating conditions. However, there are numerous physical systems which have strong nonlinear characteristic. For these systems, a linearized model is valid only for a limited range of operation, and often only at the operating point at which the linearization is done. A particular method used to linearize a nonlinear system about an operating point is the “offset derivative”. This technique makes use of the method of finite difference to approximate partial derivatives of a Taylor series expansion of the nonlinear state equations about a nominal operating point or trajectories. The difficulty encountered in using the method of finite difference is that small numerical errors can appear in partial-derivative approximations. These errors can cause large enough errors in the Jacobian matrices’ elements that the eigenvalues of the system can be changed. Since, the trajectories have different slopes in different directions about the operating points; these errors can be reduced or eliminated through positive and negative perturbation about the operating point. Once the full state linear model of the system has been determined, it may be possible to reduce the order of the linear models while still retaining the important dynamic characteristics of the system. This is desirable for two reasons. First, a lower order system is easier to handle mathematically; second, controls developed using optimal control techniques require that all specified states of the system be used to derive the optimal control. However, some of these states are not measurable and therefore cannot be used for control purposes. Furthermore, if all the states of a large-order system were measurable, the control derived would be too complex to implement. Once the linear models are generated, they should be evaluated as to how well the approximate the dynamics of the nonlinear system about the operating points for which they were generated. As an example this evaluation is made for the third order
Bernnan and Leake Jet Model by comparing the full-state linear model and nonlinear engine model.

**Linearization Mathematics**

Represent a nonlinear system by the following vector-matrix state equations:

\[
\frac{dX(t)}{dt} = X'(t) \equiv f[X(t), U(t), t] \quad (1)
\]

\[
Y(t) = g[X(t), U(t), t]
\]

Where \(X(t)\) represents the \(nx1\) state vector, \(U(t)\) the \(px1\) input vector, \(f[X(t), U(t), t]\) denotes an \(nx1\) function, \(g[X(t), U(t), t]\) represents \(1xk\) function, and \(Y(t)\) is \(1xK\) vector. In general \(f\) and \(g\) are a function of state vector and the input vector. The state variables \(X(t)\) cannot change instantaneously with time. While output variables \(Y(t)\) may. It should be clear that no single linear model can accurately represent the system because of generally wide operating range and nonlinear characteristics. Thus, linear model must be derived at various conditions particularly, if the Jacobian, \(J\), is experiencing a large variation in time and space throughout the operating envelope.

Let the nominal operating trajectory and the output be \(X_0, Y_0\) respectively, which correspond to the nominal input \(U_0\) and some fixed initial conditions. Expanding the nonlinear state equation (1) into a Taylor series about \(X(t) = X_0\) yields

\[
X'_i(t) = f_i(X_0, U_0, t) + \sum_{j=1}^{n} \frac{\partial f_i(X, U, t)}{\partial X_j} \bigg|_{X_0, U_0} (X_j - X_{0j}) + \sum_{j=1}^{p} \frac{\partial f_i(X, U, t)}{\partial U_j} \bigg|_{X_0, U_0} (U_j - U_{0j}) + \text{HOT} \quad (2)
\]

Let

\[
\Delta X_i = X_i - X_{0i} \quad i = 1, 2, \ldots, n
\]

\[
\Delta U_j = U_j - U_{0j} \quad j = 1, 2, \ldots, p
\]

Then since

\[
\Delta X'_i = X'_i - X'_{0i}
\]

\[
X'_{0i} = f_i(X_0, U_0)
\]

If all the higher order terms (HOT) are negligible then, equation (2) can be written as

\[
\Delta X_i(t) = \sum_{j=1}^{n} \frac{\partial f_i(X, U)}{\partial X_j} \bigg|_{X_0, U_0} \Delta X_j + \sum_{j=1}^{p} \frac{\partial f_i(X, U)}{\partial U_j} \bigg|_{X_0, U_0} \Delta U_j
\]
At an operating point the system is assumed to be time invariant; thus, the above equation may be written in vector-matrix form

\[ \Delta \mathbf{X}' = J \Delta \mathbf{X} + G \Delta \mathbf{U} \]  \hspace{1cm} (3)

The J and G matrices are Jacobian matrices defined by

\[
J = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix} \quad G = \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_p} \\
\frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_p} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_p}
\end{bmatrix}
\]

Or in more compact form:

\[ J_{ij} = \frac{\partial f_i}{\partial x_j} \quad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, p \] \hspace{1cm} (4)

\[ G_{ij} = \frac{\partial f_i}{\partial u_j} \quad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, p \]

Thus, the nonlinear system has been linearized at a nominal operating point. The output equation of (1) may be linearized using the preceding technique by assuming small perturbations about the operating point, i.e.

\[ \Delta Y_i = Y_i - Y_{oi} \]

With \( i = 1, 2, \ldots, k \)

The output Taylor series expansion is:

\[ Y_i(t) = g_i(X_0, U_0, t) + \sum_{j=1}^{k} \frac{\partial g_i(X_0, U_0, t)}{\partial X_j} \bigg|_{x_0,u_0} (X_j - X_{oj}) + \sum_{j=1}^{p} \frac{\partial g_i(X_0, U_0, t)}{\partial U_j} \bigg|_{x_0,u_0} (U_j - U_{oj}) + \text{HOT} \] \hspace{1cm} (5)

Since \( \Delta Y_{oi} = g_i(X_0, U_0) \)
Now, if all the higher order terms (HOT) are small compared with the order terms then, equation (5) can be written as

\[
\Delta Y_i = \sum_{j=1}^{k} \left. \frac{\partial g_l(X, U, t)}{\partial X_j} \right|_{x_0, u_0} \Delta X_j + \sum_{j=1}^{p} \left. \frac{\partial g_l(X, U)}{\partial U_j} \right|_{x_0, u_0} \Delta U_j
\]

At an operating point the system is assumed to be time invariant: thus, the above equation may be written in vector-matrix form.

\[
\Delta Y = C \Delta X + D \Delta U
\]  

The \(C\) and \(D\) matrices are input Jacobian matrices defined by

\[
C = \begin{bmatrix}
\frac{\partial g_1}{\partial X_1} & \frac{\partial g_1}{\partial X_2} & \ldots & \frac{\partial g_1}{\partial X_n} \\
\frac{\partial g_2}{\partial X_1} & \frac{\partial g_2}{\partial X_2} & \ldots & \frac{\partial g_2}{\partial X_n} \\
\ldots & \ldots & \ldots & \ldots \\
\frac{\partial g_k}{\partial X_1} & \frac{\partial g_k}{\partial X_2} & \ldots & \frac{\partial g_k}{\partial X_n}
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
\frac{\partial g_1}{\partial U_1} & \frac{\partial g_1}{\partial U_2} & \ldots & \frac{\partial g_1}{\partial U_n} \\
\frac{\partial g_2}{\partial U_1} & \frac{\partial g_2}{\partial U_2} & \ldots & \frac{\partial g_2}{\partial U_n} \\
\ldots & \ldots & \ldots & \ldots \\
\frac{\partial g_k}{\partial U_1} & \frac{\partial g_k}{\partial U_2} & \ldots & \frac{\partial g_k}{\partial U_n}
\end{bmatrix}
\]

Or in more compact form

\[
C_{ij} = \frac{\partial g_i}{\partial X_j} \quad i = 1, 2, \ldots, k \quad j = 1, 2, \ldots, n
\]

\[
D_{ij} = \frac{\partial g_i}{\partial U_p} \quad i = 1, 2, \ldots, k \quad j = 1, 2, \ldots, n
\]

The approximating of matrix elements of the linearized model, equations (4) and (7) is accomplished by a finite difference method using \textbf{PC-MATLAB}. The program only requires the desired operating point, inputs, perturbation size, and the nonlinear model state equations. Then, the individual states are varied one at a time while the others are held constant. This is done to derive the \(J\) and \(C\) matrices. To generate \(G\) and \(D\) the inputs are varied one at a time while holding the others constant including the states. The biasing errors in the finite-difference calculations of \(J\) may be reduced by perturbing the state in the positive direction from the operating point to generate \(J^+\) and \(C^+\). The, the states are perturbing in the negative direction from the operating point to derive \(J^-\) and \(C^-\) matrices. Finally, the \(J^+\) and \(J^-\) matrices and \(C^+\) and \(C^-\) matrices are averaged to obtained the \(J\) and \(C\) matrices. The perturbation size must be within the linear region of the operating point, and it must be larger than the iteration tolerance for implicit calculations.
The larger perturbation size ensures easier calculations of the partial derivative as long as it does not exceed the linearity limit about the operating point.

Another important potential source of error due the linearization process is inability of the linear model to converge to the expected steady-state value. Thus, some modification of the $G$ matrix is necessary to eliminate this steady-state error. Note that at steady-state $\Delta X' = 0$ which results in

$$J \Delta X + G \Delta U = 0$$

then

$$\Delta X = -J^{-1}G \Delta U = 0$$

(8)

$\Delta X$ is calculated by making small perturbations in the inputs. Then, the term $(-J^{-1}G)$ can be evaluated by $\Delta X/\Delta U$, since $J$ is known, the modified $G_m$ is

$$G_m = -J(-J^{-1}G)$$

(9)

For the matrix $D$, inserting equation (8) in equation (6) results in

$$\Delta Y - C(-J^{-1}G \Delta U) = D_m \Delta U$$

(10)

Where the modified $D_m$ is calculated by making small changes in inputs while holding all the states constant such that equation (10) is satisfied. Once the matrices are properly determined, the linearity of these matrices must first be confirmed. This is normally done by generating a $L_1$ matrix with one perturbation size and then a $L_2$ matrix with different perturbation size. If the eigenvalues of both $L_1$ and $L_2$ are within the frequency range of interest, then both $L_1$ and $L_2$ are reasonably valid. This indicates that both perturbation sizes are within the linear region about the operating point. Second, the linear models should be evaluated to see how well they approximate the dynamics of the nonlinear system about the operating point from which they are generated. This is accomplished by comparing the simulation results of both linear and nonlinear models using a very accurate numerical integration algorithm. For a given input the linear model should maintain the dynamics of the nonlinear model, and the difference between the steady-state errors of both model responses should be as small as possible. This was done for a nonlinear system given in the next section.

**Nonlinear Jet Engine Model**

As an example of the linear-model generation procedure, the nonlinear third-order Bernman and Leaks engine model described by the following nonlinear differential equation² is considered.

$$T_3 = 0.64212 + 0.35788 N^2$$

$$W_3' = 1.3009N - 0.13982 \left[ P_4^2 - \sqrt{(P_4 + 0.41688N - 0.0899PN)} \right]$$
\[ P'_4 = W'_f \left( 0.93586P_4/P_\rho + 31.486 \right) + 21.435W'_3T_3 - 53.864P'^2_4/P_\rho \]

\[ P'_\rho = 37.78W'_3 - 38.448W'_3 + 0.66849W'_f \]

\[ N' = (1.258/N) \left( \frac{P^2_4}{P_\rho} - W'_3N^2 \right) \quad (11) \]

Where \( P_4 \) is Combustor Pressure, \( P_\rho \) is Combustor density, \( N \) is Rotor Speed, \( W'_3 \) is Compressor Discharge Mass Flow, \( T_3 \) is Compressor discharge Temperature, \( W'_f \) is fuel input rate. The system is normalized about \( W'_f = 1.0 \) with initial conditions:

\[ P_4 = 0.5831, \quad P_\rho = 1.77504, \quad N = 0.54589 \]

States:

\[ X_1 = P_4, \quad X_2 = P_\rho, \quad X_3 = N \]

And input:

\[ U = W'_f \]

The resulting linear model from the \( \pm 0.3 \) percent perturbation with \( X_0 = [1 \ 1 \ 1]^T \) as the operating point is:

\[
\begin{bmatrix}
    X'_1 \\
    X'_2 \\
    X'_3
\end{bmatrix} = 
\begin{bmatrix}
    -112.266 & 52.925 & 42.259 \\
    -48.1501 & 0 & 47.443 \\
    2.8377 & -1.258 & -4.096
\end{bmatrix}
\begin{bmatrix}
    X_1 \\
    X_2 \\
    X_3
\end{bmatrix} + 
\begin{bmatrix}
    32.422 \\
    0.668 \\
    0
\end{bmatrix}U
\]

(12)

Therefore:

\[ J = \begin{bmatrix}
    -112.266 & 52.925 & 42.259 \\
    -48.1501 & 0 & 47.443 \\
    2.8377 & -1.258 & -4.096
\end{bmatrix}, \quad \text{and} \quad \mathbf{G} = \begin{bmatrix}
    32.422 \\
    0.668 \\
    0
\end{bmatrix} \]

The eigenvalues resulting from using different perturbation sizes is tabulated in table (1). Also, figure (1) shows the simulation of this linear model compared to the actual nonlinear simulation. Adams-Bashforth two step (AB-2) integration method \( ^4 \) \( (x_{n+1} = x_n + 0.5 * h[3x'_n - x'_{n-1}]) \) with stepsize of \( h=0.002 \) sec is used to simulate the linear model and nonlinear model. Also,
### Table (1) Eigenvalues variation of linearized Brennan and Leake engine

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Perturbation</th>
<th>0.3% of operating point</th>
<th>0.3% of operating point</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>± 0.3</td>
<td>+0.3</td>
<td>-0.3</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>-81.24</td>
<td>-82.03</td>
<td>-80.34</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>-32.33</td>
<td>-31.19</td>
<td>-33.51</td>
</tr>
</tbody>
</table>

**Figure (1)** Simulation of linear and nonlinear engine model using AB-2 with h=0.002 sec.

**Conclusion**

It is critical that computer usage be integrated into problems which involve the application of basic concepts in engineering. A method to derive and validate linear models from a nonlinear digital simulation discussed in this paper can be viewed as an appropriate use of the computer. To derive a good linear model at an operating point, accurate partial derivatives must be obtained. When using the finite-difference method to approximate partial derivatives, small errors in the calculations can occur. These errors are bias errors, which must be minimized or eliminated if possible. This can be accomplished by perturbing the state in both positive and negative directions about an operating point and averaging the resulting partials. For simulations in which differential
equations are solved explicitly, this method results in repeatability of system eigenvalues for
different perturbation sizes about the operating point. For simulations in which the differential
equations are solved implicitly, exact repeatability of system eigenvalues for different perturbation
sizes is difficult to obtain because of the iterative nature of the solutions. Thus, for a given
perturbation size the linear model predicts the average steady-state value that the nonlinear
simulation would give for the same perturbation size in the plus and minus direction about the
operating point.

The accuracy of the matrices was checked by showing that linearity holds for different
perturbation sizes. Once the accuracy of the matrices was assured, the ability of the linear models
to approximate the nonlinear system at an operating point was investigated. This was done by
comparing a similar transient run with the nonlinear and linear simulations.

Finally, the methods used in this paper to generate linear models from a nonlinear digital
simulation are general. The application presented is for the third order Bernnan and Leake engine
model, but the method is not restricted to that simulation. The problem of bias errors in finite-
difference approximations of partial derivative, again are not limited to Bernnan and Leake engine
model simulation. They are universal and must be dealt with whenever deriving linear models or
comparing transients run with the nonlinear and linear simulations. The technique and the
simulation algorithm is very useful in teaching undergraduate and graduate control systems
courses.

References
2. Bernnan, T. C. and Leake, Simplified Simulation Models for Control Studies of Turbojet Engines, Technical

Appendix
The following personal computer software is written in MATLAB to linearized nonlinear systems
about a given nominal operating point using finite differences method.

function [a,b] = linearize (F,x0,u0,pert)
% THIS PROGRAM LINEARIZED THE SYSTEM DESCRIBED IN F ABOUT THE POINT
% XXXXX. IT
% RETURNS THE LINEAR a AND b MATRICES SUCH AS THAT xdot = a*x + b*u IS
% LINEAR. Pert
% IS THE RECENT PERTURBATION ABOUT THE OPERATING POINT.
% TIME IS ASSUMED TO BE INVARIENT IN F. NOTE THAT THE PERTURBATION IS
% DONE IN BOTH
% POSITIVE AND NEGATIVE DIRECTION.
% F MUST BE OF THE FORM
%
% Function Xprime = Function_name (u,x)

% WITH F = ‘function-name’ IN THE CALL TO BE LINEARIZED. Function_name IS THE % NAME YOU
% ASSIGN TO THE FUNCTION.
% THE FUNCTION MUST RETURN THE DERIVATIVE AT THE POINT.
% WHERE u IS THE INPUT AND x IS THE STATE OF THE SYSTEM. SEE THE FILE % JETS3.M AS AN
% EXAMPLE (THE FUNCTION MUST BE .M FILE). TO RUN AN EXAMPLE USING % JET3.M, AT THE
% MATLAB PROMPT TYPE:
% x0=[1;1;1]
% u0= 1
% pert=0.003
% [a,b]=lineriz(‘jet3’ , x0,u0,pert)

% CALCULATE PERTURBATIONS AND DERIVATIVE AT OPERATING POINT %
%
% delx =x0. * pert;
% delx =u0. * pert;
% xd= feval (f,u0,x0);
% %
% % RETURB X IN POSITIVE DIRECTION %
% %
% for i=1 : length (x0),
% for J=1 : length (x0),
% x= x0;
% for k=1 : length (x0),
% if k == j,
% if delx (k) == 0
% dx=eps;
% else
% dx=delx(k);
% end
% x(k) = x(k) + dx;
% end
% xt = feval (f,u0,x);
% if delx (j) == 0,
dx=eps;
else
dx=delx(j);
end
xp(I,j) = (xt(i) - xd(i)) / dx;
end
end

%%
% PERTURB X IN NEGATIVE DIRECTION %
%%
%%
for i=1 : length (x0),
for J=1 : length (x0),
x= x0;
for k=1 : length (x0),
if k == j,
if delx (k) == 0
    dx=eps;
else
    dx=delx(k);
end
x(k) = x(k) + dx;
end
xt = feval (F,u0,x);
if delx (j) == 0,
dx=eps;
else
    dx=delx(j);
end
xm (i,j) = - (xt(i) - xd(i)) / dx;
end
end

%%
% PERTURB U IN POSITIVE DIRECTION %
for i=1 : length (x0),
for J=1 : length (u0),
u= u0;
for k=1 : length (u0),
if k == j,
    if delu (k) == 0
du=eps;
else
   du=delu(k);
end
u(k) = u(k) + du;
end
end

ut = feval(F,u,x0);
if delu(j) == 0,
   du=eps;
else
   du=delu(j);
end
up(i,j) = (ut(i) - xd(i)) / du;
end
end

%%
% PERTURB U IN NEGATIVE DIRECTION %
for i=1 : length(x0),
   for J=1 : length(u0),
      u= u0;
      for k=1 : length(u0),
         if k == j,
            if delu(k) == 0
               du=eps;
            else
               du=delu(k);
            end
            u(k) = u(k) + du;
         end
      end
   end
end
ut = feval(F,u,x0);
if delu(j) == 0,
   du=eps;
else
   du=delu(j);
end
um(i,j) = (ut(i) - xd(i)) / delu(j);
end
end
% CALCULATE LINEARIZED SYSTEM (JACOBIAN ) %
a= (xp+xm) / 2;
b= (up+um) / 2;
function yprime   = jet3(u,y)
% NONLINEAR 3RD ORDER NORMALIZED BERNNAN AND LEAK TURBOFAN JET ENGINE MODEL
% yprime  IS THE DERIVATIVE OF STATE VARIABLE. FUNCTIONS IN THIS FORM ARE VERY
% USEFUL IN FUNCTION EVALUATION USING MATLAB.
wf=u;
T3=0.64212 + 0.35788*y(3,:)^2;
w3 = 1.3009*y(3,: ) – 0.139825*y (1,:)w3= w3 – 0.13982*sqrt(y(1,: )^2 + 0.41688*(y(3,:)^2 -0.899*y(1,:) *y(3,:)
yp1= (0.93586* y(1,: ) /y(2,:) + 31.486) *wf+ 21.435*w3*t3 - 53.86* (y(1,:)
yp2= 37.78*w3 – 38.488*y(1,:) + 0.66849*wf;
yp3 = 1.258* (y(1,: )^2/y(2,:) ) – w3*y (3,:) )^2)/y(3,:);
yprime = [yp1;yp2;yp3];