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Are Better Teaching Methods the Answer to Improved Math Proficiency or Are We Simply Barking Up the Wrong Tree?

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Are Better Teaching Methods the Answer to Improved Math Proficiency or Are We Simply Barking Up the Wrong Tree? (Fundamental)

Abstract

The Organization for Economic Cooperation and Development (OECD) administers its Program for International Student Assessment (PISA) study once every three years to assess the scholastic performance of fifteen-year-old students in the field of mathematics, science and reading. This study is conducted among OECD member nations and select non-member nations. Results from the PISA study consistently show that, on average, the United States ranks poorly in mathematics when compared against most of the other OECD member nations. While disparity in math proficiency may exist between states and opinions may differ on the way samples were collected from each nation for the study, the generally poor proficiency in mathematics among students at the K-12 level has serious implications as they advance into undergraduate studies, especially in STEM fields. Students who lack strong fundamentals in mathematics at the K-12 level will find themselves struggling to overcome the steep learning curve in courses discussing the concepts of calculus and differential equations when they are also already grappling with simpler concepts such as basic trigonometry and linear algebra. This constant struggle could lead to demotivation among students and retention issues in STEM colleges. Various literature highlights the difficulties that students face when learning mathematics, and numerous pedagogical discussions were made with the hope that the teaching of mathematics can be done more efficiently and more effectively. Yet the fact that the United States, as a whole, shows little improvement, if any, in the PISA rankings for math proficiency in the past decade indicates that there are underlying problems which may not be solved by just proposing new teaching methods. In this paper, we inspect the merits of emphasizing the fundamentals of mathematics and how a solid grounding in mathematics can be founded on “trivial” methods, as well as how they can be invaluable to help students segue into developing their computational thinking skills. The factors that lead to an apparent stagnation in math proficiency among students are identified and discussed to ascertain whether new teaching methods are the answer to improving math proficiency among students. We consider the observations made by instructors from Texas Tech University about the apparently poor mastery of basic mathematical skills among freshmen engineering students. These instructors teach an introductory course that covers mathematical methods to solve problems on trigonometry, systems of linear equations, quadratic equations, sample statistics, vectors, etc. These observations, coupled with higher expectations by universities on their students, provide the motivation for the discussion that will follow in this paper. Finally, we discuss a path forward for engineering undergraduate students who are struggling to keep up with the mastery of pre-college mathematical skills needed prior to pursuing their engineering core courses.

Background

Mathematics is an important tool by which the concepts of science and technology can be explained and modelled on pen and paper. The mastery of mathematics allows for the study – besides pushing the frontiers – of science and technology. Students who wish to pursue a degree in a STEM-related field must have adequate knowledge and proficiency in mathematics to be successful in their STEM career. Therefore, STEM students are expected to have the adequate preparation in mathematics prior to participating in a STEM degree program.

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Numerous pedagogical discussions were made with the hope that the teaching of mathematics can be done more efficiently and more effectively, especially at the K-12 levels. Yet the fact that the United States, as a whole, shows little improvement, if any, in the PISA rankings for math proficiency in the past decade[10,11] indicates that there are underlying problems which may not be solved by just proposing new teaching methods[3].

Individuals critical of the poor performance of mathematics among students in the United States point out that rote learning or memorization is a major factor that contributes to poor proficiency in mathematics[1]. Students are taught algorithms to perform calculations, which arguably does not develop number sense and a deeper understanding of mathematics. The appropriate response to curb rote learning seems to lie on better teaching methods and new pedagogies. Such methods can include the well-known flipped classroom model, live demonstrations of specific concepts, integrating hands-on activities and so on. However, new teaching methods and pedagogies may require instructors to undergo additional training to internalize the new methods. Additional time and other resources may be needed to develop new teaching aids. Both parents and instructors must buy in to the new teaching methods and observe their implementations patiently over time before the success of the new methods can be ascertained[5]. If parents are not invested in the new methods, or if instructors are impatient with the internalization of the new methods, then it is likely that current (perhaps ineffective) teaching methods will remain in force. The reluctance to adopt better teaching methods can be exacerbated if there is lack of incentives and resources to provide adequate training and guidance. Educationists are also critical of standardized testing because it affects the way subjects are delivered (thus indirectly dictating teaching methods and pedagogies) and the way students learn. In the quest to satisfy testing requirements and boost test scores, many K-12 instructors limit discussions over fundamentals of mathematics to maximize time to cover test materials and teach mathematical algorithms to solve specific problems related to specific topics[7,8], which further contributes to the "one mile wide and one inch deep" characterization of the K-12 curriculum. As a result, students place more emphasis on obtaining the correct answers, rather than understanding the concepts and fundamentals in the solutions that led to the correct answers.

The teaching of mathematics at the K-12 level has primarily been prescriptive. Students may be able to explain satisfactorily the concept of multiplication, e.g. 2×3 is similar to adding the number 3 for a total of two times (i.e. 3+3), or adding the number 2 for a total of three times (i.e. 2+2+2). However, many students may not be able to explain why the product of two negative numbers result in a positive number. Students can commit to memory that the product of two negative numbers produces a positive result, but they may never be able to explain the fundamental mathematical reasoning behind such a result. There seems to be a tacit expectation that mathematics can be prescribed first at early levels of education, and then the fundamental reasoning for some mathematical concepts can be covered at more advanced levels. In reality, however, the fundamental reasoning is not taught, and this gap is not filled even at tertiary levels of education.

The Finnish education system has often been lauded as one of the best systems that ensures a conducive environment for students and teachers. Students in general have shorter class times and teachers have a greater level of autonomy on how lessons should be paced, arranged and delivered[9,13,14]. Yet, students in Finland are consistently able to perform exceptionally well based on the PISA results[10,11]. While the United States can hope to emulate our Finnish counterparts whose education system delivers positive results consistently, the fact that the status quo is still in place in the American education system means that colleges and universities must work with the varying proficiency of mathematics that students currently bring with them. Even if students have found the learning of mathematics to be uninteresting, uninspiring and difficult at the K-12 level, instructors at institutions of higher learning can still hope to develop confidence and better mathematical reasoning in their students. College instructors could start by explaining the technological implications made possible through mathematics (e.g. how solving quadratic equations can lead to the design of vehicle suspensions, how calculus opened the frontiers of space exploration, how trigonometry ensured the safe and stable construction of buildings, etc.). Once mathematics is connected to tangible engineering outcomes, then a case has been presented to students that mathematics in engineering is not about knowing how to solve a mathematical problem, but rather to know when to apply a mathematical solution and why that applied solution makes physical sense or is consistent with engineering theories. Mathematics will not be another subject of little real-life use, but rather an important modelling tool of which reasonably accurate tangible results can be achieved.

Long Division and "Trivial" Methods – Are They Truly Worthless?

Upon getting elementary students to master the concept of multiplication, the mathematical discussion often proceeds to the teaching of quotients. At the elementary level, students are often taught to perform long divisions to calculate the results by hand. Figure 4 shows examples of long divisions. Some instructors consider mastery of long divisions to be trivial and have questioned the need to emphasize the teaching of dividing two whole numbers using long divisions because students who advance into higher level courses will ultimately refuse to apply long divisions and instead obtain their solutions using calculators. There are also arguments against the long division because of its algorithmic nature that does not develop number sense; however, instructors should also realize that the long division allows for a meaningful discussion about remainders, e.g. why some remainders are zero and some are not, how remainders relate to the concept of fractions, etc.

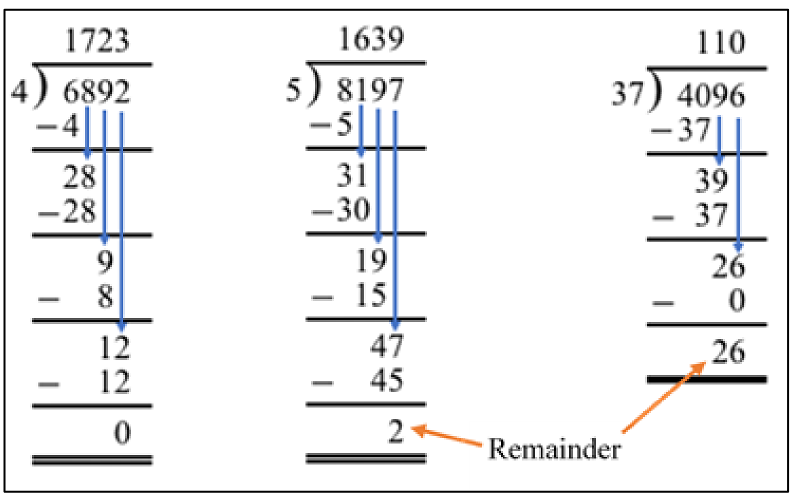


Figure 4: Examples of long division

The virtue of teaching students to perform the quotient of two integers using long divisions becomes more apparent especially when discussing the parity of numbers, i.e. whether a given integer is odd or even. Instructors can demonstrate to students that the parity of an integer can be determined using long divisions where the remainder will be zero when even numbers are divided by 2 (in other words, even numbers are completely divisible by 2), and the remainder will be 1 (or non-zero) when odd numbers are divided by 2.

Students can also learn that all even numbers are multiples of 2 and that odd numbers are never multiples of 2. However, the concept of remainders from long divisions can be applied into computer programming. Given a long array of integers, a student familiar with the concept of remainders can separate the integers into their respective even-number array and odd-number array by applying conditional statements, computer logic, and calculating remainders when each integer is divided by 2.

The remainders computed from the quotient of two numbers are also particularly important in the field of encryption. To develop the algorithms to compute remainders, a programmer will usually be first exposed to the concept of remainders through learning the long division. The concept can then be translated into a computerized algorithmic procedure that can be automated.

While some instructors argue that teaching the long division – and other methods discounted as being “trivial” – contributes no added proficiency in mathematics as they are merely algorithmic or procedural, the observations that can be made from the results of applying these “trivial” methods can assist to expand students’ knowledge in the basic property of numbers, which we argue actually enhances mathematical proficiency.

Algorithmic procedures, though probably never used in any real application, can provide a solid background for a more advanced topic later. Calculus students are taught Riemann sums and can be assigned problems to approximate an area under a curve (maybe by hand) prior to introducing the concept of integrals. After learning how to compute integrals analytically, students may never use Riemann sums to calculate areas under curves again (unless they use a computer program to approximate areas numerically, in which far more advanced methods such as the trapezoidal rule and Simpson’s rules exist). However, what students gain besides being skilled in computing integrals is the ability to connect summations with integrals, and when provided with problems that are too complex, they know that they can always fall back to the fundamental definitions for a numerical solution.

Instructors should not be too hasty to discount the value of “trivial” methods just because technology will eventually make such methods obsolete. Scientific calculators today are powerful enough to compute definite integrals, sample statistics, linear algebra, solving systems of linear equations, and so on. However, just because we have calculators to perform these computations does not “trivialize” the importance of knowing how to compute sample statistics, for example, by hand. Computational thinking relies a lot on being familiar with these “trivial” methods to create programs that can automatically execute these methods for specific problems.

Observations at Texas Tech University

As students enroll into university with varying proficiency in mathematics, an introductory course in engineering is taught to first-year students at Texas Tech University. The course introduces the concepts of computer programming and provides adequate leveling in mathematics to students before they proceed with upper level courses. Leveling is provided on topics such as trigonometry, solving systems of linear equations, quadratic equations, vectors and so on. Engineering applications are also discussed during the delivery of these topics.

The majority of students come from different parts of Texas, with the rest coming from neighboring states, and even fewer from other regions of the country or from foreign nations. Most students, therefore, would have completed their high school education in Texas.

Trigonometry is an area of mathematics that is widely used in engineering. Every engineering student who goes through courses in physics and engineering mechanics must apply trigonometry to solve engineering problems. Topics such as projectile motion, equilibrium of forces and other vector analyses will utilize trigonometry concepts to resolve a two- or three-dimensional problem into multiple one-dimensional components. Therefore, students enrolled in this introductory course to engineering must learn basic trigonometry which encompasses topics such as the Pythagorean Theorem, the Law of Sines, the Law of Cosines and how to compute the three basic trigonometric ratios – sine, cosine and tangent of an angle.

In data collected from 278 students over the course of five semesters, slightly more than one-fifth of the students who completed this introductory course scored below 70% on trigonometry questions in their first tests after the topic of trigonometry was covered. Figure 1 provides a graphical representation of the distribution of grades achieved by students specifically on trigonometry problems during their first tests.

The fact that trigonometry is ubiquitous in many engineering courses means that failure to adequately understand the concepts of basic trigonometry will result in difficulty coping with engineering courses later. Conversations with students who struggle with trigonometry reveal that some students enrolled in their first semester of engineering school do not necessarily have working knowledge of trigonometry. Faculty members may mistakenly assume that students have working knowledge of trigonometry, at which point the lessons were aimed and designed to recall and refresh these concepts. For the small group of students who see the material for the first time, this refresher may be insufficient to develop an understanding of the fundamentals.

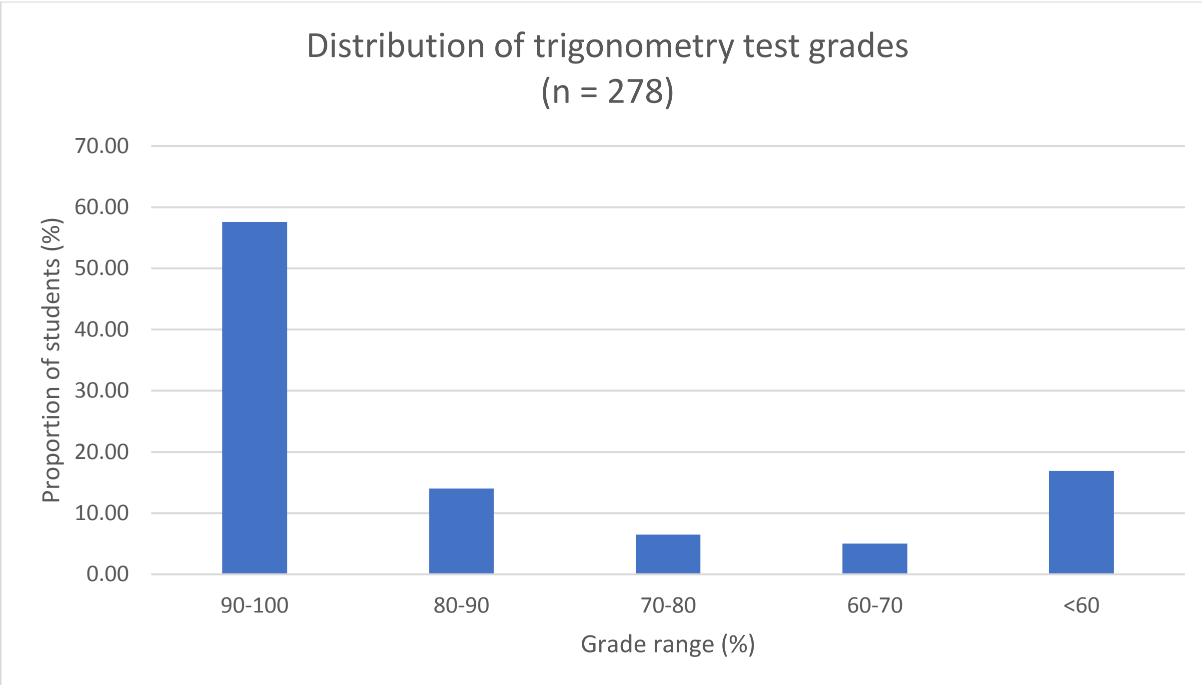
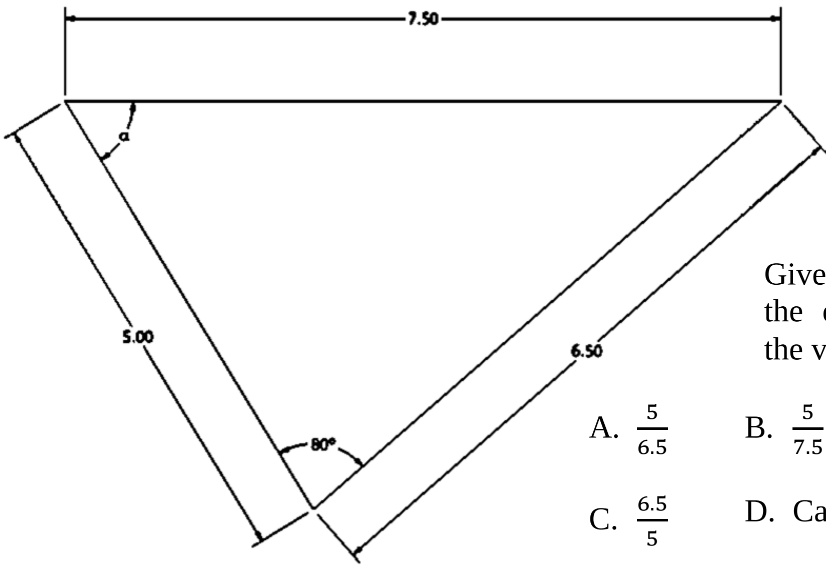


Figure 1: Distribution of grades sampled from 278 students

In a recent quiz assigned to students to check their ability to recall how trigonometric ratios are defined in the context of a right triangle, students were intentionally provided with an oblique triangle with an interior angle of 80°, as shown in Figure 2, to calculate the value of $\tan(\alpha)$, given that the lengths of all sides are known. At this point of the course, the concepts of the Law of Sines and the Law of Cosines have not been covered. As this was a multiple-choice problem, students were given a choice of the following answers: $\frac{5}{6.5}$, $\frac{6.5}{5}$, $\frac{5}{7.5}$, or that the value of $\tan(\alpha)$ cannot be calculated (bearing in mind that at this point, students were not yet equipped with the knowledge to compute the other two interior angles using the Law of Cosines before calculating the tangent of angle α). The instructor anticipated that most students will correctly identify that the value of $\tan(\alpha)$ cannot be calculated because it is not clear whether either of the two unknown interior angles is a right angle without calculating them, and one of the interior angles is not a right angle. Out of the 40 responses obtained from students, only one identified the correct answer. The other 39 students selected the incorrect answer of $\frac{6.5}{5}$.



Given the triangle in the diagram, compute the value of $\tan(\alpha)$.

A. $\frac{5}{6.5}$

B. $\frac{5}{7.5}$

C. $\frac{6.5}{5}$

D. Cannot calculate $\tan(\alpha)$

Figure 2: A trigonometry quiz question administered to students

This simple quiz provides a glimpse on students' expectations where assessments are concerned. Students generally anticipate that instructors will ask questions that mirror example problems or that instructors will ask questions such that all conditions that make the prescribed methods applicable are automatically satisfied. In this case, students did not pay attention to the fact that the triangle in Figure 2 is not a right triangle, which nullifies the validity of computing trigonometric ratios by dividing the lengths of two sides. Students may also have misconceptions that a trigonometric ratio can be calculated for any triangle if the lengths of the sides are provided. Teasing out misconceptions and emphasizing the importance of fundamental definitions allowed for a more in-depth discussion on the properties of a triangle and why the right triangle is used as the basis for defining trigonometric ratios using the length of sides.

The possibility that students' standardized test training somewhat hindered them from reasoning the mathematical constructs properly before applying a solution cannot be entirely discounted. Students are often taught strategies for choosing multiple choice answers on standardized exams, and it is possible that this prior training influenced students away from answer choice D. Such a quiz question can also serve to convey the message to students that college education aims to further advance a student's cognitive ability to analyze and evaluate a problem, not merely to memorize a method and apply it blindly to any problem of a particular topic.

Building Fundamental Understanding of Mathematics Through Proofs

Mathematical concepts are grounded on established mathematical logic, axioms and proofs. The prescriptive method of teaching mathematics often omits proofs, although axioms may be emphasized. Many engineering instructors who may also cover mathematical topics omit proofs so that they can stay on schedule with the material that they have planned to cover over the semester, and also focus on the application of formulas and equations to specific problems. Some instructors also believe that examples and applications will suffice to establish the truth of specific mathematical concepts or statements[3,4,6]. There is also the presumption that engineering instructors need not cover the mathematical fundamentals in detail because students are expected to learn the fundamentals in a separate mathematics course. However, this presumption can be flawed if the mathematics course is not a mandatory part of the degree program. In such cases, instructors should point out to students that such fundamentals are covered in a separate mathematics course and they should be strongly encouraged to enroll in that course.

Also, understandably, some proofs can take a long time to go through and, coupled with the fact that some students may not be interested in proofs at all, instructors tend to believe that class time is better utilized on applications and advancing more engineering concepts rather than placing some focus on proofs. Students who are interested to know how some equations or formulas were derived are often directed to textbooks or other literature. However, literature may also omit intermediate steps that authors consider trivial and produce the final result without realizing that students may lack the mathematical acumen and confidence to reproduce the intermediate steps. Some instructors may also prefer that students who are interested in proofs to show up during office hours so that they may further discuss the topic outside of class.

Many of the mathematical concepts discussed in the introductory course (which also may be covered in some high school syllabi) are short and simple enough that proofs can be explained adequately without taking too much of class time. The Pythagorean Theorem, for example, can be proved by utilizing the concept of geometric area (and at most, the distributive property of numbers in elementary algebra). Figure 3 illustrates the graphical proof which students will find satisfactory to explain why $a^2 + b^2 = c^2$ and enables instructors to emphasize the reason that the Pythagorean Theorem works only for right triangles.

A side benefit to the proof from Figure 3 is that students will also understand why $(a + b)^2 = a^2 + 2ab + b^2$, a formula which most students just commit to memory without realizing the distributive property of numbers or the compatibility of algebraic operations with geometric concepts. This method of teaching mathematics is arguably more comprehensive than teaching students how to apply the infamous FOIL (**F**irsts, **O**utsides, **I**nsides, **L**asts) method, which works solely for algebraic multiplication of the form $(a + b)(c + d)$.

The topic of complex numbers is taught as part of the introductory course at Texas Tech University, and instructors have observed that students who memorized the formula that:

$$(p+qi)(r+si)=(pr-qs)+(ps+qr)i$$

where $i^2=-1$, tend to commit more errors in multiplying the correct terms or keeping track of the correct signs. The method of computing the algebraic multiplication of two factors, as shown in the left portion of Figure 3, was introduced to students and extended to compute the multiplication of two complex numbers. Such instruction provides a graphical visualization of the distributive property of numbers over addition and helps to minimize errors in performing such multiplications.

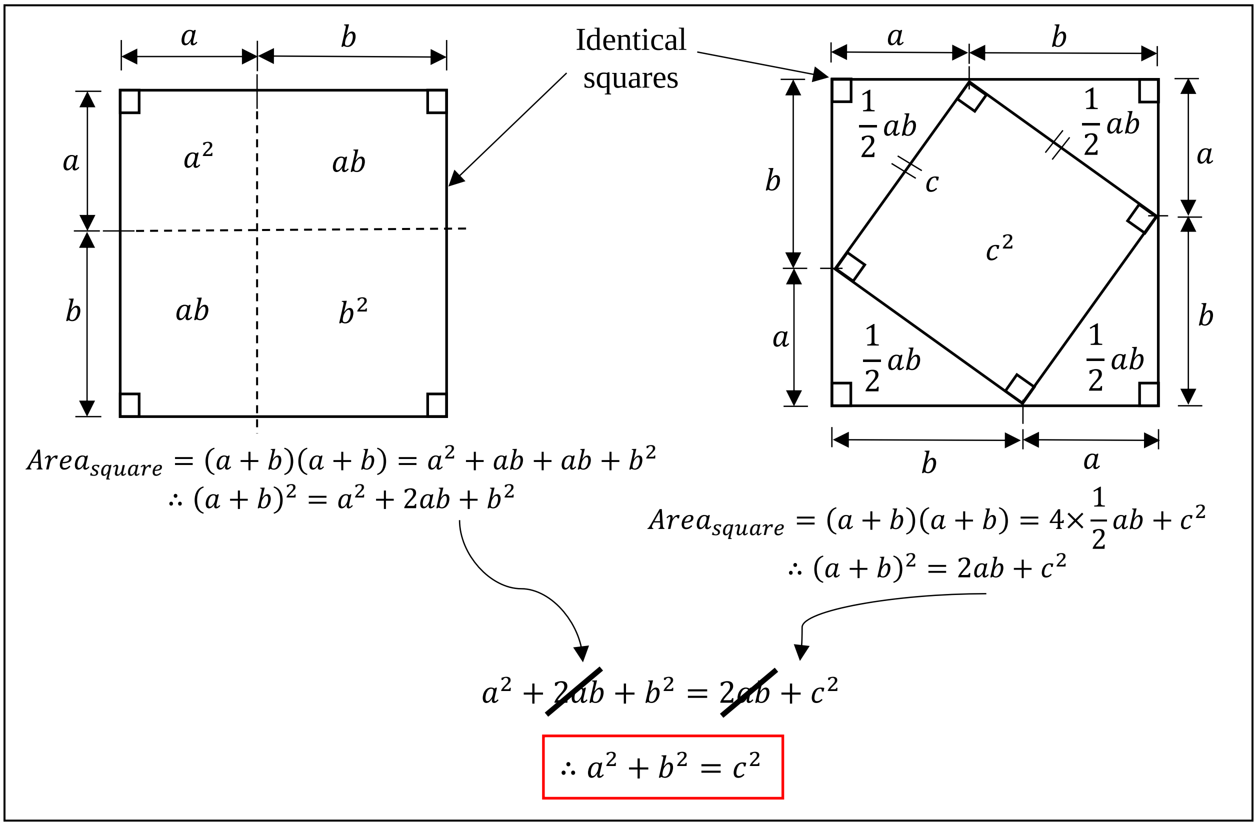


Figure 3: Graphical proof of the Pythagorean Theorem using concepts from geometry

The concept of first-order linear interpolation, which is taught in the introductory course at Texas Tech University, is also best explained through proof using trigonometry (or theory of proportions), as the proving process demonstrates a sequential and logical reasoning of interpolation and what the objective of performing interpolation is. Rather than displaying the final formula to students and then going through examples where the formula is applied, proving the formula for linear interpolation involves explaining to students that an unknown point in between two known points can be estimated assuming that a straight line passes through the two known points and the unknown point. The provision of assumptions in the discussion of linear interpolation is similar to how a lot of engineering discussions are carried out, because many engineering concepts are developed based on a list of assumptions made to simplify certain mathematical formulation, or to idealize an engineering property. In the case of first-order linear interpolation, the assumption is that the unknown point lies on the straight line connecting the two known points, which is not necessarily true in reality, but in most applications would produce estimates that are reasonably accurate.

Providing proofs to certain mathematical formulas and equations is not an unreasonable approach towards dispelling the notion that proofs are difficult or that they are of little value. Topics that involve longer proofs may be omitted for the sake of optimizing class time for other discussions, however with the repository of proofs being recorded and shown on videos (YouTube or Khan Academy, for instance), instructors could direct students towards these resources and encourage them to reinforce their fundamentals which might have been glossed over during class.

Lessons Learned and Moving Forward

The introductory course at Texas Tech University provides leveling in basic engineering mathematics because freshmen who enrolled in engineering have varying proficiency in mathematics among themselves. With that in mind, instructors should be wary about assuming students' prior knowledge in mathematics because the latter may be learning the material for the first time.

Given that many instructors provide short assessments in the form of quizzes to evaluate students' understanding about a prior topic that was taught, quiz questions on mathematics should be designed to tease out misconceptions that students may have. By teasing out some common misconceptions, a deeper discussion can take place to explain and clarify students' fundamental errors. These discussions may consume about five to ten minutes at the beginning of class to reiterate and reinforce some important concepts that were covered in the immediate preceding class. The fundamental constructs of a mathematical concept can be briefly explained, and if these explanations require substantial amount of class time to cover adequately, students could be directed to online videos and other resources to improve their conceptual understanding. Instructors can consider incentivizing students in ways they deem appropriate to encourage students to go through the supplemental materials outside of class time and to seek clarification when needed. Such discussions are valuable because the opportunity to learn from mistakes – especially those that most students make – enables students to appreciate the importance of understanding a concept rather than learning through rote memorization, for instance. These discussions could also send a message to engineering students that at college level, the ability to reason through the steps towards the correct solution is more valuable than the mere ability to apply algorithmic procedures blindly. The recapitulation of preceding mathematical concepts can also help instructors to segue into the next mathematical topic (e.g. bringing trigonometry concepts and Cartesian coordinates into the topic of interpolation), which demonstrates the connectivity of topics as is common in engineering courses and provides a logical flow in the development of subsequent lessons. Instructors may consider taking these low stakes assessments for little or no credit to draw students out of their "correct answer" mindset. Doing so can help instructors to communicate the message that "I am trying to help you" rather than "I am trying to trick you." Even if it involves a shift in pedagogy, we argue that such a shift is minimal. Various literature exists to highlight examples of mathematical misconceptions, which instructors can emulate or utilize to evaluate students' comprehension of a specific topic[2].

Universities can consider a one-credit course or seminar that covers proofs on specific topics which form the fundamentals that are applied in advanced level mathematics. These seminars can first focus on proofs that can be delivered easily before gradually progressing towards more advanced proofs. A course in proofs can help students to develop strong, logical mathematical reasoning. By understanding the mathematical constructs and reasoning that lead to the proof of a specific formula, students who complete the proving process will realize that proofs are not necessarily esoteric and filled with advanced mathematical jargons. Being skilled in proofs also enables students to identify potential errors in literature, to have a better understanding of course material, and may even lead to self-development of solutions[12]. With a deeper understanding in the fundamentals of mathematical constructs and reasoning, we believe that students will not merely perceive mathematics as another subject that they must pass to pursue engineering, but rather perceive mathematics as a useful and powerful tool by which the physical context of an engineering concept can be modelled and explained. This improved reasoning skill should translate into better assessment scores, as well as transforming the way students' understanding of specific concepts are assessed (sample problem is shown in the Appendix, with a brief explanation on what common misconceptions exist).

Understandably, time limitations can restrict the depth of material that instructors could cover in a lesson. The supplementation of videos and external multimedia resources for students to watch and understand outside of class hours helps to reinforce concepts covered in class and can provide an expanded repository of examples and applications.

Much has been said about how a lot of curricula can be characterized as "one mile wide and one inch deep". By spending some time and resources to cover simple fundamentals more completely and thoroughly, students can gain depth in their understanding and retention of knowledge. Going through material more thoroughly also demonstrates to students the level of thinking required in the evaluation and synthesis of knowledge – the two highest level of learning in the cognitive domain following Bloom's taxonomy.

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Appendix: Sample assessment problem related to solving systems of linear equations with examples of student misconceptions

The following problem was provided to students during a test:

A florist makes five identical bridesmaid bouquets for a wedding. She has $610 to spend on all bouquets and wants 24 flowers for each bouquet. Roses cost $6 each, tulips cost $4 each, and lilies cost $3 each. She is required to have twice as many roses as the sum of the other two types of flowers combined in each bouquet. Calculate the number of roses in each bouquet using Cramer's Rule. (For this entire problem, ignore sales tax.)

Students are expected to realize that this is a problem involving a system of linear equations. Students are not provided the system of linear equations explicitly, but are expected to synthesize the equations themselves from the given word problem. After the system of linear equations is identified, students will solve for the number of roses in each bouquet using Cramer's Rule.

Instructors may expect the following logical reasoning towards obtaining the correct set of equations:

Instructor's reasoning:
The following unknowns are defined:

$x = number\ of\ roses\ per\ bouquet$
$y = number\ of\ tulips\ per\ bouquet$
$z = number\ of\ lilies\ per\ bouquet$

In theory, as there are three unknowns in the problem, there needs to be three sets of linear equations to solve for the unknowns. The first equation in the system of linear equations is obtained through knowing that the sum of all three types of flowers in each bouquet must add up to 24, i.e.:

$$x + y + z = 24$$

If $610 is the total expenditure for all bouquets, then one bouquet of flowers will cost $122 (from dividing $610 by 5). Given the cost of each type of flower, the second equation in the system of linear equations is:

$$6x + 4y + 3z = 122$$

(Instructor's reasoning cont'd)
The third and final equation is obtained by understanding that there are more roses than there are tulips and lilies combined. In fact, the number of roses is two times more than the sum of tulips and lilies, which means that dividing the number of roses by the number of tulips and lilies combined will yield:

$$\frac{x}{y+z} = 2$$

which can be algebraically simplified and finally expressed as:

$$x - 2y - 2z = 0$$

The final system of linear equations is as follows:

$$\begin{gathered} x + y + z = 24 \\ 6x + 4y + 3z = 122 \\ x - 2y - 2z = 0 \end{gathered}$$

Students will then express the system of linear equations in matrix form before solving using Cramer's Rule.

An alternative solution exists where the unknowns represent the number of roses, tulips and lilies respectively in all five bouquets. The student must then recognize that there are 120 flowers in total (i.e. $x + y + z = 120$) and that the cost for all five bouquets is $610 (i.e. $6x + 4y + 3z = 610$). There will be an additional step required to divide the final answer of *x*, *y*, and *z* by 5 to obtain the number of flowers <u>in each bouquet</u>.

While many students can construct the three equations correctly and solve for the number of roses in each bouquet, the following sample response from students is not uncommon:

Sample response from a student:
The following unknowns are defined:

$x = roses$
$y = tulips$
$z = lilies$

The sum of roses, tulips and lilies must add up to 24:

$$x + y + z = 24$$

The cost of each flower multiplied by their respective unit cost will add up to $610:

$$6x + 4y + 3z = 610$$

Roses are two times more than tulips and lilies combined:

$$2x = y + z$$

The system of linear equations required to solve for "roses" is:

$$\begin{gathered} x + y + z = 24 \\ 6x + 4y + 3z = 610 \\ 2x - y - z = 0 \end{gathered}$$

The sample student response highlights multiple issues of flawed mathematical reasoning. Instead of using *x*, *y* and *z* as a container which represents the **quantity** of a specific type of flower, some students perceive such variable names as mere labels rather than numbers that are not known prior to solving. As students did not define a basis (whether *x* represents the number of roses in all five bouquets or just in one bouquet) for which the system of linear equations is derived, the first two equations that the student came up with are inconsistent with one another (the first equation is on a one-bouquet basis, the second is on a five-bouquet basis). Considering that some students did not comprehend that the variable names in the system of linear equations represent a number that is yet to be calculated, they do not realize that the third equation effectively reduces the actual number of roses by a factor of 4. Had they understood that the variable names represent values that satisfy the system of linear equations, they could have asked the question: "If the sum of tulips and lilies equal to 10, will solving my third equation yield 20 roses?” Such a check would reveal the fact that the third equation is incorrect because if the sum of tulips and lilies equal to 10, then solving for x in the third equation would yield five roses, and not 20.

While students understand that there are more roses than there are tulips and lilies combined, students think that by performing the product of 2 and x, they have essentially doubled the number of roses. Such a misconception is quite common, and it requires some time and explanation by the instructor to instill deeper mathematical meaning to the letters x, y and z as representation of specific quantities that are unknown.

Upon formulating the three sets of equations (whether they are correct or otherwise), students generally have little problem implementing the procedures following Cramer’s Rule to solve for the unknowns, therefore students generally have developed cognitive skills in remembering and applying procedures. The biggest hurdle lies on their ability to analyze a problem and synthesize the three equations from the problem, both of which are cognitive skills engineering schools aim to develop in their students.