Session 1320

Clarification of Partial Differential Equation Solutions Using 2-D and 3-D Graphics and Animation

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Abstract

The work discussed here demonstrates the use of two- and three-dimensional graphics and animation to clarify various solutions to partial differential equations describing a variety of dynamic physical problems. These graphical representations allow students to visualize the simultaneous variation of the dependent variable in space and time.

1. Introduction

There are a host of dynamic problems in the engineering disciplines that are described by partial differential equations. In the typical engineering math sequence the mathematics associated with their solutions often obscures the meaning and physical nature of the solutions. These problems arise in electromagnetics, vibrations, fluid dynamics, heat transfer and chemical mass and energy transport. The extensive graphics capabilities of MATLAB™ make the illustration of these solutions a reasonable task. The idea of using graphics to illustrate the solutions to hyperbolic partial differential equations has been published but animation was not employed.[1] Animations of bending vibrations in beams and longitudinal vibration of bars has been illustrated by Gramoll et al.[2,3] The response of a plucked string employing implicit methods in time has been demonstrated previously but animation was not used there either.[4] The ability to animate lumped parameter dynamic system behavior employing the handle graphics of MATLAB has been illustrated by Watkins et al.[5] The response of a Bernoulli-Euler cantilever beam has been calculated and animated using central spatial differencing.[6] A recent article illustrates the transport of pollutants employing web-based computer graphics.[7]

MATLAB is perhaps the most widely used general-purpose scientific and engineering software package in engineering education and engineering practice. It is thus appropriate to develop software for the purpose given here in that computing environment. The array computation ability and readily available graphics make such software development a reasonable task.

The authors have written MATLAB m-files to solve the following problems and illustrate their solutions.

1. Voltage waves on a sinusoidally driven lossless electrical transmission line.
2. The drawdown of well water in a pumped aquifer model.
3. Free and forced vibrations of a Bernoulli-Euler cantilever beam.
4. Conduction heat transfer in a slab with differing boundary temperatures.

5. Vibrations of a plucked string via the D'Alembert solution and by the Fourier series solution.
6. Steady-state oscillations on a sinusoidally driven string.

Because of the wide variety of systems considered, the animations are not intended for a single class nor even a single engineering major. The intent of the authors is that the software developed be used as demonstrations by the instructor in appropriate classes and also made available to students for their personal edification. In the case of the transmission line animation there is a GUI allowing the line and load parameters to be modified and the animation rerun. The most significant part of this work demonstrates the animation of the dynamic systems given above; however, it is unfortunate that the written material must contain only still images.

2. General Strategy

Consider a physical problem governed by a linear partial differential equation of the form

$$L_t[y(x,t)] + L_x[y(x,t)] = f(x,t) \qquad (1)$$

where $L_t[.]$ is a linear temporal differential operator in t and $L_x[.]$ is a linear spatial differential operator in x and f(x, t) is a known forcing function. It will be assumed that the problem is well posed such that there are sufficient boundary and initial conditions to give a meaningful solution. Table 1 illustrates these operators for the physical problems discussed in this work.

Table 1
Partial Differential Equation Operators for Problems Considered

Problem	$L_t[.]$	$L_x[.]$
Transmission Line	$-\dfrac{\partial^2}{\partial t^2}$	$\dfrac{1}{L'C'}\dfrac{\partial^2}{\partial x^2}$
Groundwater	$-\dfrac{S}{T}\dfrac{\partial}{\partial t}$	$\dfrac{\partial^2}{\partial x^2}+\dfrac{1}{x}\dfrac{\partial}{\partial x}-\dfrac{K'}{Tb'}$
Vibrating Beam	$\dfrac{\partial^2}{\partial t^2}$	$\dfrac{EI}{\mu}\dfrac{\partial^4}{\partial x^4}$
Heat Conduction	$-\dfrac{\partial}{\partial t}$	$\dfrac{k}{\rho c}\dfrac{\partial^2}{\partial x^2}$
Vibrating String	$-\dfrac{\partial^2}{\partial t^2}$	$\dfrac{T}{\mu}\dfrac{\partial^2}{\partial x^2}$

Whether the solution is approximate, exact or numerical, it will be a function of the variables x and t. This combination of space and time variables makes inferences by students about the physical nature of the solution y(x,t) difficult. The following graphical presentations can aid in the interpretation of the solution.

1. Plots of $y(x,t_i)$ as a function of x for selected values of t_i (freeze frame).

2. Plots of $y(x_j,t)$ as a function of t for selected values of x (measured values of y(x, t) at various locations x_j).
3. A plot of y(x,t) as a function of x and t in three dimensions.
4. Using animation to plot and erase y(x,t) versus x for a series of closely spaced values of t such that a movie of y(x, t) is attained.

The first three are relatively easy to accomplish in MATLAB but the fourth has a problem of speed. For animation of reasonable quality a frame rate of about 20 frames per second (the movie rate) is required. The handle graphics of MATLAB make many animations very efficient. Most problems are such that the solution can be generated on-line in real time and simply played back on request for animation purposes. In computationally intensive cases the solution can be generated off-line and saved as a .mat file and then played back from the previously generated file. The MATLAB code in none of the cases examined exceeds 150 lines.

To accomplish the animation the response is evaluated for nt values of time, each over nx values of the spatial variable and the results are stored in an nt×nx array called framedata. For animation the rows of array framedata are played back by using the handle graphics wherein the portion of the image that changes with each time step is defined as a graphics object. The object can be either a plot or a patch. Table 2 is a segment of MATLAB code that animates a plot.

Table 2
Segment of MATLAB Code for Animation

```
% assume xmin, xmax, fmin, fmax, nx and nt are previously defined
igure(1);clf;
range=xmax-xmin;
axis([xmin-0.2*range xmax+0.2*range  fmin fmax]); %Keeps plot size constant during
    animation
hold on;
x= linspace(xmin,xmax,nx); %Generate spatial variable
P=plot(x,framedata(1,:),’EraseMode’,’xor’); %Defines the handle for the plot object
pause; % Wait for any key to be pressed to continue
or t=2:nt; %Start animation loop
    set(P,’Ydata’,’framedata(t,:)); %Redefine Ydata for the object with handle P
    pause(0.04); %Holds animation frame for 40 milliseconds
end; % End of animation loop
```

The data stored in the framedata array can also be used for parametric 2-D plots and an overall 3-D plot of response as a function of location and time.

3. The Lossless Transmission Line

Consider the lossless transmission line shown in Figure 1 with respective inductance and capacitance per unit length L’ and C’. The line is governed by the telegrapher’s equations which are[8]

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$$\frac{\partial v}{\partial z} = -L' \frac{\partial i}{\partial t}$$
$$\frac{\partial i}{\partial z} = -C' \frac{\partial v}{\partial t} \qquad (1)$$

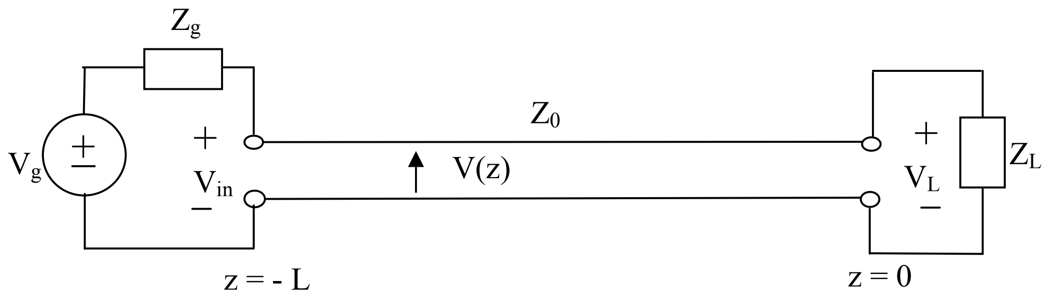

Figure 1. Lossless Transmission Line with Phasor Voltages Defined

or alternatively by the wave equation in voltage or current. Here the spatial variable z is employed as is common in the literature. When the line is driven by a sinusoidal source with radian frequency ω these equations can be simplified by the phasor method to give a pair of ordinary differential equations in the phasor voltage and current

$$\frac{dV}{dz} = -j\omega L' I$$
$$\frac{dI}{dz} = -j\omega C' V \qquad (2)$$

It is assumed that the line is driven by an ideal phasor source V_g with a series impedance Z_g and that the line is terminated with a load of impedance Z_L. The input impedance at the driven end (z = - L) is

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta L}{Z_0 + jZ_L \tan\beta L} \qquad (3)$$

where Z_0 is the characteristic impedance of the line, Z_L is the load impedance and $\beta = \omega\sqrt{L'C'}$ is the phase constant. The phasor voltage at the source end of the line is

$$V_{in} = V_g \left[\frac{Z_{in}}{Z_{in} + Z_g} \right] \qquad (4)$$

The phasor voltage along the line is

$$V(z) = V_g \left[\frac{Z_{in}}{Z_{in} + Z_g} \right] \left[\frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{j\beta L} + \Gamma e^{-j\beta L}} \right] \qquad (5)$$

where $\Gamma = (Z_L - Z_0)/(Z_L + Z_0)$ is the complex reflection coefficient. The time domain solution for the voltage at any location on the line is

$$v(z,t) = |V(z)| \cos(\omega t + \angle V(z)) \qquad (6)$$

Due to the complicated nature of relations (3) and (5) computer assistance is necessary to have a clear picture of what is happening. A careful look at these relations will indicate that, in general, the situation is that of forward and backward traveling waves or alternatively a forward traveling wave superimposed on a standing wave. The only cases when there is only a standing wave is when the load impedance is zero (short) or infinity (open) or purely imaginary (reactive). When the load is matched to the line, the load impedance equals the characteristic impedance, $Z_L = Z_0$ ($\Gamma = 0$), there will be a forward traveling wave only. All this is easily discovered with the

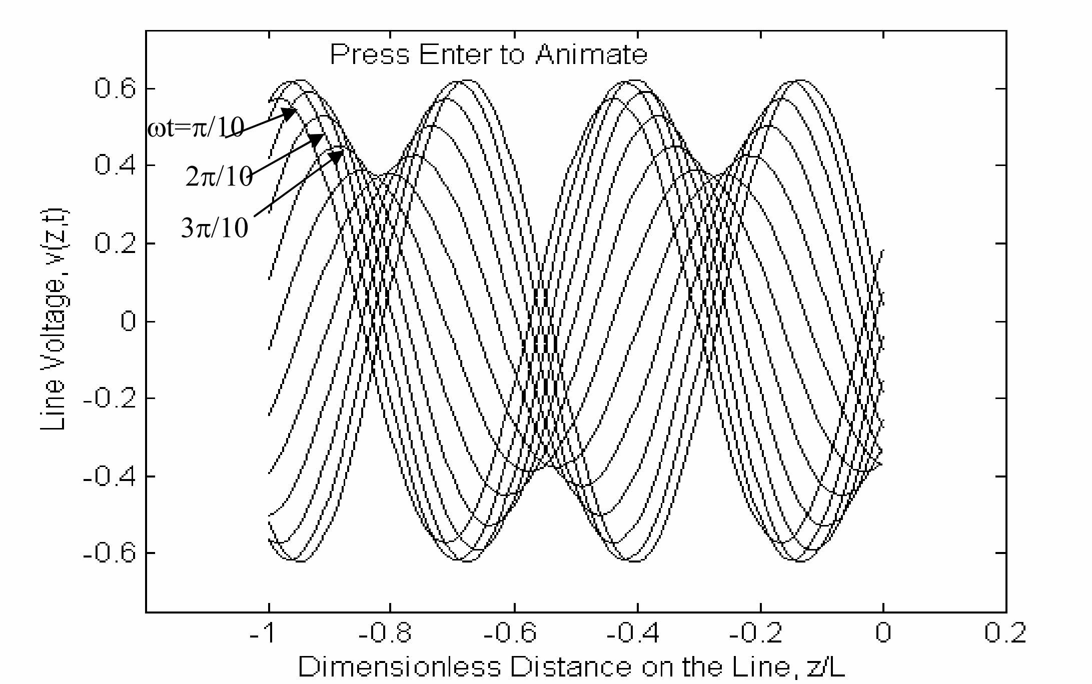

Figure 2. Voltage Waves on the Transmission Line for a Series of Times.

animation software developed. Figure 2 illustrates the voltage versus distance for a series of times for a case of Z_0= 300 Ω and Z_L = 500 Ω while Figure 3 is a plot of voltage versus time and distance along the line.

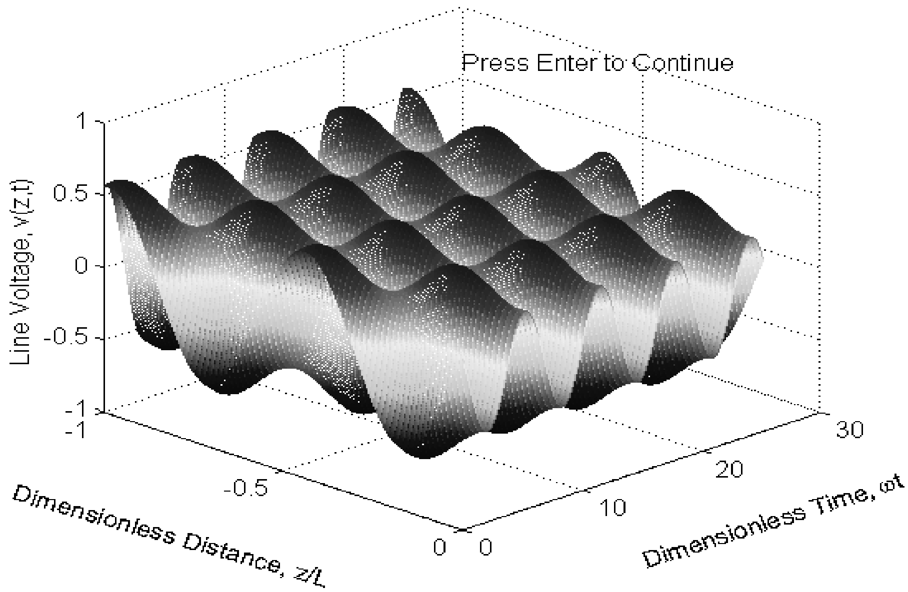

Figure 3. Line Voltage as a Function of Time and Location

4. Leaky Aquifer Groundwater Problem

This problem of the leaky aquifer shown in Figure 4 is governed by the diffusion-like partial differential equation in the drawdown s(r, t)[9]

$$\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}-\frac{K'}{Tb'}s=\frac{S}{T}\frac{\partial s}{\partial t} \qquad (7)$$

where T = Kb is the transmissivity of the pumped aquifer. K' and K are respectively the hydraulic conductivities of the aquitard and pumped aquifer. Parameters b' and b are the associated thicknesses and S is storage coefficient of the aquifer. Here the spatial variable r has been employed because of radial symmetry.

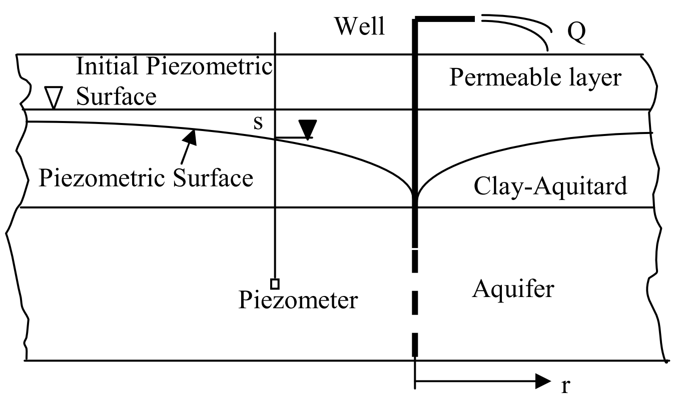

Figure 4. Water Well in a Leaky Aquifer

The initial condition is

$$s(r,0) = 0 \tag{8}$$

and the pumping boundary condition is

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2\pi T} \tag{9}$$

where Q is the volumetric pumping rate. There is a boundedness condition on s as $r \to \infty$. The solution is given by a Boltzman transformation to be

$$s(r,t) = \frac{Q}{4\pi T}\int_u^\infty \frac{\exp\left(-x - \frac{r^2}{4B^2x}\right)}{x}dx \tag{10}$$

where $u = \frac{r^2 S}{4Tt}$ and $B = \sqrt{\frac{Tb'}{K'}}$. The complicated nature of (10) dictates that computer assistance is necessary in the solution and interpretation thereof. The software developed presents the drawdown of the piezometric surface in several forms and the most significant is shown in Figure 5 while the steady-state cone of depression is shown in Figure 6.

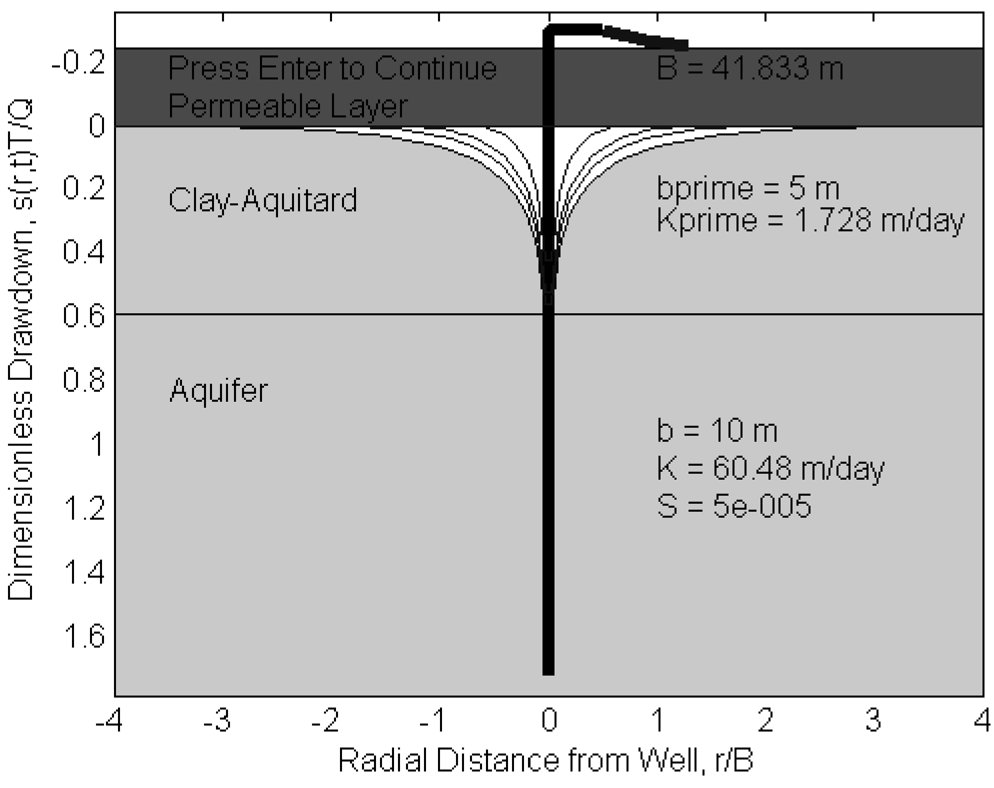

Figure 5. Animation of Piezometric Surface Drawdown for a Water Well.

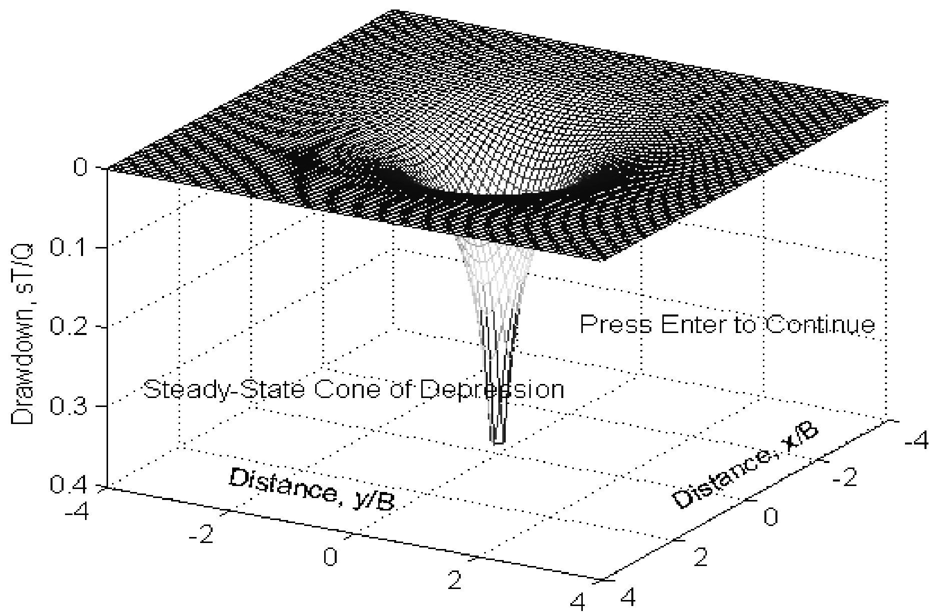

Figure 6. Steady-State Cone of Depression for a Water Well.

5. Vibrating Cantilever Beam

Consider the Bernoulli-Euler cantilever beam shown in Figure 7 with bending stiffness EI and mass per unit length μ

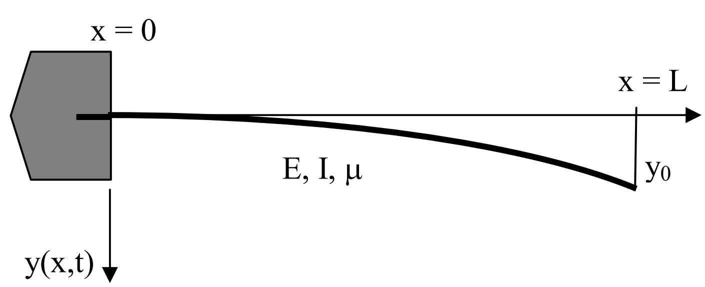

Figure 7. Free Vibration of a Bernoulli-Euler Beam

The motion is governed by the beam equation

$$EI\frac{\partial^4 y}{\partial x^4} + \mu\frac{\partial^2 y}{\partial t^2} = 0 \qquad (11)$$

with boundary conditions

$$y(0,t) = \frac{\partial y(0,t)}{\partial x} = \frac{\partial^2 y(L,t)}{\partial x^2} = \frac{\partial^3 y(L,t)}{\partial x^3} = 0 \qquad (12)$$

and initial conditions

$$y(x,0) = f(x) \text{ and } \frac{\partial y(x,0)}{\partial t} = 0 \qquad (13)$$

The solution to this problem can be given as a generalized Fourier series in the orthogonal beam functions $\varphi_n(x)$ and is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \cos(\omega_n t)\, \varphi_n(x) \qquad (14)$$

where

$$\varphi_n(x) = \cosh(\beta_n x) - \cos(\beta_n x) - \alpha_n[\sinh(\beta_n x) - \sin(\beta_n x)] \qquad (15)$$

are the orthogonal beam functions and the α_n and $\beta_n L$ are constants determined in the solution to the boundary value problem. The n<u>th</u> radian natural frequency is

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\mu}} \qquad n = 1,2,\cdots \qquad (16)$$

The b_n are given by expansion of the initial displacement f(x) to be

$$b_n = \frac{1}{L}\int_0^L f(x)\, \varphi_n(x) dx \qquad (17)$$

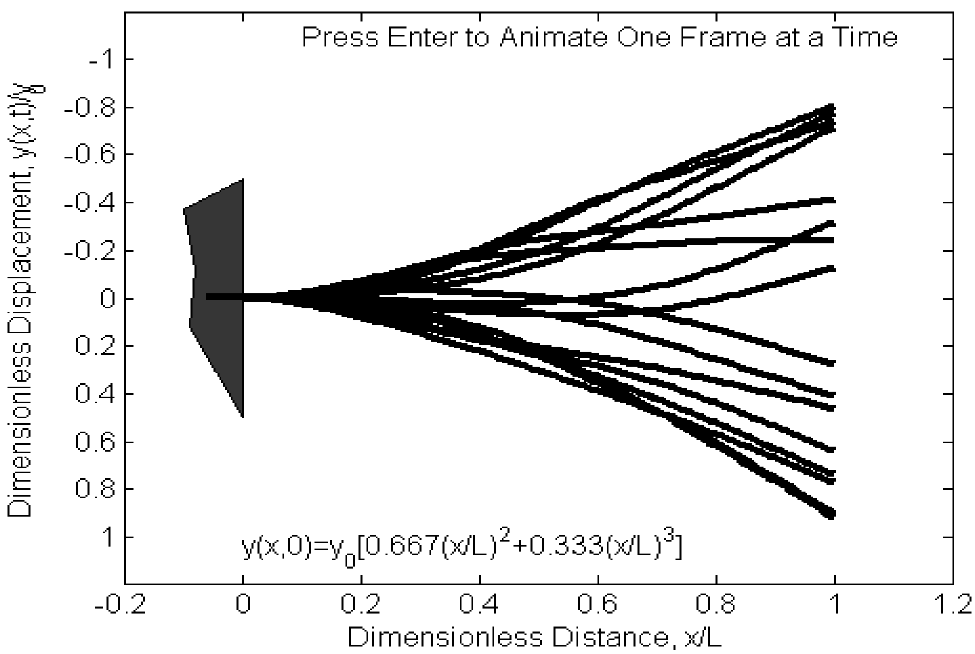

Figure 8. Animation of One First Natural Period of Cantilever Beam Vibration.

If the initial value of the displacement is of the form

$$f(x) = a_2\left(\frac{x}{L}\right)^2 + a_3\left(\frac{x}{L}\right)^3 \qquad (18)$$

the coefficients b_n are

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$$b_n = a_2 \frac{4(-1)^{n+1}\alpha_n}{(\beta_n L)^3} + a_3 \frac{12(-1)^n (1-\alpha_n\beta_n L)}{(\beta_n L)^4} \qquad (19)$$

The animation of the first natural period of the beam vibration is given in Figure 8. It is clear that the higher modes are contributing to the irregular vibratory pattern.

6. Transient Heat Conduction Problem

Consider the case where a slab of infinite extent and thickness L in the x-direction as illustrated in Figure 9. The conduction is governed by the one-dimensional diffusion equation[10]

$$k\frac{\partial^2 T}{\partial x^2} = \rho c \frac{\partial T}{\partial t} \qquad (20)$$

where ρc is the heat capacity per unit volume and k is the thermal conductivity. The associated boundary conditions are

$$T(0,t) = T_0, \qquad T(L,t) = 0 \qquad (21)$$

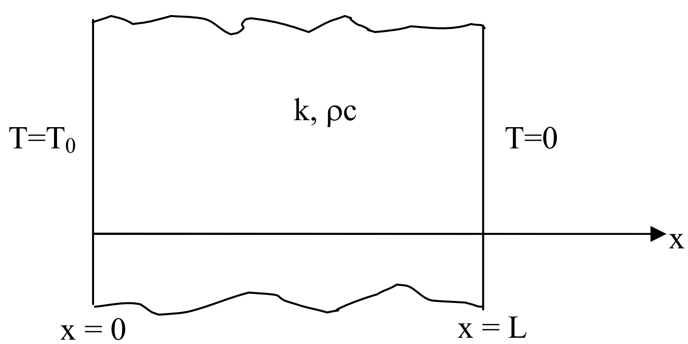

Figure 9. Heat Conducting Slab

with initial condition

$$T(x,0) = 0 \qquad (22)$$

The solution to this problem is

$$T(x,t) = T_0\left\{\left(1-\frac{x}{L}\right) - \sum_{n=1}^{\infty}\frac{2}{n\pi} e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)\right\} \qquad (23)$$

where $\kappa = k/\rho c$ is the diffusivity. With terms in both the variables x and t and a Fourier series to sum it is not clear what is the nature of the solution. Animation illustrates how the temperature changes with time and space as illustrated in Figure 10.

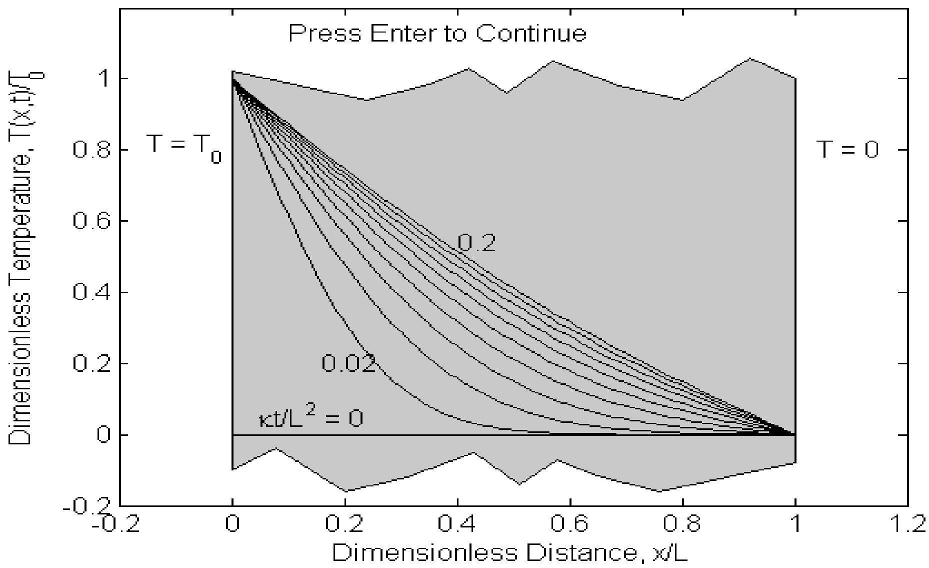

Figure 10. Transient Heat Conduction in a Slab

7. Plucked String Problem

Consider the taut string plucked at point x = a as shown in Figure 11.

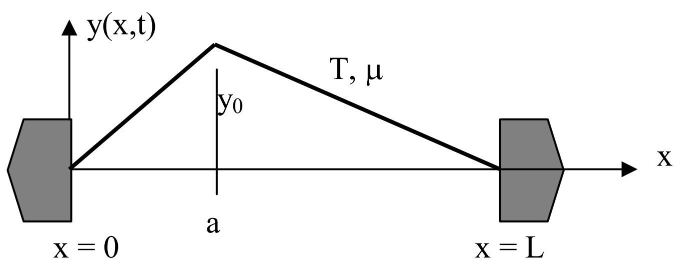

Figure 11. Taut Plucked String

The resulting vibration is governed by the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{24}$$

The square of the wave velocity is $c^2 = T/\mu$. and the boundary conditions are

$$y(0,t) = y(L,t) = 0 \tag{25}$$

The initial displacement is given by

$$y(x,0) = f(x) = \begin{cases} y_0\left(\dfrac{x}{a}\right) & 0 \le x \le a \\ y_0\left(\dfrac{L-x}{L-a}\right) & a \le x \le L \end{cases} \tag{26}$$

with zero initial velocity. The D'Alembert solution to the problem is a combination of forward and backward traveling waves.

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] \tag{27}$$

A separation of variables solution to this same problem gives the following solution to the problem

$$y(x,t) = \sum_{n=1}^{\infty} q_n \cos(\omega_n t)\sin\left(\frac{n\pi x}{L}\right) \tag{28}$$

where the q_n are the Fourier sine expansion coefficients for the initial condition and are

$$q_n = \frac{2y_0}{(n\pi)^2 \frac{a}{L}(1-\frac{a}{L})}\sin\left(\frac{n\pi a}{L}\right) \tag{29}$$

and the n<u>th</u> radian natural frequency is

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} \qquad n = 1,2,\cdots \tag{30}$$

Displacement of the string for one cycle of motion is illustrated in Figure 12.

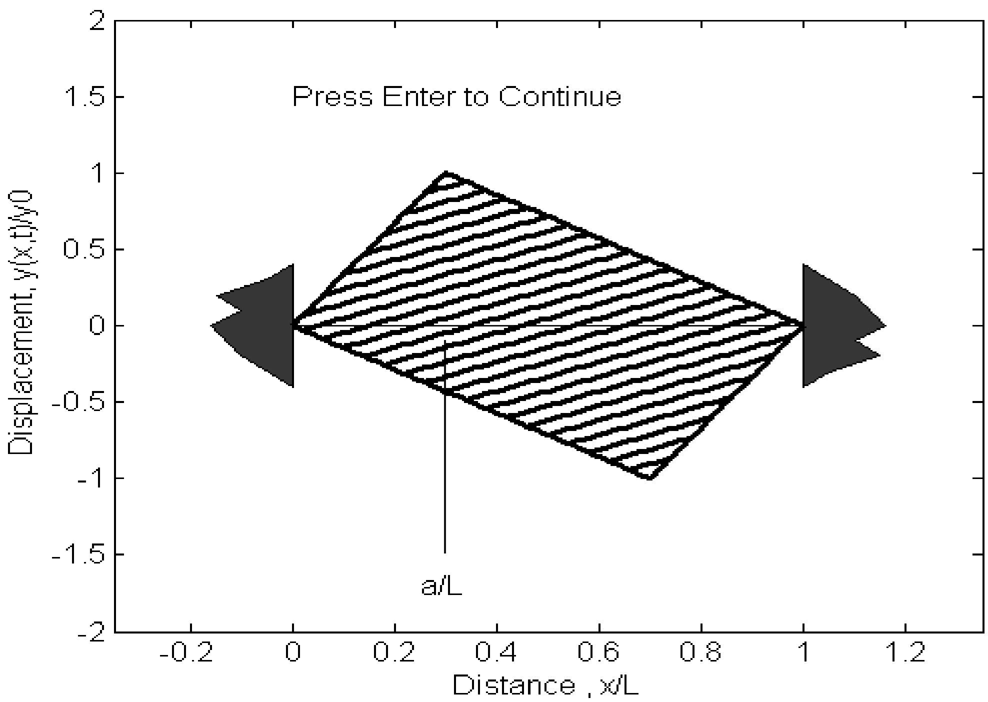

Figure 12. One Cycle of the Motion of a Taut String Plucked at a/L = 0.3.

8. Oscillating Taut String Problem

Consider the steady–state vibration of a taut string which undergoes a prescribed sinusoidal motion at x = 0 as illustrated in Figure 13. The motion is governed by the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{31}$$

where $c^2 = T/\mu$. The boundary conditions are

$$y(0,t) = y_0 \sin \omega t, \qquad y(L,t) = 0 \tag{32}$$

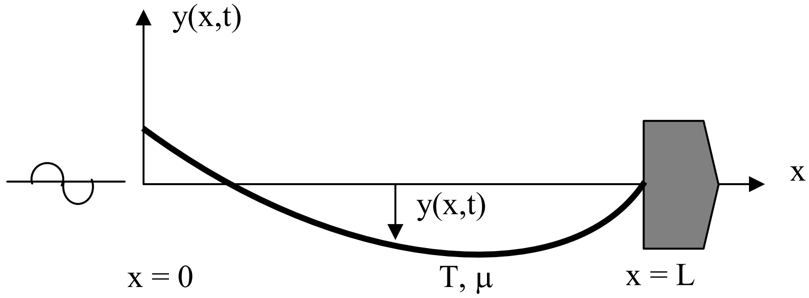

Figure 13. Driven Oscillations in a Taut String

The steady-state solution to this problem is

$$y(x,t) = y_0\left[\cos\beta x - \cot\beta L \sin\beta x\right]\sin\omega t \tag{33}$$

where the parameter $\beta = \omega/c$ and can further be written as

$$\beta = \frac{\pi}{L}\frac{f}{f_1} \tag{34}$$

with f_1 being the first natural frequency of the string given by.

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \tag{35}$$

Clearly this is a standing wave pattern on the string, the shape of which is controlled by the parameter β. This solution looks simple but the uninitiated have trouble interpreting (33) to tell what is happening in a physical sense. One complete cycle of the motion of the string is illustrated in Figure 14.

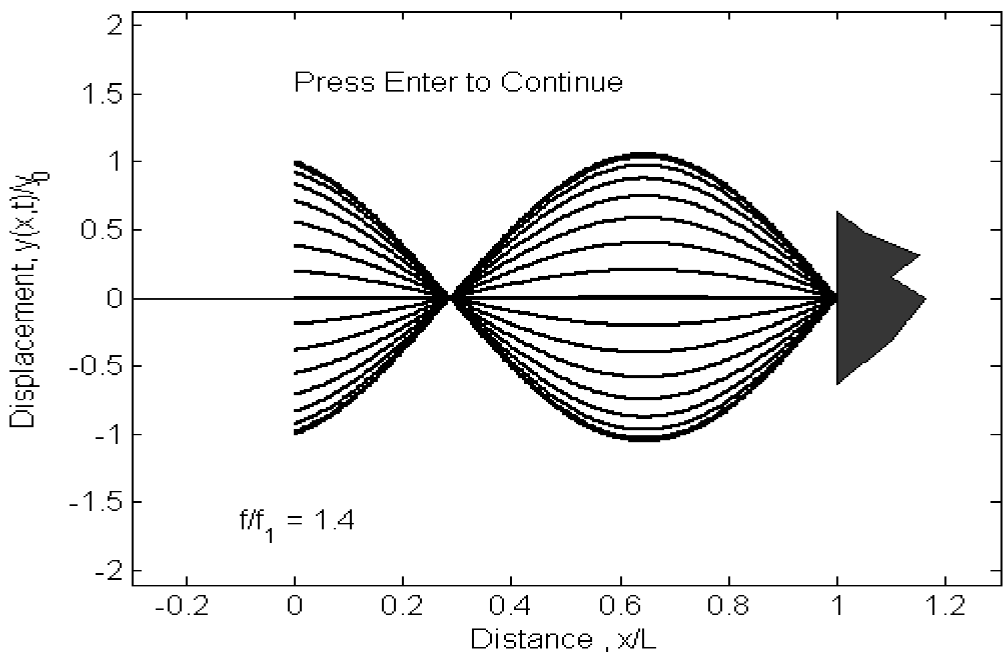

Figure 14. String Displacement when Driven at 1.4 Times the First Natural Frequency.

10. Discussion

The authors had such great success when animating distributed parameter dynamic systems with only one spatial variable they went on to explore the animation of problems with two spatial variables such as vibrating membranes and plates and two-dimensional heat transfer. Unfortunately the three-dimensional graphics of MATLAB are not sufficiently fast to make such animations reasonable on a 1.8 GHz Pentium 4 PC. As computer clockrates increase we can look forward to a wider range of problems that can be animated.

11.Conclusion

The authors have found the m-files developed to be useful in the explanation of physical problems described by partial differential equations. Any or all of them are available from the first author on request at quot@uwyo.edu.

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