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## **AC 2011-14: ASSESSING THE RELIABILITY OF SOME CLASSICAL MECHANICAL VIBRATION DESIGNS VIA SIMULATION SOFTWARE**

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# Assessing the Reliability of some Classical Mechanical Vibration Designs via Simulation Software

## Abstract

This work is part of a series on problems which aid students in achieving a better understanding of underlying engineering principles and a better appreciation for the limitations of linear physical modeling in dynamics. Another issue worthy of attention is how robust some designs are based on linear modeling. The problems treated here (3 in all) do not have analytical solutions and have only become tractable due to the widespread availability and early exposure in introductory mathematics classes to simulation software such as MAPLE<sup>®</sup>, MATLAB<sup>®</sup> etc. MAPLE<sup>®</sup> is employed here. The first problem is meant to enhance students understanding of stability. It concerns a spring-mass system vibrating in a slot in a horizontal disk rotating with a prescribed motion. It is shown that for certain spin-up speeds, instabilities can develop if the system parameters are not chosen properly. Effects of spring non-linearity on these instabilities are explored. An area that students should be aware of is the reliability of designs based on linear models. A passive vibration absorber is revisited and it is shown that the classical choice of system parameters may not work if spring non-linearities are included. Choices that do work are given. Finally a problem involving "vibration cancellation" is studied. The response of a linear single degree of freedom spring-mass system to a pulse can be made identically zero for all times greater than a certain one by the application of a second pulse with a suitable phase difference. Some effects of spring non-linearities on the linear model predictions are given. Assessment was achieved by noting students better and fuller understanding of the basics.

## Introduction

Other articles on the use of simulation in engineering education exist. See for example, the work of Fraser et al. <sup>1</sup> on simulation in fluid mechanics. Questions from the Fluid Mechanics Concepts Inventory <sup>2</sup> (FMCI) identified some student conceptual difficulties. A simulation involving these concepts was developed and its efficacy was addressed using a second administration of the FMCI. A recent work of Kieffer et al. <sup>3</sup> explored the use of simulation in helping students achieve a better understanding of materials science concepts. They used a survey and student performance to assess impact. This latter point is also the main assessment of the current work. It is the authors' experience that exposure to simulation, such as the ones at hand, leads to a better and fuller understanding of the basics.

This paper is one of an ongoing series (see references <sup>4, 5, 6, 7, 8</sup>) on the role of mathematical software in furthering the depth of understanding of the dynamics of mechanical systems.

A major theme of the current work is the effect of non-linearities. In particular, one of the issues addressed is how robust are design parameters obtained from linear models.

The problems treated here (3 in all) do not have analytical solutions and have only become tractable due to the widespread availability and early exposure in introductory mathematics

classes to simulation software such as MAPLE<sup>®</sup> 9, MATLAB<sup>®</sup> 10 etc. MAPLE<sup>®</sup> is employed here.

The first problem is meant to enhance students understanding of stability. It concerns a spring-mass system vibrating in a smooth slot in a horizontal disk rotating with a prescribed motion. It is shown that for certain spin-up speeds, instabilities can develop for certain values of the system parameters. Effects of spring non-linearity on these instabilities are explored.

An area that students should be aware of is the reliability of designs based on linear models. A passive vibration absorber is revisited and the classical choice of system parameters is investigated for a case where spring non-linearities are included.

Finally a problem involving "vibration cancellation" is studied. The response of a linear single degree of freedom spring-mass system to a pulse can be made identically zero for all times greater than a certain one by the application of a second pulse with a suitable phase difference. Some effects of spring non-linearities on the linear model predictions are given.

## Physical Examples

### *Spin-up Stability*

An interesting and informative example is that of a particle, restrained by a spring, vibrating in a smooth slot in a rotating platform. Intuitively, there is competition between the stabilizing spring force and the destabilizing centrifugal force and a basic question is how that scenario plays out.

Shown in Figure 1 is a mass moving in a slot and connected to a spring, the whole system rotating in a horizontal plane at a constant rate. Applying Newton's law expressed in polar coordinates leads to:

$$-F_s \vec{u}_r + N \vec{u}_\theta = m[(\ddot{R} - R\dot{\theta}^2) \vec{u}_r + (2\dot{R}\dot{\theta} + R\ddot{\theta}) \vec{u}_\theta] \quad (1)$$

where  $F_s$  is the spring force and  $N$  a slot normal reaction.

Equating the  $\vec{u}_r$  components gives, assuming the spring is linear and unstretched when  $R=L$ :

$$\ddot{R} + \frac{k}{m}(R - L) - R\dot{\theta}^2 = 0 \quad (2)$$

Students in dynamics should be more aware of the merits of using dimensionless variables (just as in fluid mechanics). Introducing the dimensionless time  $\tau = (\sqrt{k/m})t$  and the dimensionless distance  $\delta = R/L$  the equation of motion becomes:

$$\frac{d^2\delta}{d\tau^2} + (\delta - 1) - \delta\left(\frac{d\theta}{d\tau}\right)^2 = 0 \tag{3}$$

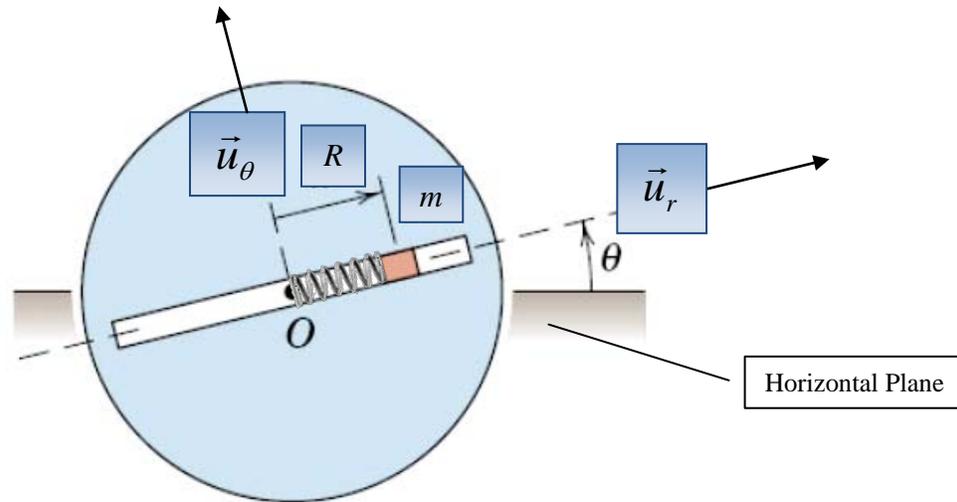


Figure 1 – Particle vibrating in a smooth slot on a rotating platform

An interesting question is: what effect does the effective spin-up rate  $d\theta/d\tau$  have on the response? A relatively simple model of the spin-up is:

$$\frac{d\theta}{d\tau} = \sqrt{p}H(\tau) \tag{4}$$

where  $\sqrt{p}$  is a measure of the spin-up rate and  $H$  denotes the Heaviside unit step function. Equation (3) is a linear differential equation with time-dependent coefficients which could be numerically challenging but poses no difficulties to codes such as MAPLE<sup>®</sup>.

Shown in Figure 2 is the response for a slow spin-up speed  $p=0.1$ . Stable motion is seen. However Figure 3, shows that for  $p=2$ , a fast spin-up speed, unbounded motion is predicted (the centrifugal force overcomes the spring force).

An item that should be emphasized to students is that the underlying linear spring model breaks down when large motions are predicted. A non-linear spring model must be employed to determine what actually occurs. A hardening spring model is used here in which the spring force is given by:  $F_s = k(R - L) + k_1(R - L)^3$ . The equation of motion now is:

$$\frac{d^2\delta}{d\tau^2} + (\delta - 1) + C(\delta - 1)^3 - \delta\left(\frac{d\theta}{d\tau}\right)^2 = 0$$

(5)

where  $C$  is a dimensionless quantity given by  $C = \frac{k_1}{k} L^2$ ; its value is assumed to be 0.1 (weakly non-linear spring).

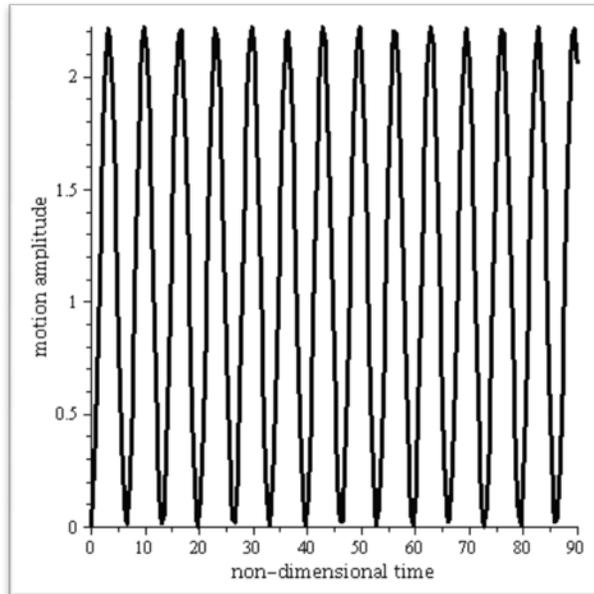


Figure 2 – Response for slow spin-up speed

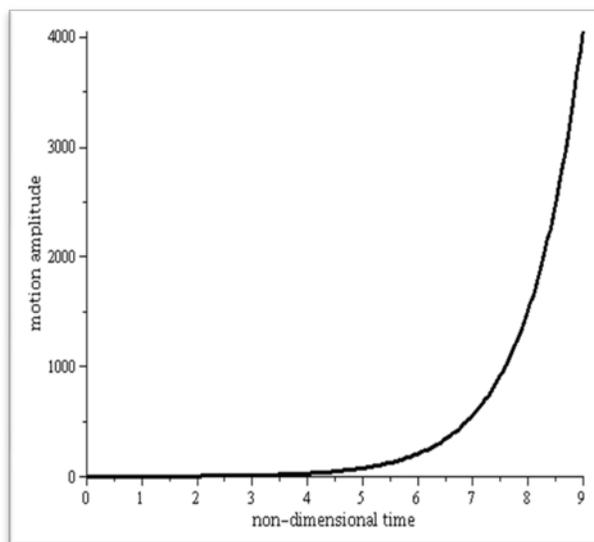


Figure 3 – Response for fast spin-up speed

With the same spin-up model as before, the equation of motion now becomes, for the fast spin-up speed  $p=2$ :

$$\frac{d^2\delta}{d\tau^2} + (\delta - 1) + 0.1(\delta - 1)^3 - 2\delta H^2(\tau) = 0 \quad (6)$$

Figure 4 shows that the response is actually bounded. The predictions of the linear model are not accurate. Note though that the motions are large when compared to the bounded linear case.

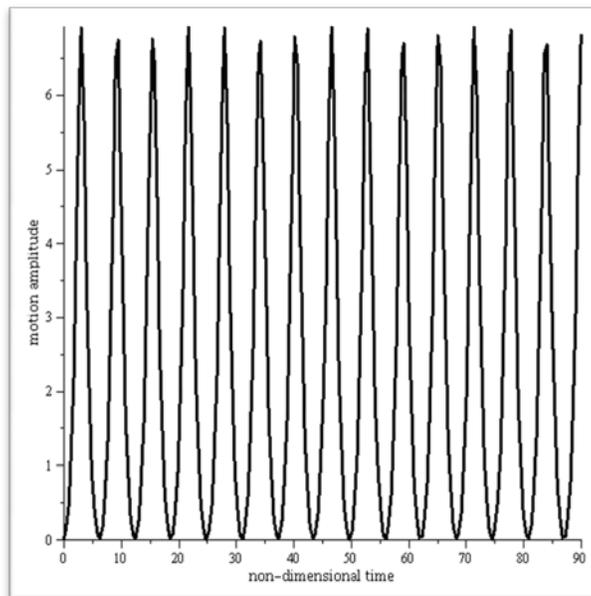


Figure 4 – Non-linear response for fast spin-up speed

### ***Vibration absorber***

A classical solution to vibration suppression is the use of a passive vibration absorber. In the design of such absorbers the system parameters are based on linear lumped parameter models. An instructive question is how robust such designs are. Here some effects of system nonlinearities are explored.

Consider the two-degree-of-freedom system shown in Figure 5.

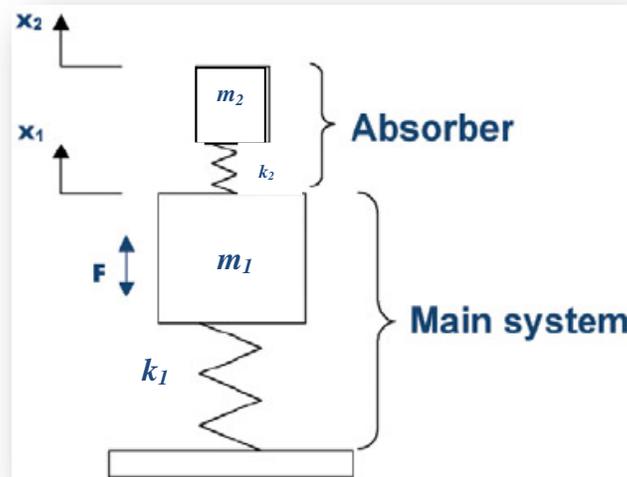


Figure 5 – Two-degree-of-freedom system

A classical scenario is one in which the mass  $m_1$  is undergoing excessive vibrations under the action of the harmonic force  $F$  and a “sacrificial” mass  $m_2$  is added whose role is to absorb the vibrations.

The equations of motion for this system can be shown to be (this example is geared toward intermediate level students):

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2 (x_2 - x_1) = F = F_{1,0} \sin(\omega t)$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0$$

(7)

Seeking steady-state solutions, one takes  $X_1 = A \sin(\omega t)$  and  $X_2 = B \sin(\omega t)$ , which gives, after substituting into equations (7):

$$A = \frac{F_{1,0} (k_2 - m_2 \omega^2)}{[(k_1 + k_2 - m_1 \omega^2)(k_2 - m_2 \omega^2) - k_2^2]}, \quad B = \frac{F_{1,0} (k_2)}{[(k_1 + k_2 - m_1 \omega^2)(k_2 - m_2 \omega^2) - k_2^2]}$$

(8)

It can readily be seen from equations (8) that  $X_1 \equiv 0$  on choosing  $k_2/m_2 = \omega^2$  (absorber condition). Other conditions on the choice of  $k_2$  and  $m_2$ , such as limiting the amplitudes of  $X_2$ , and / or restrictions on the allowed new natural frequencies, are then applied.

Consider the following numerical values, in which the forcing frequency is close to the original natural frequency:  $m_1 = 10 \text{ Kg}$ ,  $k_1 = 2000 \text{ N/m}$ ,  $F_{1,0} = 100 \text{ N}$ ,  $\omega = 13.5 \text{ rad/s}$ . Note that the original frequency is  $\sqrt{k_1/m_1} = 14.14 \text{ rad/s}$  which is quite close to  $13.5 \text{ rad/s}$ . The absorber condition gives:  $\sqrt{k_2/m_2} = 13.5 \text{ rad/s}$ . The amplitude of  $X_2$  is  $F_{1,0}/k_2$  and this is restricted to be  $0.2 \text{ m}$ . Then  $k_2 = 500 \text{ N/m}$ ,  $m_2 = 2.74 \text{ Kg}$  (this gives a mass ratio of  $2.74/10 = 0.274$ , within the range recommended by Inman <sup>11</sup>).

Numerical integration of the differential equations is now used to verify the above parameter value choices. The differential equations are:

$$10\ddot{x}_1 + 2500x_1 - 500x_2 = 100\sin(13.5t)$$

$$2.74\ddot{x}_2 + 500x_2 - 500x_1 = 0$$

(9)

A MAPLE<sup>®</sup> worksheet for the numerical solutions of these equations is given in the Appendix. Note that a “numerical damping” term  $2\dot{x}_1$  has been added to suppress the transient. Figure 6 shows that indeed  $X_I$  is identically zero as predicted.

Suppose now that the first spring is non-linear (hardening) and develops a force:  $k_1x_1 + k_3x_1^3$ . The equations of motion (with “numerical damping” added) become:

$$10\ddot{x}_1 + 2\dot{x}_1 + 2500x_1 + 2000x_1^3 - 500x_2 = 100\sin(13.5t)$$

$$2.74\ddot{x}_2 + 500x_2 - 500x_1 = 0$$

(10)

Here  $k_3$ , was set to  $2000 \text{ N/m}$  (large non-linear effect). However the effect on the vibration absorption seen on the main mass has not been affected. This can be seen in Figure 7, which shows the amplitude of vibrations for mass  $m_1$  with the non-linear spring change.

This is evidence that the design based on parameters obtained from a linear model is robust.

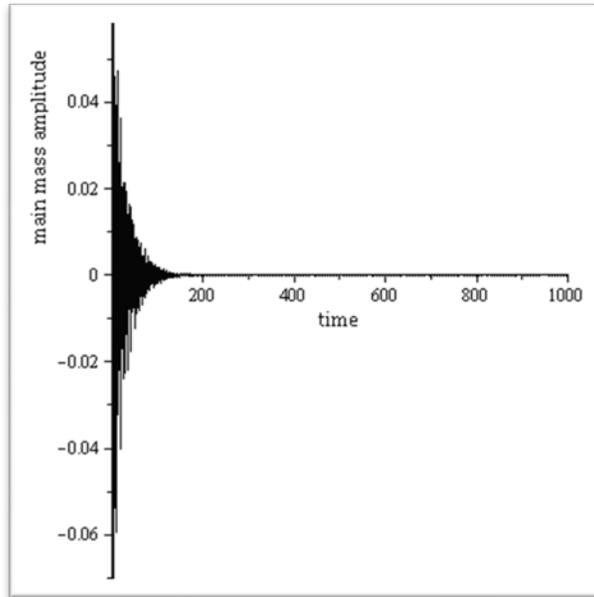


Figure 6 – Amplitude of vibrations for main mass (linear vibration absorber)

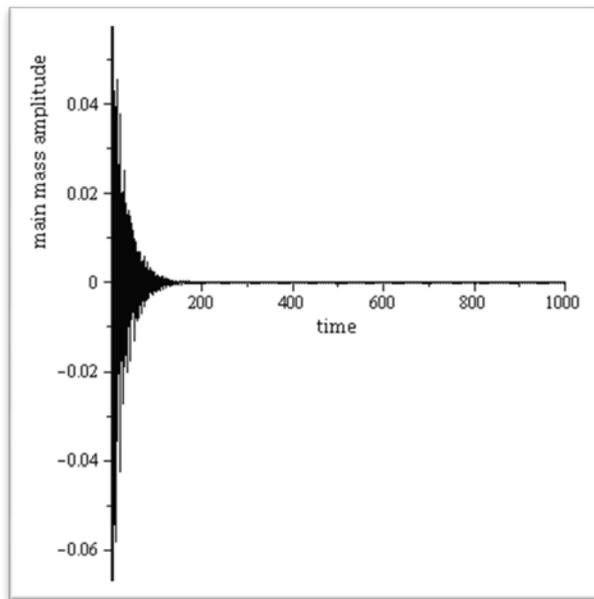


Figure 7 – Amplitude of vibrations for main mass (non-linear vibration absorber)

## Vibration Cancellation

An area of considerable technical interest is the suppression of vibration and noise. One possible strategy for achieving this is the addition of signals with a phase such that they cancel out the original motion. The problem treated here, which is again directed at intermediate dynamics students, is to determine to what degree is a strategy based on linear modeling effective. This issue is addressed by comparing results from a linear and non-linear spring-mass system.

Consider a single-degree-of-freedom linear system subject to two impacts modeled by Heaviside unit step functions. The equation of motion is:

$$\ddot{x} + \omega_n^2 x = \frac{f_0}{m} [H(t - t_2) - H(t - t_1)] \quad (11)$$

where  $\omega_n$  is the undamped natural frequency of the linear system.

The central question is whether a relationship between  $t_2$  and  $t_1$  can be found such that  $x$  is identically zero for  $t > t_2 > t_1 > 0$ . Equation (11), with zero initial conditions, can readily be solved using the Laplace transform capability in MAPLE<sup>®</sup> (or more traditional Laplace transform methods). The solution, for  $t_2 > t_1 > 0$ , is:

$$x = \frac{2f_0}{m\omega_n^2} \left\{ \left[ \sin \frac{\omega_n(t - t_2)}{2} \right]^2 - \left[ \sin \frac{\omega_n(t - t_1)}{2} \right]^2 \right\} \quad (12)$$

Using sum/difference trigonometric identities equation (12) can be reduced to:

$$x = -\frac{2f_0}{m\omega_n^2} \left\{ \sin \frac{\omega_n(2t - t_1 - t_2)}{2} \sin \frac{\omega_n(t_2 - t_1)}{2} \right\} \quad (13)$$

Thus  $x \equiv 0$  for  $\omega_n(t_2 - t_1)/2 = n\pi$ ,  $n = 1, 2, \dots$ ; zero response is achieved for  $t_2 = t_1 + 2\pi/\omega_n$ .

For purposes of illustration, consider the specific system:  $m = 10$  Kg,  $k = 1000$  N/m and  $f_0 = 1000$  N. The natural frequency is  $\omega_n = 10$  rad/s.

Choose  $t_1 = 2$  seconds. Then  $t_2 = 2 + 2\pi/10 = 2.6283$  seconds. Thus  $x$  should be identically zero for  $t > 2.6283$  s. This is confirmed numerically using MAPLE<sup>®</sup>. See Figure 8.

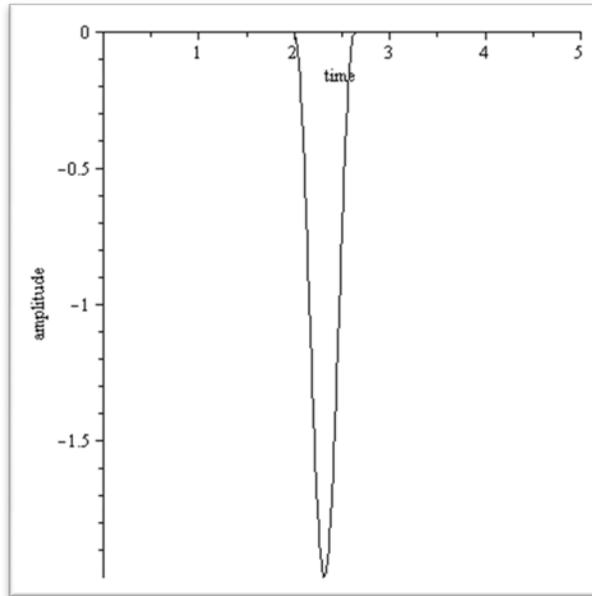


Figure 8 – Linear system response: amplitude versus time

Now consider a non-linear (weakly) system in which the spring is modeled as a hardening one. The spring force is given by:  $kx + k_1x^3$  and  $k_1$  is taken to be 10% of  $k$ . Then the differential equation becomes:

$$\ddot{x} + \omega_n^2 x + \frac{k_1}{m} x^3 = \frac{f_0}{m} [H(t - t_2) - H(t - t_1)]$$

(14)

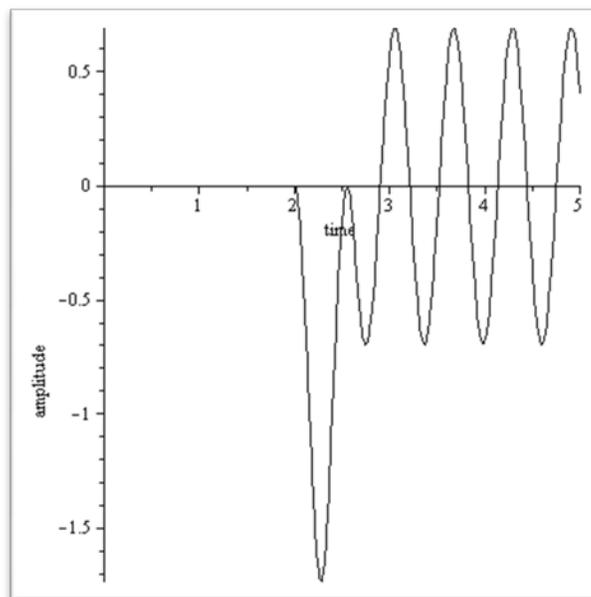


Figure 9 – Non-linear system response: amplitude versus time using values from linear model

Equation (14) is non-linear and MAPLE<sup>®</sup>'s numerical capability must be employed. Figure 9 shows that the phase choice for the linear system does not work. However Figure 10 shows that a value of  $t_2 = 2.55$  s does work. The effect of the non-linearity is a 3% change. The original design is judged to be reasonably robust.

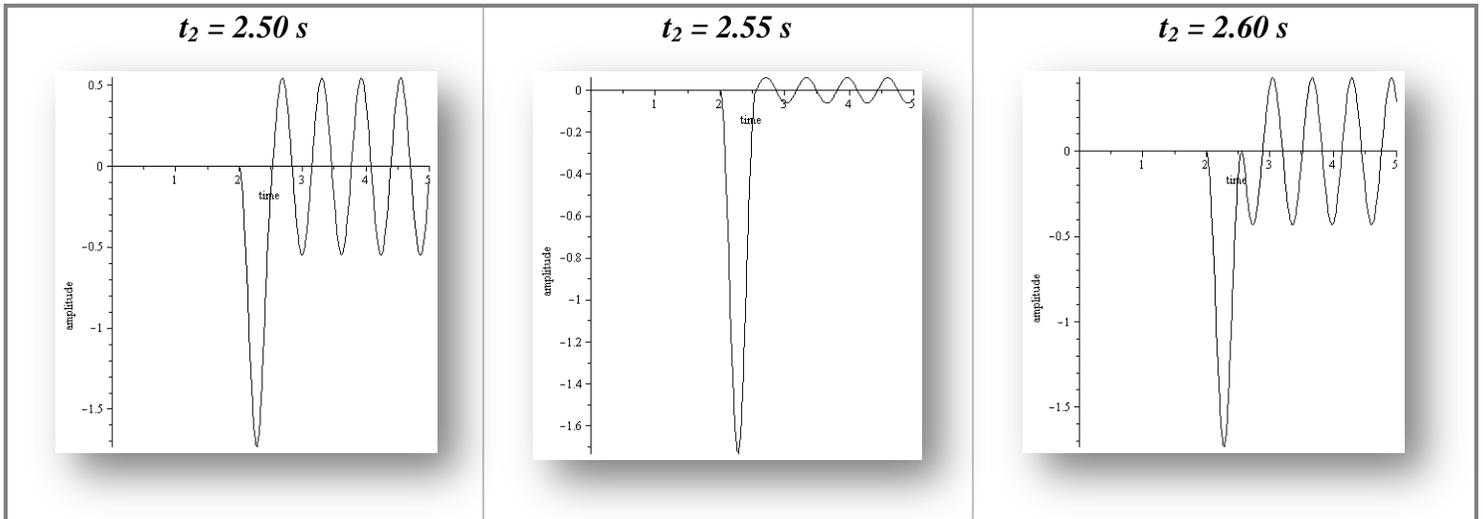


Figure 10 – Non-linear system response: amplitude versus time for  $t_2 = 2.50, 2.55$  and  $2.60$  s

One final item was examined, namely the effect of the initial amplitude  $f_0$ . Figure 11 shows that the value of  $f_0 = 1000$  N leads to “zero response” after  $t_2 = 2.30$  s, a value different from the  $f_0 = 100$  N case. Thus amplitude effects are also present in the choice of parameters.

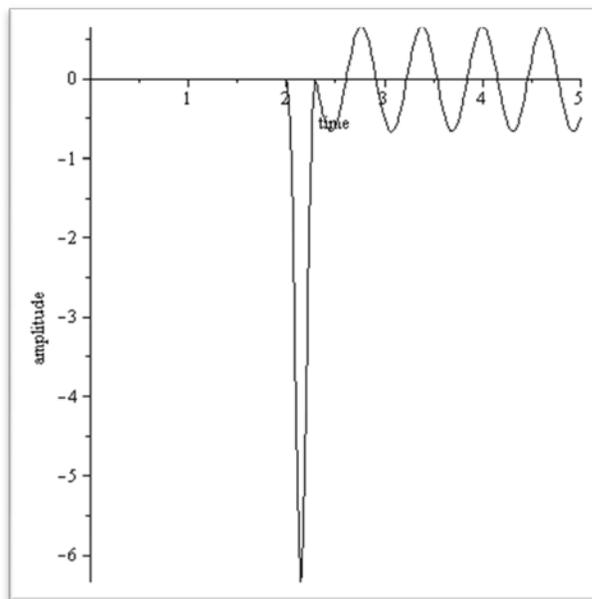


Figure 11 – Non-linear system response: amplitude versus time for  $t_2 = 2.30$  s

## Summary

In the preceding text examples were developed to demonstrate how numerical simulation software can be used to investigate some non-linear effects in mechanical systems.

Three illustrative problems were discussed. The first involved a particle, restrained by a spring, vibrating in a smooth slot in a rotating platform. It was shown that, for large enough rotational speeds, the motion becomes unstable and infinite amplitudes are predicted. A non-linear model shows that the amplitudes are large but finite.

The second example was based on a vibration absorber. It was found that the parameters obtained from the linear model also worked well for a non-linear spring model. The linear approach to the problem is robust.

The final example dealt with vibration cancellation. In this case, it was found that the phase lag employed for the linear model did not work in general for the non-linear model. However, a phase lag that worked was found for the non-linear system. The effect of the non-linearity was an increase in the phase lag of about 3%. Applied force amplitude effects were also observed.

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9. [www.maplesoft.com](http://www.maplesoft.com)
10. [www.mathworks.com/](http://www.mathworks.com/)
11. D. J. Inman, *Engineering Vibration*, 3<sup>rd</sup> ed: Pearson Prentice Hall, 2008.

## Appendix – MAPLE® Worksheets

1.

### #spin up problem

```
#here s=x, tau=t
restart;
with(DEtools):
with(plots):
eq:=diff(x(t),t$2)+(x(t)-1)-x(t)*p*(Heaviside(t))^2=0;
eq1:=subs(p=1,eq);
ic:=x(0)=0,D(x)(0)=0;
sol1:=dsolve({eq1,ic},numeric);
list01:=["Times,Roman,10"];
odeplot(sol1,[t,x(t)],0..90,numpoints=1000,thickness=3,color = black, numpoints = 200, labels = ["non-dimensional
time", "motion amplitude"], labeldirections = [horizontal, vertical],labelfont=list01);
#p=.1 slow spin up. looks stable
eq2:=subs(p=2,eq);
sol2:=dsolve({eq2,ic},numeric);
odeplot(sol2,[t,x(t)],0..9,thickness=3,color = black, numpoints = 200, labels = ["non-dimensional time", "motion
amplitude"], labeldirections = [horizontal, vertical],labelfont=list01);
#p=2 fast spin up. instability predicted!
#here s=x,tau=t
restart;
with(DEtools):
with(plots):
eq:=diff(x(t),t$2)+(x(t)-1)+.1*(x(t)-1)^3-x(t)*p*(Heaviside(t))^2=0;
eq1:=subs(p=2,eq);
ic:=x(0)=0,D(x)(0)=0;
sol1:=dsolve({eq1,ic},numeric);
list01:=["Times,Roman,10"];
odeplot(sol1,[t,x(t)],0..90,numpoints=1000,thickness=3,color = black, numpoints = 200, labels = ["non-dimensional
time", "motion amplitude"], labeldirections = [horizontal, vertical],labelfont=list01);
```

2.

### #vibration absorber

```
restart;
with(plots):
with(DEtools):
eq1:=10*diff(x1(t),t,t)+2500*x1(t)-500*x2(t)+2*diff(x1(t),t)-100*sin(13.5*t);
eq2:=2.74*diff(x2(t),t,t)+500*x2(t)-500*x1(t)=0;
ic:=x1(0)=0,x2(0)=0,D(x1)(0)=0,D(x2)(0)=0;
sol1:=dsolve({eq1,eq2,ic},numeric,method=rkf45,maxfun=7000000);
list01:=["Times,Roman,10"];
pL:=odeplot(sol1,[t,x1(t)],0..1000,numpoints=1000,color = black, numpoints = 1000, labels = ["time", "main mass
amplitude"], labeldirections = [horizontal, vertical],labelfont=list01);
display(pL);
#ok x1 identically zero
#effect of nonlinearity
eq3:=10*diff(x1(t),t,t)+2500*x1(t)+2000*x1(t)^3-500*x2(t)+2*diff(x1(t),t)-100*sin(13.5*t);
sol2:=dsolve({eq3,eq2,ic},numeric,maxfun=10000000);
pN:=odeplot(sol2,[t,x1(t)],0..1000,numpoints=1000,color=black, labels = ["time", "main mass amplitude"],
labeldirections = [horizontal, vertical],labelfont=list01);
display(pN);
#plots[display]({pL,pN});
```

```
#still works even with a strong nonlinearity(transient is different)
#design quite robust
```

### 3.

#### #vibration cancellation

```
restart; with(linalg); with(DEtools); with(plots); digits := 20;
t1 := 2.1;
for i from 1 to 20 do;
eqc := diff(x(t), t, t)+100*x(t)+10*x(t)^3 = 1000*(Heaviside(t-c)-Heaviside(t-2));
ic := x(0) = 0, (D(x))(0) = 0;
eq2p5 := subs(c = t1, eqc);
sol2p5 := dsolve({ic, eq2p5}, numeric, method = rosenbrock);
odeplot(sol2p5, [t, x(t)], 0 .. 5, color = black, numpoints = 200, labels = ["time", "amplitude"], laheldirections =
[horizontal, vertical]);
t1 := t1+0.5e-1;
end do;
restart;
with(plots); with(plots); with(DEtools);
t1 := 2.3;
for i from 1 to 20 do;
eqc := diff(x(t), t, t)+100*x(t)+10*x(t)^3 = 100*(Heaviside(t-c)-Heaviside(t-2));
ic := x(0) = 0, (D(x))(0) = 0;
eq2p5 := subs(c = t1, eqc);
sol2p5 := dsolve({ic, eq2p5}, numeric, method = rosenbrock);
odeplot(sol2p5, [t, x(t)], 0 .. 5, color = black, numpoints = 200, labels = ["time", "amplitude"], laheldirections =
[horizontal, vertical]); t1 := t1+0.25e-1;
end do;
```