AC 2009-2517: CALCULATION OF THE GENERAL IMPEDANCE BETWEEN ADJACENT NODES OF INFINITE UNIFORM N-DIMENSIONAL RESISTIVE, INDUCTIVE, OR CAPACITIVE LATTICES

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Calculation of the general impedance between adjacent nodes of infinite uniform *N*-dimensional resistive, inductive, or capacitive lattices

Introduction

All undergraduate Electrical Engineering (EE) students study basic electric circuit theory where they get introduced to the fundamental concepts and principles of electric circuits (e.g., Ohm's law, Kirchhoff's circuit laws, superposition, symmetry, and the concept of equivalent circuits). To help them better understand and gain the ability to apply these concepts and principles, the students are exposed to numerous and relatively straightforward electric circuit problems involving impedance networks comprised of a finite number of circuit elements. In addition, some more complex examples of electric circuits with a special configuration are used to provoke student's motivation and curiosity and to engage them in the subject matter. One such well-known problem is finding the effective resistance seen between opposite corners of a cubic resistor network[1] comprised of twelve equal resistors each of value R resulting in an effective resistance given by $R_{eff} = 5R/6$. Another well-known problem is finding the effective resistance of the one-dimensional, semi-infinite resistive-ladder network[1] comprised of equal resistors each of value R which is found to be $R_{eff} = R(1+5^{0.5})/2$.

Although two-dimensional (2D) network problems are generally considered to be more difficult and challenging for undergraduate EE education, there are exceptions to this view. For example, Aitchison[2] considered the problem of finding the effective resistance between any two adjacent nodes of the infinite uniform 2D Liebman resistive mesh (i.e., the infinite 2D square resistive lattice comprised of identical resistors each of value R) and, using a simple solution based on superposition and symmetry, he found it to be $R_{eff} = R/2$. Aitchison pointed out that, in general, undergraduate EE students found this class of problems with such solutions to be more instructive and satisfying, rather than exclusively analyzing simpler, standard circuit problems. Similarly, again using superposition and symmetry, Bartis[3] calculated the effective resistance between any two adjacent nodes of three other cases of infinite 2D resistive lattices including the triangular, Honeycomb and Kagomé lattices for which the effective resistances were found to be $R/3$, $2R/3$, and $R/2$, respectively. Bartis provided these problems as counter-examples to those who believe that such complex resistive network problems can not be treated at the elementary level. He also elaborated that these types of problems help teach the students how to approach more complex network problems and tackle their solutions. Indeed, these more complex infinite resistive lattice problems can serve as excellent pedagogical vehicles for teaching and motivating EE students to appreciate the power of superposition and symmetry in electrical circuit analysis. (Note that this class of 2D infinite resistive lattice problems was subsequently extended to the much more difficult calculation of the effective resistance between any two arbitrary, non-adjacent nodes using more advanced mathematical techniques including finite difference equations, random walks, and the lattice Green's function[4-9]. However, the authors consider this class of problems [4-9] too advanced and, therefore, well beyond the reach of undergraduate EE students.)

Recently, the authors observed that the straightforward analysis of the Aitchison and Bartis class of 2D infinite resistive lattice circuit problems[2,3], above, can be extended to the most general case of calculating the effective impedance between any two adjacent nodes of any infinite uniform N-dimensional resistive, inductive, or capacitive lattice (where N = 1, 2, or 3), and remain within the reach of undergraduate EE students using familiar and fundamental concepts including Ohm's law, Kirchhoff's laws, superposition, and symmetry. This most general class of network problems clearly and simply demonstrates the power of superposition and symmetry in solving infinite R, L, or C network problems. (For reference, Figs. 1 and 2 show the infinite 2D square and Honeycomb lattices, respectively. In each figure, all branches are identical resistors, inductors, or capacitors with equal values of R, L, or C, respectively.) The authors recently published the calculation of the total effective resistance R_{eff} between any two adjacent nodes of any infinite uniform N-dimensional resistive lattice, where $N = 1, 2$, or 3, and the lattice is identically periodic and infinite in extent in all N dimensions with a zero-potential boundary condition at infinity[10]. In this publication, they formally derived the general expression for the effective resistance, R_{eff}, of the infinite resistive lattice case, and briefly included (without proof) the similar general expressions for L_{eff} and C_{eff} for the infinite inductive and capacitive lattice cases, respectively.

The goal of this paper is to expand the authors' previous work[10] to the full detailed analysis of the most general problem of finding the total effective impedance (R_{eff}, L_{eff} or C_{eff}) between any two adjacent nodes of any infinite uniform N-dimensional resistive, inductive, or capacitive lattice, where $N = 1, 2$, or 3, and the lattice is identically periodic and infinite in all N dimensions with a zero-potential boundary condition at infinity. Using the very familiar and fundamental undergraduate-level electrical engineering principles of superposition and symmetry along with Ohm's law for a resistor, or the magnetic flux/current relationship for an inductor, or the electrical charge/voltage relationship for a capacitor, a general and easy-to-remember solution is obtained for each of these cases given by: $R_{eff} = 2R/M$, $L_{eff} = 2L/M$, and $C_{eff} = MC/2$, where M is the total number of elements connected to any particular node of the lattice. These most general solutions are of significant pedagogical interest to electrical engineering students since they generalize the previous specific works on this topic into three simple equations. Additionally, they are excellent illustrations of the elegance of superposition and symmetry to simply understand and solve such general problems, while keeping the analysis simple, fundamental and within the reach of undergraduate students. The authors have recently experimented with adding circuit problems of this type to their first-semester undergraduate EE circuits course with good success. Therefore, the authors propose that this class of problems be considered for formal addition to the introductory undergraduate EE curriculum. We will now present the detailed analysis of the infinite uniform N-dimensional resistive lattice case, followed by the inductive and capacitive lattice cases.

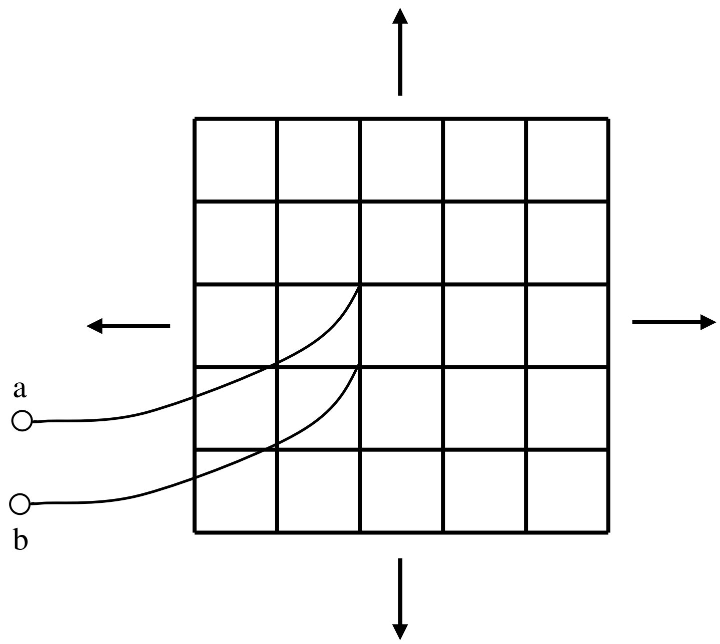

Fig. 1. Infinite 2D square lattice

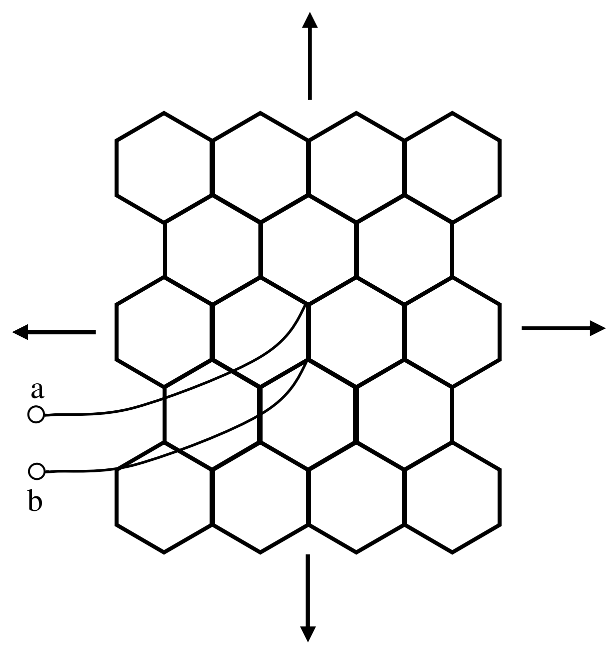

Fig. 2. Infinite 2D Honeycomb lattice

Case 1: Infinite N-dimensional Resistive Lattice

For the purpose of illustration, consider the two-port test circuit shown in Fig. 3 containing the infinite 2D square resistive lattice shown in Fig. 1. In the infinite resistive lattice, each branch corresponds to a single resistor of value R and the number of resistors connected to each node is denoted by M (where $M = 4$ in Fig. 1). Based on Fig. 3, we use superposition, symmetry, and Ohm's law along with two identical test current sources ($I_a = I_b = I$) to calculate the effective resistance R_{eff} seen between any two adjacent nodes. First, we inject a current $I_a = I$ (supplied by a test current source connected to the zero-potential boundary at infinity) into any single node "a" on the N-dimensional resistive lattice. Then we extract another test current $I_b = I$ (supplied by a second identical test current source flowing in the opposite direction) out of the lattice from node "b", adjacent to node "a". Next, we calculate the total resulting voltage $V_{a\text{-}b}$ due to both test current sources using Ohm's law, $V_{a\text{-}b} = IR_{eff}$. Finally, we determine the effective resistance as $R_{eff} = V_{a\text{-}b}/I$. By using Kirchhoff's current law and symmetry, we find that each of the M resistors connected to node "a" will receive I/M portion of the injected current I flowing away from node "a". Similarly, each of the M resistors connected to the adjacent node "b" will receive I/M portion of the extracted current I flowing towards node "b". Using superposition and Ohm's law, the total voltage, $V_{a\text{-}b}$, across the resistor R connected between nodes "a" and "b" in the test circuit in Fig. 3 is:

$$\begin{aligned} V_{\text{a-b}} &= \left(\frac{I_a}{M}\right)R + \left(\frac{I_b}{M}\right)R \\ &= 2\left(\frac{I}{M}\right)R \\ &= R_{eff}\, I \end{aligned} \tag{1}$$

And, therefore, the effective resistance between any two adjacent nodes "a" and "b" of an infinite resistive lattice is:

$$\begin{aligned} R_{eff} &= \frac{V_{\text{a-b}}}{I} \\ &= \frac{2R}{M} \end{aligned} \tag{2}$$

Equation (2) is a new and remarkably simple, elegant, and powerful result that applies to any infinite uniform N-dimensional resistive lattice. For example, for an infinite 2D Honeycomb resistive lattice (see Fig. 2) where $M = 3$, the effective resistance between any two adjacent nodes is simply $R_{eff} = 2R/3$, where R is the value of each resistor on each branch of the lattice. Similarly, $M = 6$ for an infinite 3D cubic resistive lattice and, therefore, $R_{eff} = 2R/6 = R/3$.

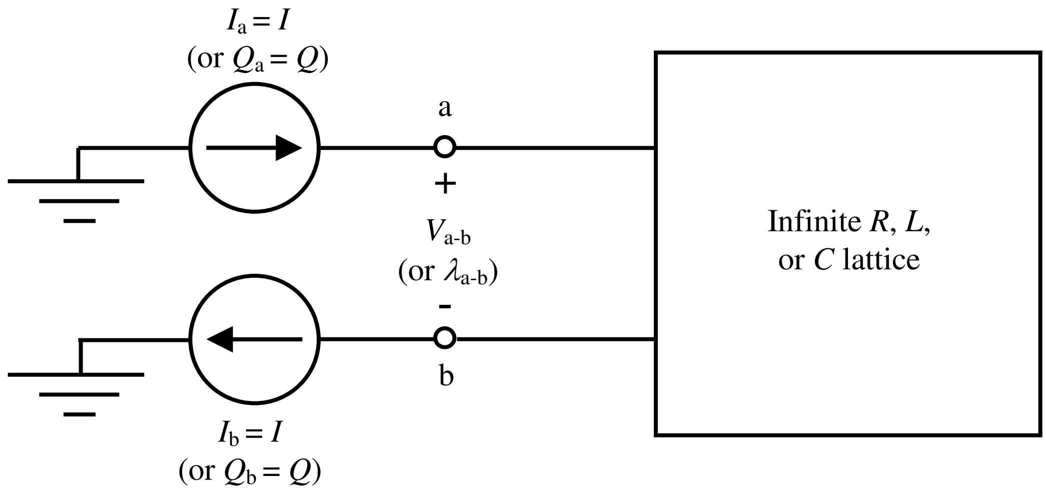

Fig. 3. Test circuit used to calculate R_{eff}, L_{eff}, or C_{eff}

Case 2: Infinite N-dimensional Inductive Lattice

Similar to Case 1, consider the two-port test circuit shown in Fig. 3 containing the infinite 2D square inductive lattice shown in Fig. 1. In the infinite inductive lattice, each branch corresponds to a single inductor of value L and the number of inductors connected to each node is denoted by M (where $M = 4$ in Fig. 1). Based on Fig. 3, we use superposition, symmetry, and the magnetic flux/current relationship for an inductor ($\lambda_{a\text{-}b} = L_{eff} I$) along with two identical test current sources ($I_a = I_b = I$) to calculate the effective inductance L_{eff} seen between any two adjacent nodes. First, we inject a current $I_a = I$ into any single node "a" on the N-dimensional inductive lattice. Then we extract another test current $I_b = I$ out of the lattice from an adjacent node "b". Next, we calculate the resulting total magnetic flux due to both sources, $\lambda_{a\text{-}b} = L_{eff} I$. By using Kirchhoff's current law and symmetry, we find that each of the M inductors connected to node "a" will receive I/M portion of the injected test source current I flowing away from node "a". Similarly, we observe that each of the M inductors connected to the adjacent node "b" will receive I/M portion of the extracted current flowing towards node "b". Using superposition and the magnetic flux/current relationship for an inductor, the total magnetic flux linking the inductor L connected between nodes "a" and "b", $\lambda_{a\text{-}b}$, in the test circuit in Fig 3 is:

$$\begin{aligned}\lambda_{a\text{-}b} &= \left(\frac{I_a}{M}\right)L + \left(\frac{I_b}{M}\right)L \\ &= 2\left(\frac{I}{M}\right)L \\ &= L_{eff} I\end{aligned} \tag{3}$$

And, therefore, the effective inductance between any two adjacent nodes "a" and "b" of an infinite inductive lattice is:

$$
\begin{aligned}
L_{\text{eff}} &= \frac{\lambda_{\text{a-b}}}{I} \\
&= \frac{2L}{M}
\end{aligned}
\tag{4}
$$

Similar to Equation (2), Equation (4) is a new and simple result that applies to any infinite uniform N-dimensional inductive lattice. Therefore, for an infinite 2D Honeycomb inductive lattice, the effective inductance between any two adjacent nodes is $L_{\text{eff}} = 2L/3$, where L is the value of each inductor on each branch of the lattice. Similarly, for an infinite 3D cubic inductive lattice, $L_{\text{eff}} = L/3$.

Case 3: Infinite N-dimensional Capacitive Lattice

Again, similar to Case 1, consider the two-port test circuit shown in Fig. 3 containing the infinite 2D square capacitive lattice shown in Fig. 1. In the infinite capacitive lattice, each branch corresponds to a single capacitor of value C and the number of capacitors connected to each node is denoted by M (where $M = 4$ in Fig. 1). Based on Fig. 3, we use superposition, symmetry, and the electrical charge/voltage relationship for a capacitor ($Q = C_{\text{eff}}V_{\text{a-b}}$) along with two identical test charge sources ($Q_{\text{a}} = Q_{\text{b}} = Q$) to calculate the effective capacitance C_{eff} between any two adjacent nodes "a" and "b". First, we inject a test charge $Q_{\text{a}} = Q$ into any single node "a" on the N-dimensional capacitive lattice from the zero-potential boundary at infinity. Then we extract an identical test charge $Q_{\text{b}} = Q$ from an adjacent node "b" out of the lattice to the zero-potential boundary at infinity. Next, we calculate the resulting total voltage due to both sources, $V_{\text{a-b}} = Q/C_{\text{eff}}$. By using symmetry, we find that each of the M capacitors connected to node "a" will receive Q/M portion of the injected test charge Q. Similarly, we find that each of the M capacitors connected to the adjacent node "b" will receive Q/M portion of the extracted test charge Q. Using superposition and the electrical charge/voltage relationship for a capacitor, the total voltage, $V_{\text{a-b}}$, across the capacitor C connected between adjacent nodes "a" and "b" in the test circuit in Fig. 3 is:

$$
\begin{aligned}
V_{\text{a-b}} &= \frac{(Q_{\text{a}}/M)}{C} + \frac{(Q_{\text{b}}/M)}{C} \\
&= 2\frac{(Q/M)}{C} \\
&= \frac{Q}{C_{\text{eff}}}
\end{aligned}
\tag{5}
$$

And, therefore, the effective capacitance between any two adjacent nodes "a" and "b" of an infinite capacitive lattice is:

$$
\begin{aligned}
C_{\text{eff}} &= \frac{Q}{V_{\text{a-b}}} \\
&= \frac{MC}{2}
\end{aligned}
\tag{6}
$$

Again, similar to Equation (4), Equation (6) is also a new and simple result that applies to any infinite uniform N-dimensional capacitive lattice. Therefore, for an infinite 2D Honeycomb capacitive lattice, the effective capacitance between any two adjacent nodes is $C_{\text{eff}} = 3C/2$, where C is the value of each capacitor on each branch of the lattice. Similarly, for an infinite 3D cubic capacitive lattice, $C_{\text{eff}} = 3C$.

Conclusion

This paper extended previous work[2,3,10] by presenting the full detailed analysis and results for the most general problem of finding the total effective impedance (R_{eff}, L_{eff}, or C_{eff}) between any two adjacent nodes of any infinite uniform N-dimensional resistive, inductive, or capacitive lattice, where $N = 1$, 2, or 3, and the lattice is identically periodic and infinite in extent in all N dimensions with a zero-potential boundary condition at infinity. Using the basic principles of Kirchhoff's laws, superposition and symmetry, along with Ohm's law for a resistive lattice, or the magnetic flux/current relationship for an inductive lattice, or the electrical charge/voltage relationship for a capacitive lattice, a general, elegant, and easy-to-remember solution is determined for each of these cases given as follows: $R_{\text{eff}} = 2R/M$, $L_{\text{eff}} = 2L/M$, and $C_{\text{eff}} = MC/2$, where M is the total number of elements connected to any particular node of the lattice. As previously stated, these general solutions are of significant pedagogical interest to undergraduate EE students since they generalize the previous specific works on this topic[2,3] to three simple equations. The analysis of these infinite R, L, or C lattice networks are easily within the reach of undergraduate students due to their reliance on fundamental laws and principles of electrical engineering. They also serve as excellent illustrations of the power and simplicity of superposition and symmetry used with appropriate fundamental physical principles to facilitate more intuitive understanding and simpler solution of such general problems. The authors have recently added electrical circuit problems of this type to their first-semester undergraduate electrical circuits course and received good feedback from the students. Therefore, the authors propose that this class of problems be considered for formal addition to introductory undergraduate EE curricula.

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