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Challenges and Innovations in Teaching Linear Algebra

Abstract

A solid grounding in linear algebra is essential in many fields of engineering. As a result, at our institution, linear algebra is a required course for our pre-engineering majors and is taught at the sophomore level. The students enrolled in this course have not usually had a proof methods course; however, the course focuses not only on computation but also on reasoning and theory. The content in this course presents a specific challenge in that it is difficult to cover the required material in one semester while at the same time devoting a sufficient amount of class time for homework questions and in-depth problem solving. In addition, it is difficult to develop test questions that can be completed within one class period while adequately assessing the students' skills in problem solving. Further, homework collection and grading can become burdensome very quickly. Collectively, these challenges can reduce the course's effectiveness in teaching students to properly apply linear algebra in solving real-world problems and can make it difficult for the instructor to evaluate student progress and provide timely feedback.

In this paper, we describe some of the challenges particular to teaching linear algebra as well as specific methods and techniques that we have used to help meet these challenges. We will describe some of the pedagogical innovations that we have employed in teaching linear algebra. These innovations include writing assignments in the form of student journals and class activities geared toward team-formation and enhancing student participation and learning. We will also discuss techniques, including using rubrics and brief writing assignments, used to evaluate students' understanding of linear algebra concepts. We will conclude with a discussion of the benefits, drawbacks and effectiveness of each of these methods in meeting the challenges associated with teaching linear algebra.

Introduction

The topics and techniques of linear algebra are especially important for engineering majors, mathematics majors, chemistry majors, and computer science majors. Engineering, chemistry, and computer science majors have an inherent need for an understanding of the practical applications of the theory in a particular discipline. In fact, many of the ABET program criteria specifically list the knowledge and application of linear algebra as a necessary skill.[1] In addition to applications, the theory and mathematical reasoning skills developed in an introductory linear algebra course will be invaluable to math majors as they progress to more proof oriented content courses.

At Georgia College & State University, there are approximately 110 mathematics majors. Roughly half of these majors are enrolled in the pre-engineering program. This program is a transfer program offered in conjunction with The Georgia Institute of Technology. The pre-engineering majors complete first and second year math courses at Georgia College & State University (GCSU). The linear algebra course at GCSU is taught at the sophomore level, and it is a required course for the pre-engineering, mathematics, and computer science majors. Occasionally, majors in other disciplines such chemistry, economics, and middle grades education

take linear algebra as an elective. The course is three credit hours per week over a fifteen week semester, and there is no lab associated with the course.

The challenge in teaching this particular linear algebra course lies in the problem of balancing applications and theory within the time allotted for class meetings. The current course focuses on both computation and reasoning skills since both are desired outcomes for the students that the linear algebra course serves. The authors have tried to focus on balancing class time and creating assignments to address the diverse needs of the students enrolled in GCSU's linear algebra course.

Classroom Innovations

In an effort to help meet the challenges described above, rather than traditional lecturing, we have developed a *modified lecture with activities* approach in delivering the linear algebra course content. In this section, we describe this approach which centers around interactive worksheets, cooperative learning and discovery activities, and individual writing assignments.

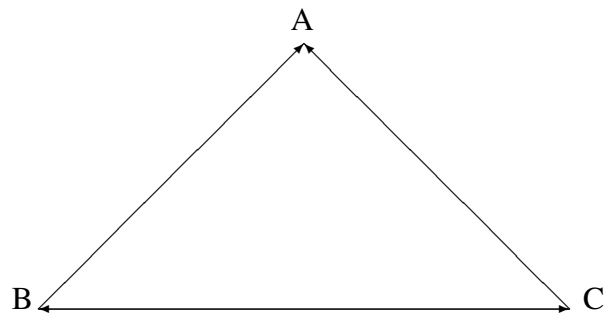
Interactive Worksheets: We have designed worksheets to complement our presentation of the subject matter. The worksheets contain definitions, theorems, and procedures as well as the example problems (without solutions) that we plan to discuss in class. Frequently we ask students to attempt a problem prior to a discussion on it. This approach allows students to take an active role in doing linear algebra problems rather than be passive observers. In the interactive worksheets, the examples the students are asked to work include basic computations, applications of theory, and applications in specific contexts. The following are excerpts taken from the interactive worksheets illustrating these three kinds of examples.

- **Example (basic computation):**
  1. *Find x and y so that the following two matrices are equal.*

$$A = \begin{bmatrix} 2 & x-4 \\ 3 & 0 \\ 7 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 6 \\ y-1 & 0 \\ 7 & 0 \end{bmatrix}$$

- **Example (application of theory):**
  1. *Plot a triangle in $\mathbb{R}^2$ with vertices $F = (x_1, y_1)$, $U = (x_3, y_3)$ and $N = (x_2, y_2)$ where x_1, x_3, x_2 are all positive and $x_1 \leq x_3 \leq x_2$.*
  2. *Find a formula for the area of this triangle by using trapezoids. Recall that the area of a trapezoid is half the distance between the parallel sides times the sum of the lengths of the parallel sides.*
  3. *Express this area in terms of the determinant of a particular matrix.*
  4. *Find the area of the triangle with vertices $(0, 0)$, $(2, 1)$ and $(-1, 6)$ using the determinant formula you found above.*
- **Example (application in specific context):** *Three military outposts are linked by communication devices as shown in the graph below. Assume that each station communicates with itself and that communication with other stations takes place only in the direction of the arrows. Use a 1 to denote that one*

station can communicate directly with another, and a 0 to mean it cannot; that is a 1 in the ith row and jth column means that i can communicate directly with j. Write a matrix that displays this communication network.



Cooperative Learning and Discovery Activities: In addition to the interactive worksheets, we use a combination of cooperative learning and discovery activities in the classroom to engage the students and to enhance their learning of linear algebra. The following are examples of commonly used classroom activities.

- **Identifying Team Members/Collaborative Assignments:** To make the students adopt an even more active approach in the learning process, we occasionally turn the process of team selection into a mathematical exercise that requires the students to apply the current topic that is being taught. Once the students are in their teams, they undertake an in-class team assignment, and the content of this assignment is related to the content emphasized in the team-forming activity. For example, after row operations and reduced row-echelon form were introduced, each student in the class drew a card which contained a 3×4 matrix. Each student's task was to find the other students in the classroom whose cards contained a matrix that was row-equivalent to the matrix on his or her card. For this activity, the group size varied from 3 to 4 students. Once the teams were formed, each team worked on a collaborative in-class assignment. The first part of the assignment dealt with the specific teams' matrices. The team had to put their row-equivalent matrices in a logical order and give the appropriate corresponding row operations at each step. The second part of the assignment dealt with common questions dealing with the theory of linear systems. For example, the teams were asked to construct, if possible, a 3×3 inconsistent, homogeneous system, and if not possible to explain why not.

- **Mixed-up Proofs/Definitions:** Focusing on precise definitions is new for most students in our linear algebra course. One method we have used to encourage students' retention of important definitions involves students collaborating as an entire class to formulate or recall lengthy definitions. For example, the definition of vector space involves many properties and quantifiers. One group challenge that we have assigned the class is to write the definition of vector space on the board. The students take turns, and on a student's turn, the student may either add a "line," modify a previous line (for example, if a quantifier was left out, the student may choose to insert that), or erase a previous line (if the student thinks it is incorrect). If a student thinks the definition is complete on his turn, he says so. Play continues until each student has judged the definition on the board correct and complete.

In addition to using the "definition scramble" described above, we also use the more traditional approach of definition quizzes to assess a student's knowledge. In fact, we frequently give a quiz over the definition of vector space in a class immediately following the one in which the students completed the vector space definition scramble.

Since our linear algebra course is at the sophomore level, most of the students enrolled have not had our proof writing course; however, we feel that the math majors need some exposure to proofs of linear algebra theorems. In an effort to meet this challenge, we scramble steps of a proof, and ask students to arrange the steps in the correct logical order. Below is an example of a mixed-up proof problem that we have used.

> **Directions:** Below is a claim along with a proof with its lines in random order. Write a careful step by step proof of the claim by rearranging the lines back to their correct order.
>
> * $A(2\mathbf{v} + 3\mathbf{w}) = A(2\mathbf{v}) + A(3\mathbf{w})$ (by the distributive property)
> * $= 2(\mathbf{0}) + 3(\mathbf{0})$ (since $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$)
> * By definition of solution, $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$.
> * $= 2(A\mathbf{v}) + 3(A\mathbf{w})$ (by a property of scalar multiplication)
> * Observe the following:
> * Thus, $A(2\mathbf{v} + 3\mathbf{w}) = \mathbf{0}$.
> * Therefore, by definition of solution, $2\mathbf{v} + 3\mathbf{w}$ is a solution.
> * Suppose $\mathbf{v}$ and $\mathbf{w}$ are solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.
> * Consider $A(2\mathbf{v} + 3\mathbf{w})$. We will show $A(2\mathbf{v} + 3\mathbf{w}) = \mathbf{0}$.
>
> **Claim:** If $\mathbf{v}$ and $\mathbf{w}$ are solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, then $2\mathbf{v} + 3\mathbf{w}$ is a solution.

The scrambled proof exercise serves as an interactive method for presenting well-written proofs.

Writing Assignments: The ability to write mathematical proofs is essential for mathematics majors, and the ability to communicate mathematical ideas in writing is important for all linear algebra students regardless of major. Towards this end, we have used a variety of writing assignments in our linear algebra classroom. We outline two types of writing activities below.

- **Test Corrections:** Occasionally, we have our students turn-in quiz or test corrections. We have introduced writing into this assignment by having students explain their errors for each problem missed. In addition, the students are to provide a correct solution in which they explain their reasoning. Furthermore, to receive credit for the corrections, each student schedules a fifteen minute appointment with the instructor during office hours. At this appointment, the instructor gives an oral quiz to the student over the missed questions.

- **Journals and Modified Homework:** We have used both journals and "modified homework" to foster students' writing and mathematical reasoning skills. A typical journal

assignment is for a student to describe a process or to apply a definition. For example, we have asked students to write a paragraph describing how to determine if a matrix is in reduced row-echelon form and to describe how reduced row-echelon form differs from row-echelon form. We use the journals to dialogue with the students by making notes and comments on their papers. The responses to the journal questions give the instructors insight into the students' understanding of a particular topic and can, in turn, direct how time class time is spent. We have used a simple evaluation rubric to grade the journals. The intent was not to penalize students for misunderstanding concepts but rather to open a dialogue to inform and formatively assess student understanding.

"Modified homework" is a similar assignment which we have assessed summatively. In other words, grades are assigned based on student performance. In general, homework assignments consist of two problem types: computational and theoretical. Computational problems were submitted and graded once. However, some theoretical problems were submitted, graded, and re-submitted multiple times. We provided feedback on students' papers, as with the journal entries, but then assigned a performance grade based on the work submitted. Students then had the opportunity to rework and resubmit the homework to improve the grade.

Observations

Some aspects of the modified lecture with activities format have been successful in terms of both student learning and student attitudes. Based on student opinion surveys and informal interviews, students have responded favorably to the interactive worksheets, modified homework, the scrambled proofs, collaborative assignments, and some of the writing activities. Based on test performance, the definition quizzes and definition scrambles have been effective in improving the students' basic linear algebra terminology.

The modified lecture with activities format has evolved through the continuous process of iterative course development. The current implementation is shaped, in part, by the authors' reflections on previous class activities, assignments, class examples, test results, student feedback, and peer reviews. Each time we have taught the class, we have changed the format slightly; however, each linear algebra class that we have taught has always incorporated some attributes of the modified lecture with activities format. In fact, one activity the authors have consistently implemented is the interactive worksheets. Over the last three iterations of the course, over 90% of the students enrolled wrote comments on the university-wide student opinion surveys. Of the students who commented, over 50% made favorable comments regarding this activity. In addition, none of the students who commented made negative remarks about the interactive worksheets. It should be noted that the students' comments were not specifically solicited. The instructors did not ask the students to directly comment on any particular aspect of the modified lecture format. The students provided these comments regarding the worksheets as evidence to support their opinions on a standard set of questions administered in every course across campus.

The development of course materials through the process of informed continuous improvement makes semester to semester comparisons of student performance based on these materials challenging, especially given the multitude of additional variables which can affect students'

course performance. Although we have not yet made a formal study of this method by comparing a control class to a class in which the modified lecture method was used, the students' exam grades and comments on the effectiveness of certain aspects of the modified lecture format could form the basis of a formal pedagogical research question that the authors could rigorously investigate. Because the mathematics department at Georgia College & State University is currently rethinking the linear algebra course structure (see the section "Further Considerations" below), the authors are waiting to implement any plans for such an investigation.

One challenge in the modified lecture with activities format is in time management. The students have indicated they would like to spend more time on collaborative activities. The instructors would also like to spend more time on these activities which present opportunities to highlight applications. In addition, meeting the demands of this course using this method can be quite time-consuming outside of class. In particular, reading and responding weekly to 25 to 40 student journals in a timely fashion is demanding and sometimes not realistic. The quiz and test corrections followed by the oral presentations in the office is also time intensive.

As part of the iterative course development process, the authors also investigated several methods of teaching linear algebra (some of which are described below) and used a variety of sources to develop the materials for the modified lecture with activities approach.[2,3,4] The authors have tried to meet the time management challenge by carefully choosing what content to present in lecture and which activities to use in-class and which to assign as an out-of-class task. Increasing the number of assignments by requiring journals, resubmitted homework, or test corrections also increases the amount of time spent grading. To meet this demand, the authors have selectively chosen which assignments to grade summatively and which assignments to grade more formatively. In general, tests and quizzes are graded very formally. Journals and participation in class activities are graded more holistically, perhaps by assigning a grade of zero, one or two. This method of evaluation still gives the students timely feedback but cuts down on the instructor's grading time.

Other Approaches

One of the methods the authors investigated is inquiry-based learning (IBL). This method can focus heavily on proof writing. The authors felt that using only this method could leave out too many of the computational skills and applications. Another pedagogy we considered is problem-based learning (PBL). A PBL course can be centered around a complex problem which requires the techniques of the course to solve. Students work in groups and learn the content as they need it to solve the problem. The authors felt using solely this approach could lead the students to focus on "just solving the problem" and perhaps shortchange the theory behind the application. Another approach is using technology as a basis for teaching the content. At this point, the authors feel that using this method alone may also lean too heavily toward application and away from theory. In developing the modified lecture with activities format, the authors tried to choose assignments and divide class time to reflect the needs of the students. Currently, the modified lecture with activities format contains elements of traditional lecture, IBL and PBL.

The authors are continually seeking ways to improve the modified lecture with activities format.

The challenge for improvement rests in the desired outcomes for this particular linear algebra course. The pre-engineering majors need computational skills as well as the ability to know when to apply linear algebra techniques. In essence, linear algebra should be a problem solving tool for them. In all likelihood, the theory will be secondary for the pre-engineering majors as they progress through the engineering curriculum. The math majors need the computational skills as well, but they will also need to understand the proofs of the theorems to be able solve problems as their curriculum progresses. The authors believe that both skills are important for each student of linear algebra. However, the challenge of balancing these demands often means the needs of the different student populations may not be fully met. Both authors have a background in pure mathematics, and as a result they often feel the course could be strengthened with more discipline-specific applications.

Further Considerations

The authors' experiences as well as the experiences of other mathematics faculty at GCSU have led to a departmental discussion of the course. There are a number of possible changes that have arisen as a result of the discussion. One possibility is to have two separate linear algebra courses: one at the freshman/sophomore level and one at the junior/senior level. The freshman/sophomore level course would focus on computation and application while the upper level course would focus more on theory and proof writing. The computer science department is considering offering their own version of linear algebra at the sophomore level. In that case, the mathematics department may consider making linear algebra an upper level theory course which would seem not to meet the needs of the pre-engineering and chemistry majors. Another possibility is to increase the current linear algebra course from three credit hours to four credit hours which would give the instructors of the course an opportunity to more evenly address computation, application, and theory. An extra class meeting per week would allow more time to cover homework problems, to introduce additional applications, to implement additional collaborative activities, or to enhance the content with the use of technology.

Conclusions

The authors' view is that the current linear algebra course has to meet the needs of a diverse group of students. As we have described above, meeting the desired outcomes for each group of students is challenging, and it can often feel that the emphasis on application or theory, the content coverage, or the sanity of the instructor is at risk. The modified lecture with activities format has been successful; however, despite teaching the course using this format multiple times, we feel there is still room for improvement. The department has not yet come to a consensus about changing the course, so the linear algebra course will continue as is. As a result, we are continually seeking innovations to meet the specific challenges raised by this particular linear algebra course.

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