



Applications of Quantum Entanglement in Modern Physics

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Introduction

Entanglement is a fundamental property of quantum mechanics. Certain physical properties of electrons can exist in entangled states. It is also possible for the polarization of photons to become entangled. In some cases, even macroscopic objects like mechanical resonators can exhibit entanglement. Entangled particles show certain correlations in specific measured properties. Entanglement has caused consternation since quantum theory was developed nearly a hundred years ago. The EPR “paradox” is a long standing argument that either quantum mechanics invokes “spooky actions at a distance” or there are parameters that are hidden so that the current theory is incomplete. We use an example of entangled states in a thought experiment developed by Mermin to introduce quantum mechanics to engineering students.[1,2] We also use it as a gateway to motivate the introduction of Dirac notation into the engineering curriculum.[3,4]

In this paper we provide some examples of useful applications of quantum entanglement that can be simulated in an upper level modern physics course. We describe some simple algorithms that students are guided to develop in the MATLAB environment. We also present the results of some simulations. These projects are well received by students and taking the time to introduce them does not harm their performance on a research-based assessment instrument.

Schrödinger introduced “entanglement” into the scientific vocabulary in 1935.[5] It is interesting that the phrase was essentially absent from publication until around 1990. Figure 1 shows a screen capture of the results of search of a Google database as was shown by [6]. The search utilized Google’s ngram viewer for the phrase *quantum entanglement*.[7]



Fig. 1
A screen capture of Google’s ngram viewer.

This paper is organized in the following manner. First we describe the course in which the curriculum is introduced. That is followed by a brief discussion of the student population and demographics. Next we present a very broad, low-level discussion of entanglement to provide context for the reader. Subsequently we describe how MATLAB is used to simulate quantum measurements and provide insights into the nature of an entangled state. We conclude with some quantitative data on student performance on the Quantum Mechanics Conceptual Survey (QMCS).[8] The QMCS was developed to assess student's conceptual understanding of quantum mechanics. It is interesting in this context because the authors used engineering students as part of the cohort that provided the validation data.

Modern Physics with Device Applications

The College of Engineering & Science at the University of Detroit Mercy has one of the nation's oldest co-operative education programs and a long history of successfully placing graduates into the workforce. Located in Detroit, a city with a rich tradition of engineering and manufacturing, the college has close ties to the automobile industry, defense contractors, and numerous suppliers.

The physics department at Detroit Mercy offers a 3 credit hour, junior-level course—Modern Physics with Device Applications (PHY 3690). The course is required for electrical engineering students and is offered during the winter term. In order to enroll in the course, students must successfully complete one year of a calculus-based general physics sequence of courses along with the associated laboratories. At Detroit Mercy the first physics course is mechanics and the second covers topics in electromagnetism. As juniors, students have taken a course in differential equations and linear algebra. Engineering students are exposed to the MATLAB environment during their freshman year.[9] Electrical engineering students use MATLAB in their engineering courses and we use a scaffolding approach to build on their prior knowledge and introduce new topics.

We introduce quantum mechanics by discussing Mermin's quantum device. This approach has been described previously.[10] In the winter term of 2019 students developed an additional simulation, one that reproduced the results of Mermin's quantum device.

Simulation of an Entangled State—the Measurement Problem

Suppose that we have a system consisting of two qubits one is Alice's and one is Bob's. Each qubit can be in the state $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or the state $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The basis states are formed by taking the Kronecker tensor product, \otimes , of the individual basis states. Students implement this in MATLAB with the *kron* function. For example, with 2-component column vectors x and y the tensor product is

$$|x\rangle \otimes |y\rangle = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \\ x_1 \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} x_0 y_0 \\ x_0 y_1 \\ x_1 y_0 \\ x_1 y_1 \end{pmatrix}.$$

This tensor product of the basis states makes the representation of Alice's and Bob's qubits take the form: $|0\rangle_{Alice} \otimes |0\rangle_{Bob} = |00\rangle$, $|0\rangle_{Alice} \otimes |1\rangle_{Bob} = |01\rangle$, $|1\rangle_{Alice} \otimes |0\rangle_{Bob} = |10\rangle$, and $|1\rangle_{Alice} \otimes |1\rangle_{Bob} = |11\rangle$. We suppress the subscripts and remember that the first qubit in the ket represents Alice and the second Bob. Of course it does not matter where they are located.

Students need to generate operators, i.e. matrices, to operate on the column vectors. The two operators that are necessary are the Hadamard operator and the Controlled NOT or CNOT operator. The Hadamard operator is given by $\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. It is easy to show that

$$\hat{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \text{ which is an equal superposition of } |0\rangle \text{ and } |1\rangle; \text{ and } \hat{H}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Operators that operate on specific qubits are needed. They are identified with a superscript in parentheses, the Hadamard operator acting on qubit 1 is $\hat{H}^{(1)} = \hat{H} \otimes \hat{I}$, where \hat{I} is the 2×2 identity matrix. If instead we wish to operate only on the second qubit we need $\hat{H}^{(2)} = \hat{I} \otimes \hat{H}$. The 4×4 matrix representing the Hadamard operator acting on qubit 1 is given by

$$\hat{H}^{(1)} = \hat{H} \otimes \hat{I} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & -1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

The other operator we need is the controlled NOT operator. This gate acts like a classical exclusive OR gate, XOR gate, except one qubit controls what happens to the other qubit. The notation for a CNOT gate where the first qubit is the control and the second qubit is the target is $\hat{C}_{NOT}^{(1,2)}$. The numbers in the superscript indicate the control and target qubits, respectively. As a quantum gate it is shown below in Fig. 2.

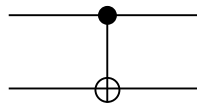


Fig. 2. A controlled NOT gate where the first or top qubit controls the second or bottom one.

If the control is $|0\rangle$ then the target qubit is unchanged, if the control is $|1\rangle$ then the target is flipped ($|0\rangle \leftrightarrow |1\rangle$). The operator does the following: $\hat{C}_{NOT}^{(1,2)} |00\rangle = |00\rangle$, $\hat{C}_{NOT}^{(1,2)} |01\rangle = |01\rangle$, $\hat{C}_{NOT}^{(1,2)} |10\rangle = |11\rangle$, and $\hat{C}_{NOT}^{(1,2)} |11\rangle = |10\rangle$. The 2-qubit representation is given by:

$$\hat{C}_{NOT}^{(1,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix operators needed to generate the entangled state are $\hat{C}_{NOT}^{(1,2)} \hat{H}^{(1)} |00\rangle$. They are shown below in Fig. 3. using the notation of quantum information theory. The meters represent a measurement of the resulting ket. The code required to generate a simulated measurement is shown later in the paper and the reader is referred to [11].

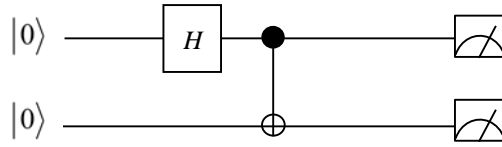


Fig. 3. A quantum circuit to simulate an entangled state.

The initial state $|00\rangle = |0\rangle_{Alice} \otimes |0\rangle_{Bob}$ is operated on by the Hadamard operator followed by the $\hat{C}_{NOT}^{(1,2)}$ gate where Alice's qubit controls Bob's. The Hadamard operator puts the first qubit into the superposition $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$. The second operation with the $\hat{C}_{NOT}^{(1,2)}$ gate results in

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, which is an entangled state. Note what happens when this state is measured.

Either the result is $|00\rangle = |0\rangle_{Alice} \otimes |0\rangle_{Bob}$ or the result is $|11\rangle = |1\rangle_{Alice} \otimes |1\rangle_{Bob}$. Whenever Alice measures $|0\rangle$, Bob measures $|0\rangle$; when Alice measures $|1\rangle$ then Bob measures $|1\rangle$. We could generate different correlations by setting the initial state to different values such as $|01\rangle = |0\rangle_{Alice} \otimes |1\rangle_{Bob}$, $|10\rangle = |1\rangle_{Alice} \otimes |0\rangle_{Bob}$, or $|11\rangle = |1\rangle_{Alice} \otimes |1\rangle_{Bob}$. This is a good exercise for students. Note the following:

$$\hat{H}^{(1)} |00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle), \text{ so that } \hat{C}_{NOT}^{(1,2)} \hat{H}^{(1)} |00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \text{ and}$$

$$\hat{H}^{(1)} |01\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle), \text{ so that } \hat{C}_{NOT}^{(1,2)} \hat{H}^{(1)} |01\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \text{ and}$$

$\hat{H}^{(1)}|10\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$, so that $\hat{C}_{NOT}^{(1,2)}\hat{H}^{(1)}|10\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, and

$\hat{H}^{(1)}|11\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$, so finally that $\hat{C}_{NOT}^{(1,2)}\hat{H}^{(1)}|11\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

MATLAB code

Below is an example of MATLAB code to perform a simulated measurement of a ket. A column vector is passed to the `ket_measure` function, the function returns an integer from 1 to 4 corresponding to the amplitudes of the ket. If the ket is $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, then the probability of returning a 1, corresponding to the state $|00\rangle$ is $a^*a = |a|^2$. The probability of returning a 2, corresponding to the state $|01\rangle$, is $b^*b = |b|^2$. The a, b, c, d could be complex numbers but here they are all real. An example of MATLAB code to generate the simulated measurement of a ket is given by:

```
function [phi_index] = ket_measure(ket)
%The function takes a complex column vector, a ket and performs a
%simulated quantum measurement.
% Input is a ket, a 4 element complex vector
% Output the index of the state
% 1 is |00>, 2 is |01>, 3 is |10>, 4 is |11>
%rng('shuffle') %initialize the random number generator
R=rand(1); % random number between 0 and 1
phi_index=0;
Temp_sum=0;
for m=1:length(ket)% loop through each element of the ket
    %increment by absolute square of amplitude
    Temp_sum=Temp_sum+ket(m)*conj(ket(m));
    if R<Temp_sum
        phi_index=m;
        break
    end
end
end
```

A sample of MATLAB code that will generate a ket that has two entangled qubits is shown below. The formation of the entangled state is achieved by performing the operations shown in Fig. 3. The code shows how to generate the Hadamard operator to operate on qubit 1 named *Hadamard1*. It goes on to show how to generate the controlled NOT gate where qubit 1 is the control and qubit 2 is the target. This operator is named *CNOT12*. The ket is initialized to $|00\rangle$

which corresponds to the column vector, $ket = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. The matrix operations can be worked out in

a step-wise fashion. First the Hadamard operator on qubit 1 gives:

$$\hat{H}^{(1)}|00\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Next the controlled NOT gate yields:

$$\hat{C}_{NOT}^{(1,2)} \hat{H}^{(1)}|00\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

In Dirac notation the result of these multiplications is $\hat{C}_{NOT}^{(1,2)} \hat{H}^{(1)}|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. This

resulting column vector is subsequently measured numerous times and the result is displayed in a tabular form. The code to generate the operators, perform simulated measurements, and display the results is shown below:

```
clear all
clc
tic; %initialize a timer
% Assign Alice qubit 1 and Bob qubit 2
ket=[1;0;0;0];%initialize ket to |00>
H=1/sqrt(2)*[1 1;1 -1]; % Hadamard matrix
I=[1 0;0 1]; % Identity Matrix
% Generate Hadamard operator to operate on qubit 1
Hadamard1=kron(H,I);%Hadamard operator on qubit 1
% Generate CNOT12 operator
CNOT12=zeros(4,4);
% these entry's make CNOT12 a controlled NOT gate with qubit 1
% as control and qubit 2 as the target. If qubit 1 is |0> it leaves
% qubit 2 alone. If qubit 1 is |1> it flips qubit 2
CNOT12(1,1)=1;
CNOT12(2,2)=1;
CNOT12(3,4)=1;
CNOT12(4,3)=1;
% The next operation operates on the initial state |00> with Hadamard1
% putting the qubit in a state that is a linear combination
% 1/sqrt(2)(|00> + |10>). The CNOT12 then flips qubit 2 if qubit 1 is
% in the state |1> and leaves qubit 2 alone of qubit 1 is |0>. The result
% is the entangled Bell state 1/sqrt(2)(|00> + |11>). This state is
% measured.
% The function ket_measure returns an integer 1 to 4 corresponding to
% the result of the measurement, which is always one of the basis states
% 1=|00>, 2=|01>, 3=|10>, 4=|11>
%measure the ket Num_max times
cindex=[0;0;0;0];
precent_state=[0;0;0;0];
Num_max=10000;
v=0;
State_Name={'|00>','|01>','|10>','|11>'};
```

```

%measure ket, Num_max times and record in cindex
%cindex is an array that holds the number of times the index occurred during
%the measurement
%percent_state is an array that holds the percentage of times the index
%occurred during the measurement
for i=1 : Num_max; %loop through the measurement Num_max times
    v=ket_measure(CNOT12*Hadamard1*ket); %measure state and return an integer
    cindex(v)=cindex(v)+1; %count the number of times a state is measured
end
for j=1:4;
    cindex(j)/Num_max;
    percent_state(j)=100*cindex(j)/Num_max;
end
t=table(State_Name,cindex,percent_state)
toc %stop the timer

```

The output of 10,000 runs of this simulation is shown in Table I.

State_Name	cindex	percent_state
$ 00\rangle$	4928	49.28
$ 01\rangle$	0	0
$ 10\rangle$	0	0
$ 11\rangle$	5072	50.72

Table I

The results of running the simulated measurement 10,000 times on the ket given by

$$\hat{C}_{NOT}^{(1,2)} \hat{H}^{(1)} |00\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle). \text{ The result is always either state } |00\rangle \text{ or state } |11\rangle.$$

Assessment of Student Learning

We have administered the Quantum Mechanics Conceptual Survey (QMCS 2.0) to students in the modern physics course for four years. The 12-question multiple choice survey of student understanding is conceptual; there is no need to memorize formulas or procedures. The multiple choice distractors are believed by the authors to be useful for checking students' preconceptions. The authors point out that most faculty believe that the survey is too easy—our experience is that students find the assessment very difficult. Faculty that teach modern physics classes have absolutely no consensus about which topics are important. The concepts that showed up the most frequently were, in decreasing order: wave function and probability, wave-particle duality, Schrödinger equation, quantization of states, uncertainty principle, superposition, operators and observables, tunneling, and measurement. In a modern physics course for engineers the need to be selective is apparent. The QMCS is given during the last week of the class and does not count negatively toward the student's final grade. Administering an assessment that does not count toward the final grade does suffer from the possibility that students will not take the test seriously and therefore do not give their best effort.

While the QMCS does not specifically address conceptual issues associated with entanglement; we felt that as more student projects on simulated quantum computation were incorporated into the curriculum, it was valuable to determine how it affects their overall performance on a valid assessment. Below in Table II we show the percentage of correct student responses to the 12 questions on the QMCS for different student cohorts. The simulated quantum computational projects were started in 2017. In 2019 an additional project on the simulation of Mermin's quantum device was added. It is apparent from the data in Table II that student performance has not been adversely affected.

	2016	2017	2018	2019
% correct	46	48	49	47
number	12	7	6	20

Table II

Table containing the percentage correct responses to survey questions on QMCS 2.0 for different PHY 3690 cohorts of students. The second row shows the number of students taking the survey.

Conclusion

Introducing quantum mechanics and Dirac notation via simulated quantum computation provides an interesting and exciting context for engineering students. Students are engaged in the projects and find quantum entanglement to be a topic of interest. We hope to further explore this area of current research by including quantum teleportation as an additional project.

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