Breguet's Formulas for Aircraft Range & Endurance
An Application of Integral Calculus

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Introduction

At the United States Military Academy, faculty attempt to expose cadets to highly integrated learning scenarios. In an effort to reinforce that the world is not compartmentalized similar to the academic environment, the Department of Mathematical Sciences conducts sessions known as Interactive Lively Applications Projects (ILAPs). In particular, the engineering curriculum provides an excellent base of support to highlight the applications of single and multivariable calculus.

This paper focuses on one such ILAP using the Breguet range and endurance equations as the foundation for some insight into the physical significance of integral calculus. In a recent semester, members of the Department of Civil & Mechanical Engineering sponsored an ILAP where cadets learned how integral calculus supports airplane design.

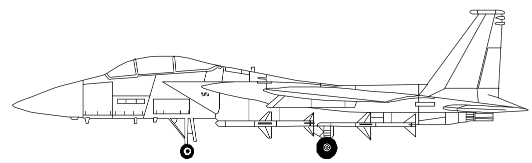

Figure 1. Typical US Air Force Airplane

Aircraft Performance Parameters

Knowledge of the requisite aircraft velocity for maximum range and maximum endurance operations is essential for air crews to optimize flight performance. Engineers focus on maximum range and maximum endurance characteristics to optimize the aircraft design for a specific mission. This information is also pivotal in the marketing of the aircraft. The need to understand maximum range and maximum endurance characteristics of an aircraft cannot be understated. Specific examples of aircraft designed for these conditions include Burt Rutan's Voyager (maximum range), Unmanned Aerial Vehicles (maximum endurance), U-2 reconnaissance aircraft (maximum range and altitude), and the P-3 Orion submarine hunter (maximum endurance).

At the most basic level, the range and endurance of an aircraft depend directly on the quantity of fuel available and the rate at which the fuel is consumed per distance traveled or per hour in the air. Given an unlimited amount of fuel, any aircraft could fly continuously until, of course, the aircraft experiences a failure. This is the concept of aerial refueling used by the U.S. Air Force. However, fuel availability for practical aircraft is limited. Therefore, the characteristics of the aircraft and the environment that affect the maximum range and endurance must be found. Such a situation involving several variables is a natural application of multivariable calculus. In addition, the total range and total endurance are functions of the aircraft weight, which is changing continuously as fuel is consumed.

Table 1. Basic Parameter Definitions.

c = Specific Fuel Consumption =	$\dfrac{\text{lbs of fuel used}}{\text{lbs of thrust * hr}}$
Non-dimensional Lift Coefficient:	$C_L = \dfrac{Lift}{\frac{1}{2}\rho V^2 S}$
Non-dimensional Drag Coefficient:	$C_D = \dfrac{Drag}{\frac{1}{2}\rho V^2 S}$
Where: S = Wing Planform Area	ρ = Air Density

In an introductory calculus or engineering course, a simple model of the problem is most appropriate. Accordingly, several assumptions help to simplify the range and endurance problem while retaining the physical significance. This ILAP included the following assumptions:

1. For a particular operating point, the aircraft consumes fuel at a constant rate per pound of jet thrust or per engine horsepower. This rate is known as the Specific Fuel Consumption (SFC), and this example highlights the thrust of a turbine aircraft. The same type analysis applies to propeller driven aircraft.

2. The drag coefficient varies as the square of the lift coefficient. A parabolic drag relationship assumes that the skin friction drag coefficient between the air and the aircraft surface is constant and the non-dimensionalized drag created by the lift force depends quadratically on the non-dimensionalized lift. The equation for what is commonly referred to as a parabolic drag polar becomes $C_D = C_{D_0} + k * C_L^2$.

3. The flight profile for maximum range and maximum endurance requires the aircraft to be in a cruise configuration. While in cruise, an aircraft is in level or near-level flight. This means that the lift vector is equal and opposite to the weight vector. Also, the thrust and drag vectors are equal and opposite, and both are normal to the lift (Figure 2). Using the relationships shown in Table 1, this assumption establishes the relationship between the aircraft velocity and weight, $V = \sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_L}}$.

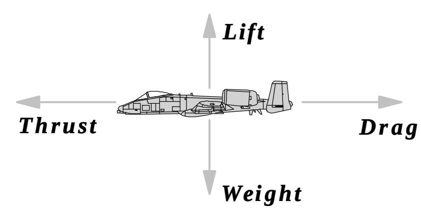

Figure 2. Cruise configuration

Such assumptions for the mathematical model permit insight into the real world problem without significantly altering the problem. The engineering application then also serves to demonstrate the utility of calculus and motivates cadets to better understand the importance of calculus. Ultimately, cadets learn that the world is an integrated environment where they must draw on their knowledge of many subjects in order to solve an actual problem.

The Breguet Equations

Once the simplifying assumptions have been made, the basis of this problem is the change of the aircraft weight over the change in time. The weight of the aircraft decreases by the weight of the burned fuel. Stated mathematically, using distance and then time as the independent variable, one obtains:

$$\frac{dx}{dW} = -\frac{V}{c}\frac{C_L}{C_D}\frac{1}{W} \qquad \frac{dt}{dW} = -\frac{1}{c}\frac{C_L}{C_D}\frac{1}{W}$$

Combining these physical facts with other known relations shown in Table 1, one derives the well known Breguet equations for range and endurance. These equations are differential in form and relate the change of aircraft weight to other aircraft parameters. The range and endurance of the aircraft depends on the sum of these small differential weight changes. Thus, integrating these equations, one finds the total range and total endurance.

$$X_{destination} - X_{takeoff} = Range = \int_{W_{TO}}^{W_{Land}} \frac{V}{c} \frac{C_L}{C_D} \frac{1}{W} dW$$

$$T_{destination} - T_{takeoff} = Endurance = \int_{W_{TO}}^{W_{Land}} \frac{1}{c} \frac{C_L}{C_D} \frac{1}{W} dW$$

At this point, there are several variables inside of the integral sign and the relationship of each of these variables to weight must be defined before the integration can be completed. The most appropriate assumption is to hold two of the three critical parameters (ρ, V, C_L/C_D) constant over the range of the integration. Each mathematical assumption has a physical analog, and these analogies are shown in Table 2. The need to hold two variables constant is a mathematical, not a physical requirement. While the air crew can operate with exactly these variables held constant, such conditions are not mandatory. For example, the air crew can vary the aircraft's velocity, altitude and angle of attack simultaneously. If, however, the air crew varies only one parameter, the correlation between the physical results and the mathematical model is very good.

Table 2. Range and Endurance parameters

Physical interpretation	Mathematical interpretation
constant angle of attack	constant C_L/C_D
constant altitude	constant ρ
constant velocity	constant V

Table 3. Range and Endurance Solutions

<table>
<tr><td>For constant altitude (ρ) and lift coefficient (C_L):
$$Range = \frac{1}{c}\frac{2\sqrt{2}}{\sqrt{\rho S}}\frac{\sqrt{C_L}}{C_D}\left(\sqrt{W_o} - \sqrt{W_1}\right)$$
$$Endurance = \frac{1}{c}\frac{C_L}{C_D}\ln\frac{W_o}{W_1}$$</td><td>For constant velocity (V) and lift coefficient (C_L):
$$Range = \frac{V}{c}\frac{C_L}{C_D}\ln\frac{W_o}{W_1}$$
$$Endurance = \frac{1}{c}\frac{C_L}{C_D}\ln\frac{W_o}{W_1}$$</td></tr>
<tr><td colspan="2">For constant velocity (V) and constant altitude (ρ):
$$Range = \frac{V}{c\sqrt{kC_{D_o}}}\left[\tan^{-1}\frac{\sqrt{k}}{\frac{1}{2}\rho V^2 S\sqrt{C_{D_o}}}W_o - \tan^{-1}\frac{\sqrt{k}}{\frac{1}{2}\rho V^2 S\sqrt{C_{D_o}}}W_1\right]$$
$$Endurance = \frac{1}{c}\frac{1}{\sqrt{kC_{D_o}}}\tan^{-1}\left[\frac{1}{\frac{2}{\left(1-W_1/W_o\right)}-1}\right]$$</td></tr>
<tr><td>Where: W_0 = Takeoff Weight</td><td>W_1 = Landing Weight</td></tr>
</table>

The constant parameter assumption allows the engineer to obtain an analytical solution to this problem. Since there are three parameters, there are three combinations of holding two parameters constant. The three mathematical solutions are shown in Table 3. By themselves, these equations contain little significance to an air crew or an aerospace engineer. However, the physical interpretations of the mathematical solutions provide significant information of use. For example, in the case of constant velocity and constant lift coefficient, the air crew has learned that the aircraft must climb during its cruise phase in order to maximize range. This occurs because as the weight decreases, the density must also decrease (gain in altitude) so that the velocity remains constant.

Velocities for Maximum Range and Endurance

Another exercise at this point is to solve for the velocity that maximizes the range and the velocity that maximizes the endurance. This exercise can be accomplished graphically or by taking the derivatives of the range and endurance with respect to velocity.

At this point, the cadets are assigned a small project that requires them to use their new knowledge to solve a realistic and exciting problem.

Voyager Design Example

Engineering is not about the analysis of existing products and processes. Engineering is the creation of products and processes that have never existed before. Let us, therefore, consider the design of an aircraft that will fly around the world on one tank of gas (unrefueled). It is possible to use the information previously presented to gain a "back of the envelope" perspective on the feasibility of such an aircraft. We must make some assumptions based on current flying aircraft. This example will be a propeller driven aircraft which changes some of the parameters, but provides the same general method as the jet considered above. Using the parameters shown in Table 4 and the Breguet Range Equation for a propeller driven aircraft (shown to the right), one can determine the ratio of fuel to takeoff weight.

Table 4. "Around the World" Aircraft Parameters

Range	25,000 miles
Specific Fuel Consumption, c	0.41 lb./(HP hr)
Propeller efficiency, η	0.85
C_L/C_D	27

$$W_1 = \frac{W_0}{\exp\left(\frac{R(SFC)}{\eta}\frac{C_D}{C_L}\right)}$$

The result of this calculation is that 70% of the weight of the aircraft at takeoff must be fuel to complete the mission. Now, the engineer can ask the question whether this requirement is reasonable or within the realm of possibility. The answer to this question lies in the feasible ratio of structural weight to takeoff weight. If a 10,000 pound aircraft is nominally considered, when the weight of the engines, landing gear, crew, avionics, supplies, and miscellaneous equipment are considered, only about 1,000 pounds is left

Table 5. Structural Weight Ratios

Vehicle	Total Weight (lb.)	Structural Wt. (lb.)	Ratio
Bicycle	220	20	11.0
Around the World Aircraft	10,000	1,000	10.0
Hang Glider	294	64	4.6
Boeing 747	873,000	195,000	4.5
A-10A	45,560	10,200	4.5
Learjet	16,500	5,500	3.3
Sail plane	1,521	440	3.5

for the structure, or 10% of the takeoff weight. Table 5 lists some structural to total weight ratios for some common vehicles.

Now, the designer knows the requirements for a successful design. Burt Rutan understood the importance of the structural weight ratio and designed an aircraft that had the highest takeoff to structural weight ratio of any aircraft ever designed. The Voyager had a takeoff weight of 9695 pounds and a structural weight of 939 pounds for a structural ratio of 10.3.

Summary

The ILAP learning experience provides both cadets and faculty with a unique opportunity to combine their learning and instruction of mathematical principles with physical aspects of engineering. In educational institutions, learning is often compartmentalized. Actual problems are not limited by academic disciplines. Cadets must learn to apply concepts across the spectrum of disciplines to solve actual problems.

The use of the Breguet equations to demonstrate the application of multivariable integral Calculus enhances the cadets' understanding of both calculus and engineering. In addition, the ILAP provides exposure to engineering for cadets early in their learning experience. This example demonstrated how engineers use Calculus as a language and tool for analysis and design.

Additionally, the ILAP provides physical meaning to the learning of calculus. Use of generic variables such as $\int xyz \; dx$ contributes marginally to the learning and application of calculus. When compared to an actual engineering problem such as described above, cadets gain a better appreciation for the utility of Calculus and better retain the subject matter.

Feedback from cadets through survey forms and faculty responses indicate that the ILAP is a positive learning experience for the cadets. Use of specific engineering applications in introductory calculus courses benefits everyone involved. Successive ILAPs have used various vibration engineering problems as a similar foundation for demonstrating the power of this approach to learning calculus. The attempt to integrate cadet learning across multiple disciplines continues. The Department of Civil and Mechanical Engineering stands ready to conduct similar exercises with the Department of Mathematical Sciences in the future.

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