AC 2011-506: ASSESSMENT OF NAVIERSTOKES' EQUATIONS IN A FLUID MECHANICS COURSE

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DR. MYSORE NARAYANAN obtained his Ph.D. from the University of Liverpool, England in the area of Electrical and Electronic Engineering. He joined Miami University in 1980 and teaches a wide variety of electrical, electronic and mechanical engineering courses. He has been invited to contribute articles to several encyclopedias and has published and presented dozens of papers at local, regional , national and international conferences. He has also designed, developed, organized and chaired several conferences for Miami University and conference sessions for a variety of organizations. He is a senior member of IEEE and is a member of ASME, SIAM, ASEE and AGU. He is actively involved in CELT activities and regularly participates and presents at the Lilly Conference. He has been the recipient of several Faculty Learning Community awards. He is also very active in assessment activities and has presented dozens of papers at various Assessment Institutes. His posters in the areas of Bloom's Taxonomy and Socratic Inquisition have received widespread acclaim from several scholars in the area of Cognitive Science and Educational Methodologies. He has received the Assessment of Critical Thinking Award twice and is currently working towards incorporating writing assignments that enhance students' critical thinking capabilities.

Assessment of Navier–Stokes' Equations in a Fluid Mechanics Course

Abstract

Navier–Stokes' equations are indeed the foundation of fluid mechanics. Stokes' theorem is used in nearly every branch of mechanics as well as electromagnetics. Stokes' theorem also plays a vital role in many secondary theorems such as those pertaining to vortices and circulation. However, the divergence theorem is a mathematical statement of the physical fact that, in the absence of the creation or destruction of matter, the density within a region of space can change only by having it flow into, or away from the region through its boundary. This is also known as Gauss's theorem. At present, our students who take a fluid mechanics course in their junior or senior year are not learning about Stokes' theorem or Green's theorem or Gauss' theorem. They also do not have a background of vector differential calculus or vector integral calculus. The students are unable to calculate the *Gradient* of a scalar field. They are unaware of the need for determining the *Divergence* of a vector field, nor do they know the importance of calculating the *Curl* of vector field. The author wanted to correct these deficiencies and introduced the importance of these concepts to his students when he taught a course in hydraulics and fluid mechanics about 12 years ago. He also collected some assessment data and he presents an analysis of his findings in this paper.

Introduction

Many scholars believe that Navier–Stokes Equations and the Continuum Equations form the two giant cornerstones of modern day fluid dynamics. In the year 2000, the *Navier–Stokes Equation* was designated as a *Millennium Problem.* The solution for each *Millennium Problem* is worth a *Million Dollars.* Navier-Stokes Equation is one of seven mathematical problems selected by the Clay Mathematics Institute of Cambridge, Massachusetts for this special million-dollar award. It is essential to generate new techniques and modern mathematical methods for addressing and analyzing flow in complex fluids such as gels, suspensions, liquid crystals and foams. We all should however recognize the fact that classical fluid mechanics has been very successful in providing us with a quantitative understanding of turbulent flow such as shock waves.

Navier–Stokes equations are fundamentally derived by applying Newton's second law of motion to fluids. The principle is to assume that *fluid stress* is actually obtained by adding a viscous term and a pressure term. Navier–Stokes equations basically help dictate the velocity of fluid flow at any given point in space as well as time. Navier–Stokes equation is actually a generalization of the equations originally presented by Swiss mathematician Leonhard Euler (Born in 1707 in Basel, Switzerland). Euler derived a set of equations that govern the flow of incompressible and frictionless fluids. Later, in 1821, French physicist, mathematician and engineer Claude-Louis Navier went further to extend Euler's equations while including the element of viscous friction. Indeed this proved to be more useful because it would then be possible to describe the flow of fluids in a realistic environment. After this, during the nineteenth century, English physicist and mathematician Sir George Gabriel Stokes worked on improving these equations and finally arrived at a complete solution for simple two-dimensional fluid flows. Navier–Stokes' equations generate interest not only in an academic atmosphere, but also in an industrial and commercial environment. Whether it be modeling an airfoil for a jet engine or it be modeling the ocean currents for weather predictions or it be modeling a fabulous sports car, Navier–Stokes' equations are there to provide the engineer with an impressive strong mathematical foundation that is absolutely essential.

Mathematical Expressions

Consider an element of fluid that is described in a three dimensional space:

$\Delta x \times \Delta y \times \Delta z$

One can easily see that three types of forces act on this element.

The first is due to weight.

The second is due to pressure.

The third is due to viscosity.

The principle, now is to apply Newton's Second Law judiciously.

Force = Mass X Acceleration.

$$\Sigma F = ma = F_{weight} + F_{pressure} + F_{viscosity}$$

Given ρ as the fluid density, P as the pressure , u as the velocity of the fluid and ν as the kinematic viscosity, one can write the equation of incompressible fluid flow as:

$$\frac{\partial u}{\partial t} + u \bullet \nabla u = -\frac{\nabla P}{\rho} + \nu \nabla^2 u$$

Where ∇^2 is the Laplacian Operator.

Also, $-\nabla \mathrm{P}$ is the Pressure Gradient.

Unsteady acceleration is represented by $\dfrac{\partial \mathrm{u}}{\partial t}$

Convective acceleration is represented by $u \bullet \nabla u$

The viscosity term is represented by $\nu \nabla^2 u$

One should recall that Euler's original equation, in modern notation, is very much similar to the above equation.

$$\frac{\partial \mathrm{u}}{\partial t} + u \bullet \nabla u = -\frac{\nabla \mathrm{P}}{\rho}$$

Where ∇ is the vector operator, *'NABLA'* or *'DEL'* whose calculations are performed just like those with vectors. However, care must be taken to make sure that it is used as an *operator* and not just as a multiplying constant. More details about how to use the ∇ operator is provided in Appendix F.

Stokes' Theorem & Green's Theorem

Stokes' theorem is a very remarkable theorem because it is valid for *any* surface bounded by *any* given curve and furthermore, it can be of any shape. Mathematically expressed, Stokes' theorem can be described by considering a surface ***S*** having a bounding curve ***C.*** Here, V is any sufficiently smooth vector field defined on the surface and its bounding curve C. It is very important to emphasize the fact that **C** is *any* closed curve in *three dimensional* space and **S** is *any* surface bounded by the said curve **C**. Mathematically this is written as:

$$\boxed{\int_{s} (\nabla \times v) \bullet dS = \int_{c} v \bullet dx}$$

In addition, it is important to note that when one considers only a *two-dimensional* space, Stokes' theorem effectively becomes Green's theorem.

Another method of expressing Stokes' theorem is by considering a vector field **F** through a surface **r** that maps a region **S** to a surface **Σ.** Observe that the boundary of **S** is mapped to the boundary of **Σ.** Under such circumstances, one can mathematically express Stokes' theorem as:

$$\boxed{\iint_{\Sigma} Curl\,(F) \bullet dS = \oint_{\partial\Sigma} F \bullet dr}$$

In other words, the flux of the curl of **F** through **Σ** is equal to the circulation of **F** around **∂Σ.**

An alternate method to express the above equation will be:

$$\boxed{\oint F \bullet dr = \iint_{S} (\nabla \times F) \bullet n d\sigma}$$

Implementation

The students taking a course in fluid mechanics are juniors or seniors who have already completed two semesters of calculus and one semester of differential equations. They also have adequate background of statics, strength of materials, dynamics and thermodynamics. These students were first provided with a general background of partial differential equations. Later, the students were also introduced to basic principles of vector calculus such as dot products, cross products, Gradient, Divergence and Curl. Finally, they were provided with lectures on Buckingham Pi Theorem, Gauss' Theorem, Green's Theorem and Stokes' Theorem. The author believed that this was adequate and that the students were now ready to understand the importance and implications of Navier–Stokes Equations.

The author provided the students with three 50-minute lectures pertaining to above-mentioned topics. In addition, several worked out examples were given to the students by means of handouts. The author focused his attention on assessing and documenting the importance of Navier Stokes equations in an undergraduate fluid mechanics course. The students were quizzed on five topics in a 50-minute examination.

The author likes to acknowledge with thanks the importance of Washington State University's Critical Thinking Rubric. This rubric has played a very important role in conducting assessment. The author did not assign individual grade points. Instead, grading was holistic and the author utilized a rubric that is similar to *Washington State University's Critical Thinking Rubric* for purposes of assessment. This rubric is shown in Appendix B.

The author also recognizes that the complexity of each topic may pose some special problems for any instructor. Furthermore, each instructor's delivery style is different. It is also possible to arrive at two different sets of data for the same subject and topic. There are multiple variables, such as instructor delivery styles, diverse student body, different mathematical background and varied fundamental knowledge (Narayanan, 2007, 2009).

Assessment and Conclusions

After an analysis of the results gathered, the author is of the opinion that the students have shown genuine interest in learning partial differential equations. The students are capable of handling the principles of vector calculus and have a strong knowledge of dot products and cross products. Effective utilization of *MATLAB* has helped in this case. However, the students are having difficulty in understanding the other three topics, including Navier-Stokes' equations.

Appendix A show the procedure followed by the author for conducting assessment.

Appendix B outlines a set of rubrics used, courtesy of Washington State University.

Appendix C shows how holistic grading can be carried out using an Excel Spreadsheet.

Appendix D shows a bar chart analysis of the data collected.

Appendix E shows an analysis of data gathered.

Appendix F shows how the operator *NABLA* is used.

Appendix G shows how the *Laplacian operator* is derived.

Review of Differential Equations has recorded an excellent score of **5** on Likert scale. Students have taken a semester-long course in differential equations. Review exercises in this area have helped the students and it appears that they have a good understanding. The students are fairly comfortable with ODEs (Ordinary Differential Equations).

A Likert scale score of **3** may be acceptable in the next category of PDEs. However, attempts must be made to improve this to **5.** Students have shown to possess an adequate understanding of Partial Differential Equations. They seem to have a fairly good grasp of the basic principles of the topic in question. A Likert scale score of **4** will also be good start.

It seems that one lecture of 50 minutes' duration is not adequate for this topic. This is the first time the students are exposed to *Principles of Vector Calculus.* This is a new mathematical technique they have to learn and therefore one has to assume that there is plenty of room for improvement. This shows an unacceptable score of **2** on Likert scale. This must change, initially to **3** and later to **4.** It would be great to have a Likert scale score of **5,** however, this may be unrealistic.

The subject matter of Gradient, Divergence and Curl seems to pose some difficulty both for the instructor and for the students. A 50-minute lecture is not sufficient in this case. Students have difficulty in calculating and handling the necessary mathematics. The previous lecture, on the principles of vector calculus should have helped, however the instructor needs to change his teaching techniques. This, again shows an unacceptable score of **2** on Likert scale. We need to move towards **4.**

Finally, we arrive at the heart of this paper, *Navier-Stokes' Equations.* It is not surprising to see a score of **2** on Likert scale, because the necessary background is not strong. Principles of Vector Calculus, Gradient, Divergence and Curl all have scored **2** on Likert scale. This leads to the conclusion that unless the students are well trained to be very strong in partial differential equations and vector calculus, one can not expect them to understand the importance of *Navier-Stokes' Equations.*

Acknowledgements

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References:

1. Batchelor, G. K. (1967) *An Introduction to Fluid Dynamics.* Cambridge, United Kingdom: Cambridge University Press.

2. Arfken, G. (1985) *Gauss's Theorem. Mathematical Methods for Physicists, 3rd edition.* Orlando, FL: Academic Press.

3. Durran, D. R. (1999) *Numerical Methods for Wave Equations in Geophysical Fluid Dynamics.* New York, New York: -Springer-Verlag.

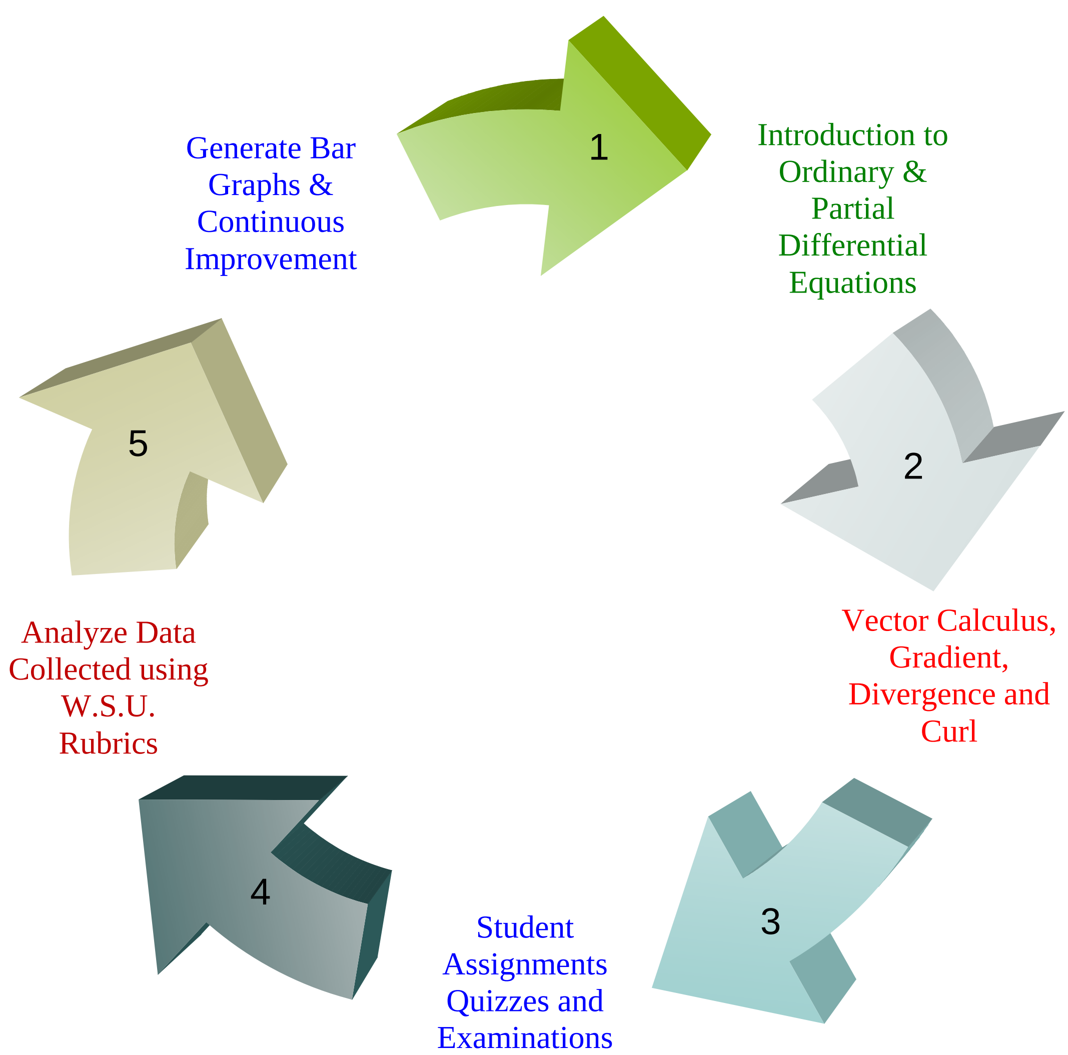
4. Narayanan, Mysore (2007) *Assessment of Perceptual Modality Styles.* Proceedings of ASEE National Conference, Honolulu, Hawaii.

5. Narayanan, Mysore (2009) *Assessment of Engineering Education based on the Principles of Theodore Marchese***.** Proceedings of ASEE National Conference, Austin, Texas..

6. Weisstein, Eric W. (2009) *Navier-Stokes Equations.* From *MathWorld*--A Wolfram Web Resource.

7. http://mathworld.wolfram.com/Navier-StokesEquations.html

APPENDIX A: Methodology used by the author.



The author has previously used this approach in other research and other ASEE publications.

APPENDIX B : Rubrics courtesy of W.S.U., Pullman, WA. (Narayanan, 2007).

	Rubrics based on Likert Scale	
5	Has demonstrated excellence. Has provided documentation. Evidence of critical thinking ability. Very good performance	Has analyzed important data precisely. Has answered key questions correctly. Has addressed problems effectively. Has evaluated material with proper insight. Has used deductive reasoning skills. Has used inductive reasoning skills. Has employed problem-solving skills. Has discussed consequences of decisions. Has been consistent with inference.
3	Has demonstrated competency. Adequate documentation. Critical thinking ability exists. Acceptable performance.	Data analysis can be improved. More effort to address key questions. Need to address problems effectively. Expand on evaluating material. Improve deductive reasoning skills. Improve inductive reasoning skills. Problem solving skills need honing. Must discuss consequences of decisions. Has been vague with inference.
1	Poor, unacceptable performance. Lacks critical thinking ability.	Absence of analytical skills. Answers questions incorrectly. Addresses problems superficially. Lacks documentation. Inability to evaluate material. Shows no deductive reasoning power. Inductive reasoning power non-existent. Poor problem solving skills Unaware of consequences of decisions. Unable to draw conclusions.

APPENDIX C : Matrix Generated using *Washington State University* Rubrics

Assessment Matrix

	Course: Hydraulics and Fluid Mechanics	A	B	C	.	.	.	.	X	Y	Z	MEDIAN	MODE	AVG.
	Washington State University RUBRICS LIKERT SCALE WEIGHT DISTRIBUTION: (1 : Strongly Disagree; 5 : Strongly Agree)													
1	Review of Differential Equations	5	5	5	.	.	.	.	5	4	5		5	
2	Introduction to Partial Differential Equations	3	3	3	.	.	.	.	3	4	3		3	
3	Principles of Vector Calculus	2	2	3	.	.	.	.	3	2	3		2	
4	Gradient, Divergence and Curl	2	2	3	.	.	.	.	2	2	3		2	
5	Navier-Stokes' Equations	2	4	3	.	.	.	.	2	2	3		2	

Example of how data can be collected.
Partial list of topics that can be observed.
Collected by Dr. Mysore Narayanan.

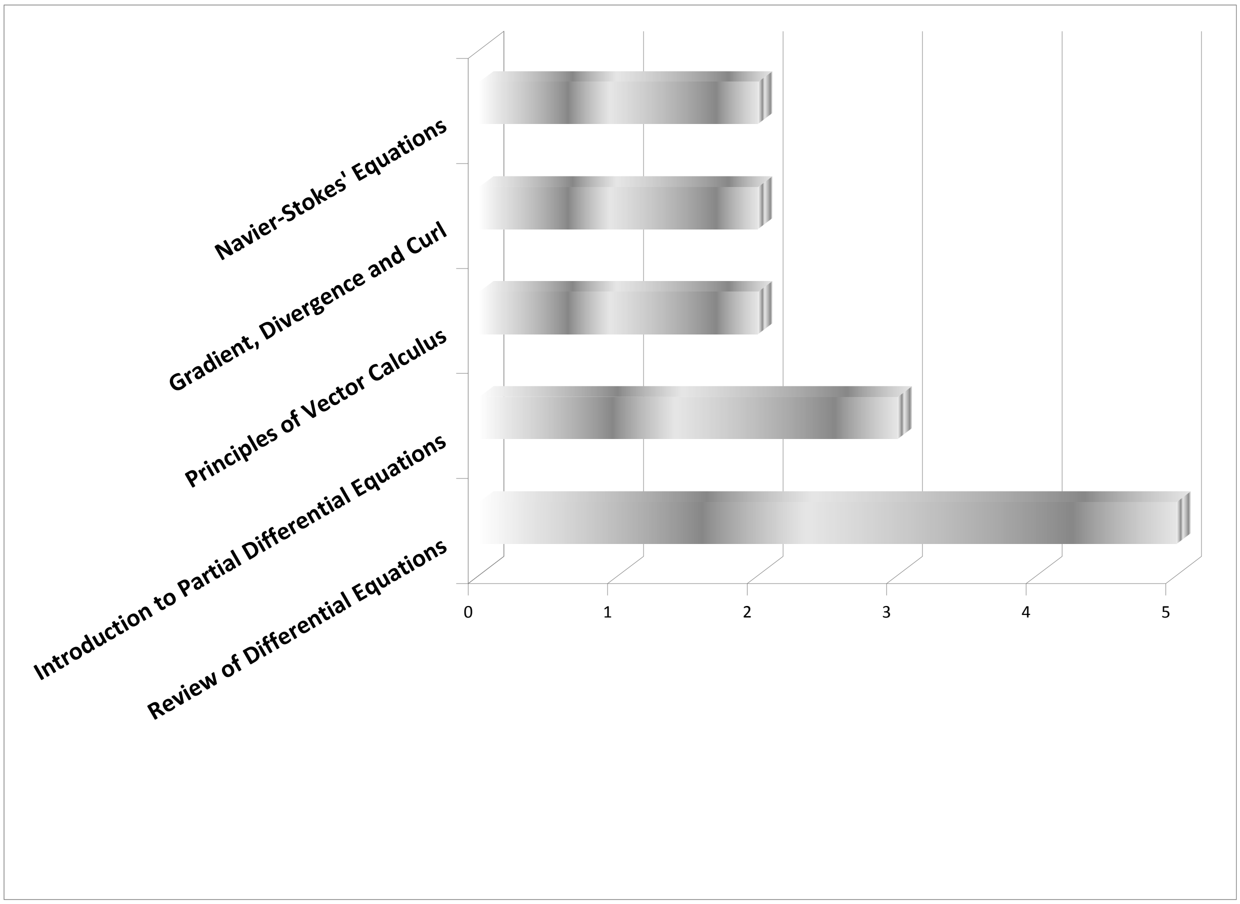
APPENDIX D : Bar Chart of Data Collected

Likert Scale Analysis

5: Strongly Agree. 1: Strongly Disagree

Rubrics courtesy of Washington State University, Pullman, WA.

Subject Studied: Fluid Mechanics



APPENDIX E : Analysis of the Bar Chart and Conclusions

Partial list of topics observed

1. **Review of Differential Equations:** Students have taken a semester-long course in differential equations. Review exercises in this area have helped the students and it appears that they have a good understanding. The students are fairly comfortable with ordinary differential equations. This shows an excellent score of **5** on Likert scale.

2. **Introduction to Partial Differential Equations:** Students have shown to possess an adequate understanding and have a fairly good grasp of the basic principles of partial differential equations. This shows a reasonable score of **3** on Likert scale. However, attempts must be made to move this to **4** at least.

3. **Principles of Vector Calculus:** It seems that one lecture of 50 minutes' duration is not adequate for this topic. This is the first time the students are exposed to this mathematical technique and therefore one has to assume that there is plenty of work remaining. This shows an unacceptable score of **2** on Likert scale. This must improve, initially to **3** and later to **4.**

4. **Gradient, Divergence and Curl:** A 50-minute lecture is not sufficient in this case. Students have difficulty in calculating and handling the necessary mathematics. The previous lecture, on the principles of vector calculus should have helped, however the instructor has to change his teaching techniques. This, again shows an unacceptable score of **2** on Likert scale. We need to move towards **4.**

5. **Navier-Stokes' Equations:** It is not surprising to see a score of **2** on Likert scale, because the background is not strong. Principles of Vector Calculus, Gradient, Divergence and Curl have scored **2** on Likert scale. This leads to the conclusion that unless the students are very strong in partial differential equations and vector calculus, one can not expect them to understand the importance of Navier-Stokes' Equations.

APPENDIX F : THE NABLA OPERATOR (THE DEL OPERATOR)

NABLA or DEL ∇ is a vector *operator*.

Since it is an *operator*, one should not just consider it as a multiplication factor.

Calculations with DEL are performed just like those with vectors.

The vector *operator* ∇ is defined as:

$$\boxed{\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$$

When the vector *operator* ∇ is operated on a scalar quantity, it is called *Gradient.*

$$\boxed{\nabla S = \hat{x}\frac{\partial S}{\partial x} + \hat{y}\frac{\partial S}{\partial y} + \hat{z}\frac{\partial S}{\partial z}}$$

Mathematically speaking, *Gradient* is actually a *vector* operation.

This vector operates on a *scalar* function. The *Gradient* can be applied to *scalars only.*

The result is to produce a *vector.*

The magnitude of the *vector* describes the rate of change of the function or the rate of ascent or descent.

The direction of the *vector* provides the direction of the steepest ascent or descent.

For example, one may find it necessary to determine the temperature *gradient* across a large piece of metal object as a part of certain heat transfer calculations.

When the vector *operator* ∇ is operated on a vector quantity and a *scalar product* is obtained, it is called *Divergence.*

$$\boxed{\nabla \bullet v = \left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right] \bullet \left[\hat{x}v_x + \hat{y}v_y + \hat{z}v_z\right]}$$

$$\boxed{\nabla \bullet v = \left[\hat{x}\frac{\partial v_x}{\partial x} + \hat{y}\frac{\partial v_y}{\partial x} + \hat{z}\frac{\partial v_z}{\partial x}\right]}$$

When the vector *operator* ∇ is operated on a *vector* quantity, and a *cross product* is obtained, it is called *Curl.* Curl basically defines the *rotation* of a given vector field. Right hand rule must be used to find the direction of the axis of rotation.

$$\boxed{\nabla \times v = \left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right] \times \left[\hat{x}v_x + \hat{y}v_y + \hat{z}v_z\right]}$$

An easier method of expressing the same would be in the *matrix form.*

$$\boxed{\nabla \times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}}$$

The matrix can be evaluated to yield:

$$\boxed{\nabla \times v = \hat{x}\left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right] + \hat{y}\left[\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right] + \hat{z}\left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right]}$$

APPENDIX G : DERIVATION OF THE LAPLACIAN OPERATOR

Some Important Observations:

1. It is possible to calculate the *Gradient* of a product of two *scalars*.

$$\nabla (ab) = a\nabla b + b\nabla a$$

2. *Divergence* can be applied to two *vectors*.

$$\nabla . (\mathbf{A} \times \mathbf{B}) = \mathbf{B} . (\nabla \times \mathbf{A}) - \mathbf{A} . (\nabla \times \mathbf{B})$$

3. *Curl* can also be applied to two *vectors*.

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} . \nabla)\mathbf{A} - (\mathbf{A} . \nabla)\mathbf{B} + \mathbf{A} (\nabla . \mathbf{B}) - \mathbf{B} (\nabla . \mathbf{A})$$

4. *Divergence* of *Gradient* yields the *Laplacian* Operator, ∇^2

$$\boxed{\nabla \bullet (\nabla S) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \bullet \left(\hat{x}\frac{\partial S}{\partial x} + \hat{y}\frac{\partial S}{\partial y} + \hat{z}\frac{\partial S}{\partial z} \right)}$$

$$\boxed{\nabla \bullet (\nabla S) = \left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} \right)}$$

$$\boxed{\nabla \bullet (\nabla S) \equiv \nabla^2 S}$$

5. *Curl* of *Gradient* is *zero.*

$$\boxed{\nabla \times (\nabla S) \equiv 0}$$

6. Observe that *Gradient* of *Divergence* will ***not*** yield the *Laplacian* Operator.

$$\boxed{\nabla(\nabla \bullet v) \neq \nabla^2 v}$$

7. Also, observe that *Curl* can be applied to two vectors.

$$\mathbf{\nabla \times (A \times B) = (B \,.\nabla)\, A - (A \,.\nabla)\, B + A\, (\nabla.B) - B\, (\nabla.\, A)}$$