Conceptual Power Series Knowledge of STEM Majors

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Mathematics & Engineering Majors' Conceptual Cognition of Power Series

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Taylor series expansion of functions has important applications in engineering, mathematics, physics, and computer science; therefore observing responses of graduate and senior undergraduate students to Taylor series questions appears to be the initial step for understanding students' conceptual cognitive reasoning. These observations help to determine and develop a successful teaching methodology after weaknesses of the students are investigated. Pedagogical research on understanding mathematics and conceptual knowledge of physics majors' power series was conducted in various studies ([1-10]); however, to the best of our knowledge, Taylor series knowledge of engineering majors was not investigated prior to this study. In this work, the ability of graduate and senior undergraduate engineering and mathematics majors responding to a set of power series questions are investigated. Written questionnaire responses of participating students and the follow-up interviews to have a better understanding of these written responses are analyzed qualitatively and quantitatively by using the Action-Process-Object-Schema (APOS) theory. Samples of the student responses to the written questionnaire and the transcribed interview data are displayed throughout this work. Written and oral interview data collected from participating STEM majors indicated a well-established knowledge of approximation, a poor knowledge of the meaning of center concept that takes place in the Taylor series expansion of functions, and a well-established knowledge of infinity as a part of infinite series concept.

Key Words: Maclaurin series, Taylor series, power series, finite series, infinite series, APOS theory, schema

Introduction

Taylor series representation of functions attracted many mathematics and physics pedagogy researchers. In mathematics education, comparison of geometric (graphical) and algebraic representations of functions as well as comparison of approximate and exact representations of functions are considered for investigating students' Taylor series knowledge. Series expansion of functions are also widely used in engineering and computer science applications. In this work, seventeen graduate and senior undergraduate students' Taylor series knowledge will be investigated based on their responses to four Taylor series questions. The participants of this study completed a calculus course that covered power series of functions in addition to the completion of a Numerical Methods course that is offered either by the School of Computer Science or the Department of Mathematics at a large Midwest U.S. university. The responses of the participants are evaluated by using the Action-Process-Object-Schema (APOS) theory. Participants are given approximately 80 minutes to respond to the written questionnaire

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that consisted of calculus questions and these participants are also interviewed for about 40 minutes. The calculus questions given in this questionnaire included function, derivative, integral, and power series questions.

Applications of Taylor series expansions include approximations, numerical computations, evaluations of definite integrals, and indeterminate limits ([5]). Assuming f is a smooth (i.e. C^{∞}) function, the Taylor series expansion of f centered at x=a is mathematically represented by the series

$$f(x) = f(a) + \sum_{n=0}^{\infty} \frac{d^n f}{dx^n}|_{x=a} (x-a)^n \tag{1}$$

The Taylor series given in (1) is called the Maclaurin series of the function in the case when a=0. The truncation of the Taylor series by using a finite upper limit is the main idea of the smooth function approximation. Approximation of differentiable functions using Taylor series requires a good understanding of the difference between several terms of the series and the series itself. Students' ability to explain the difference between the Taylor series approximation and the exact representation of smooth functions is the key point of algebraic understanding of the concept. This idea is investigated in physics and mathematics education by several researchers.

The research on Taylor series expansion of functions in mathematics education is limited and mainly focuses on series convergence of functions ([6–14]). Most of these studies' common theme is pertaining the calculus and series knowledge with a bigger scope than investigating the Taylor series knowledge of students. Students' visual and non-visual reasoning of sequence and series convergences are investigated in [8] and [9] respectively. Relating graphical representation of Taylor series to the convergence of the series is observed to be possible even for students with poor mathematical backgrounds who may not have the ability to explain the convergence analytically ([9]). In [10] the geometric (i.e. graphical representation) understanding of the Taylor series is observed to be the distinguishing factor between the experts and novice approaches for the tasks related to the Taylor series convergence. This difference between the two groups is observed by investigating their knowledge on the truncated and actual Taylor series expansion of several functions. Another pedagogical research on infinite series is implemented in [15] in which sequence knowledge of students is also observed. Hardly any pedagogical research on engineering students' conceptual power series knowledge is investigated prior to this study.

The research on Taylor series expansion of functions in physics education is also limited and emphasized mainly on applications. In [4] students' difficulties to connect the algebraic forms of individual terms in the Taylor series to the specific features of a graph of the function (e.g., the slope at a given point) are investigated and documented on the contrary to the pedagogical research implemented on Taylor series in mathematics education. The main aim of the research is to derive the Boltzmann factor using a Taylor series expansion of entropy. The findings in [4] also indicated many participating students' familiarity with the Taylor series but their difficulty in its fluent use during the physical applications. Another result observed in [4] is the benefit of a pre-tutorial homework assignment for refreshing the memory of what exactly a Taylor series is and its use for modeling physical contexts.

Action, Process, Object, and Schema (APOS) theory is a constructivist theory that deals with measuring learners' successes and failures based on the mathematical instructions and mathematical backgrounds. One of the central ideas in APOS theory is to understand learners' cognitive progress after a set of instructions and classify them to understand the success of the instructional design. An action is performing an activity based on external instructions. Cognitive development starts with the elementary "applications" as the actions that turn into processes when the learner manages to operate correlated variations. Process turns into object when new actions can be applied with the existing process. Schema is "more or less coherent collection of objects along with actions which the subject can perform on them" ([3]).

Pedagogical research on APOS theory applications of functions' series expansion is limited in the literature ([1]). The only research overlap on APOS theory and infinite series concepts' is the report that describes a three-semester calculus course developed at Purdue University with support from the U.S. National Science Foundation. The design of the course was based on APOS theory and involved students writing and running programs in a mathematical programming language and making calculations on the computer using a symbolic computing system. The pedagogical strategy consisted of cooperative student group work in a computer lab where they are expected to develop mental structures by using the mathematics instructions. Students are confronted with problem situations designed to get them use the mental structures developed in the computer lab to construct their understanding of mathematical concepts in small groups. Assignments are intended to provide practice with standard calculus problems and reinforce their understandings ([2]). The literature on engineering and mathematics majors' function series knowledge investigation prior to this study is very limited. In this work, the following four research questions are asked to the participants as a part of a larger scoped research that also included calculus questions.

12. In a few sentences legibly answer each of the following questions (a) through (d).

a) Describe the difference, if any, that exists between e^x and $1 + \frac{x}{1!} + \frac{x^2}{2!}$.

b) Describe the difference, if any, that exists between $e^1 + e^1\frac{(x-1)}{1!} + e^1\frac{(x-1)^2}{2!}$ and $e^2 + e^2\frac{(x-2)}{1!} + e^2\frac{(x-2)^2}{2!}$.

c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2\frac{(x-2)^n}{n!}$.

d) Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k}\frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

The four research questions given in (a) - (d) are designed to observe participants' responses to the following Taylor series' subject matters:

1. The difference between the Maclaurin series of exponential function and its approximation.
2. The difference between the Taylor series approximation of the exponential function centered at x=1 and x=2.
3. The difference between the Maclaurin series and Taylor series expansion (when x=2) of the exponential function.

4. The difference between finite and infinite series: the Maclaurin series of the exponential function and its approximation up to a number of finite terms.

The following exponential function information is provided to the participants in the questionnaire.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^x = \sum_{n=0}^{\infty} e \frac{(x-1)^n}{n!} = e^1 + e^1 \frac{(x-1)}{1!} + e^2 \frac{(x-1)^2}{2!} + e^2 \frac{(x-1)^3}{3!} + \ldots$$

$$e^x = \sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!} = e^2 + e^2 \frac{(x-2)}{1!} + e^2 \frac{(x-2)^2}{2!} + e^2 \frac{(x-2)^3}{3!} + \ldots$$

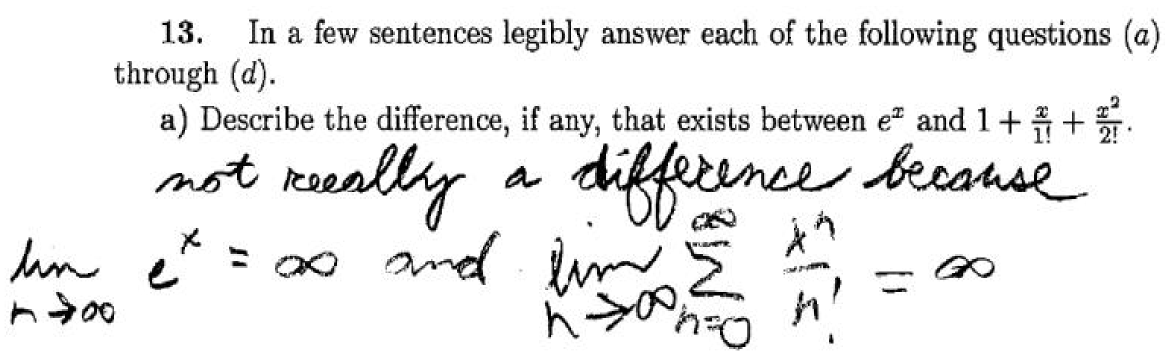
APOS theory classification of the engineering and mathematics students will be implemented by using the written and video recorded interview responses to the four subject matters. Each one of these subjects will be analyzed in different sections with the supporting written and transcribed video responses of the participants. This work is the first application of the APOS theory to classify engineering and mathematics learners' Taylor series knowledge to the best of our knowledge.

Maclaurin Series & Approximation

In this section participating students responses to the research question

a) Describe the difference, if any, that exists between e^x and $1 + \frac{x}{1!} + \frac{x^2}{2!}$.

will be evaluated based on their written questionnaire and interview responses. The purpose of this question is to understand graduate and senior undergraduate engineering and mathematics students' basic knowledge of Maclaurin series approximation. Participants' cognitive reasoning to describe the differences between e^x and $1+\frac{x}{1!}+\frac{x^2}{2!}$ is observed. Diverse explanations of the difference between these two terms is observed during the interviews.



13. In a few sentences legibly answer each of the following questions (a) through (d).

a) Describe the difference, if any, that exists between e^x and $1 + \frac{x}{1!} + \frac{x^2}{2!}$.

not really a difference because

$\lim_{n \to \infty} e^x = \infty$ and $\lim_{n \to \infty} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \infty$

Fig. 1 Response of research participant 4

Interviewer: Here we have three different ways to represent the same function and part (a) says describe the difference if any that exists between e^x and its first three terms. And you are saying not really a difference between. Can you explain me your answer?
RP 4: I can't remember really. There supposed to be a limit there I think. (writes the limit in front of the summation term

$$\lim_{n\to\infty}\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

I mean if you do enough terms they should go forever but I'm not sure.
Interviewer: …in this case when you said infinity what do you mean by that (pointing $\lim_{n\to\infty}\sum_{n=0}^{\infty}\frac{x^n}{n!}=\infty$) and you have limit of e^x is ∞.
RP 4: I'm not really sure now, I have no idea.
Interviewer: Okay. And do you know … error term?
RP 4: …not really.
Interviewer: If I tell you that there might be an error term here, can you figure out how that works?
RP 4: No, not really.

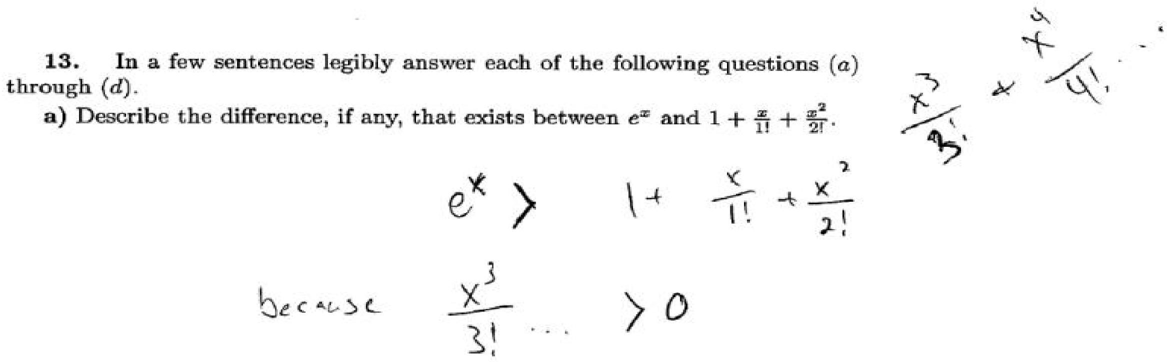


Fig. 2 Response of research participant 15

Interviewer: Here the question says, you are given three different types of representations of e^x and the question is asking to describe the difference between e^x and this (pointing the terms $1+\frac{x}{1!}+\frac{x^2}{2!}$) And you are saying e^x is bigger than that (Pointing written part $e^x > 1+\frac{x}{1!}+\frac{x^2}{2!}$)

In terms of error, if you compute error, what would that be? What would be the error term?
RP 15: …it would be just for this example, the error term is just next term in the sequence (writes $\frac{x^3}{3!}$)
Interviewer: That's it?
RP 15: Yes, it is bigger than that though. I know because I have seen it. And the rest of it (Writes $\frac{x^3}{3!}+\frac{x^4}{4!}$)
Interviewer: Here you are saying e^x is bigger than this $1+\frac{x}{1!}+\frac{x^2}{2!}$ because ($\frac{x^3}{3!}$)) and so on so forth (pointing ... of $\frac{x^3}{3!}$ +... part) is bigger than zero. …what do you mean by that?
RP 15: …I mean we know that the error is positive. Like we know that the actual function greater than our approximation.

The participants' reasoning of the approximation varied based on the majors. For example, RP 3 and RP 5 answered part (a) using the Big O notation which is a well-known concept in Computer Science to describe the error term difference between the finite and infinite series. 89.24% of the participants had the correct response to part (a).

Taylor Series Approximation Differences

The responses of the participants to the research question

b) Describe the difference, if any, that exists between $e^1 + e^1 \frac{(x-1)}{1!} + e^1 \frac{(x-1)^2}{2!}$ and $e^2 + e^2 \frac{(x-2)}{1!} + e^2 \frac{(x-2)^2}{2!}$.

will be evaluated based on their written questionnaire and interview responses. The purpose of this question is to understand graduate and senior undergraduate engineering and mathematics students' knowledge in Taylor series approximation of the exponential function centered at two different input values: x=1 and x=2. Students were not only expected to realize that these two functions are different mathematically but also they were expected to realize that $e^1 + e^1 \frac{(x-1)}{1!} + e^1 \frac{(x-1)^2}{2!}$ and $e^2 + e^2 \frac{(x-2)}{1!} + e^2 \frac{(x-2)^2}{2!}$ are Taylor series approximations of the exponential function centered around x=1 and x=2 respectively. Some of the participating students respond by thinking that the "difference" stated in the question is the mathematical difference.

Written responses of research participants 1, 4, 8, 13, 14, 16, and 17 given in Figures 3-9 reflected the mathematical difference between the given two series approximations. During the interviews the participants were shown Equation (1) given in Section 1 when $f(x) = e^x$. Participants 1, 4, and 14 could not remember the locational difference between the two Taylor series approximations given in the question whereas participants 8, 16, and 17 were able recognize the locational difference between the two terms and explain how they differ in center. Participant 13 tried to explain the difference between the finite Taylor series terms but could not succeed during the interview.

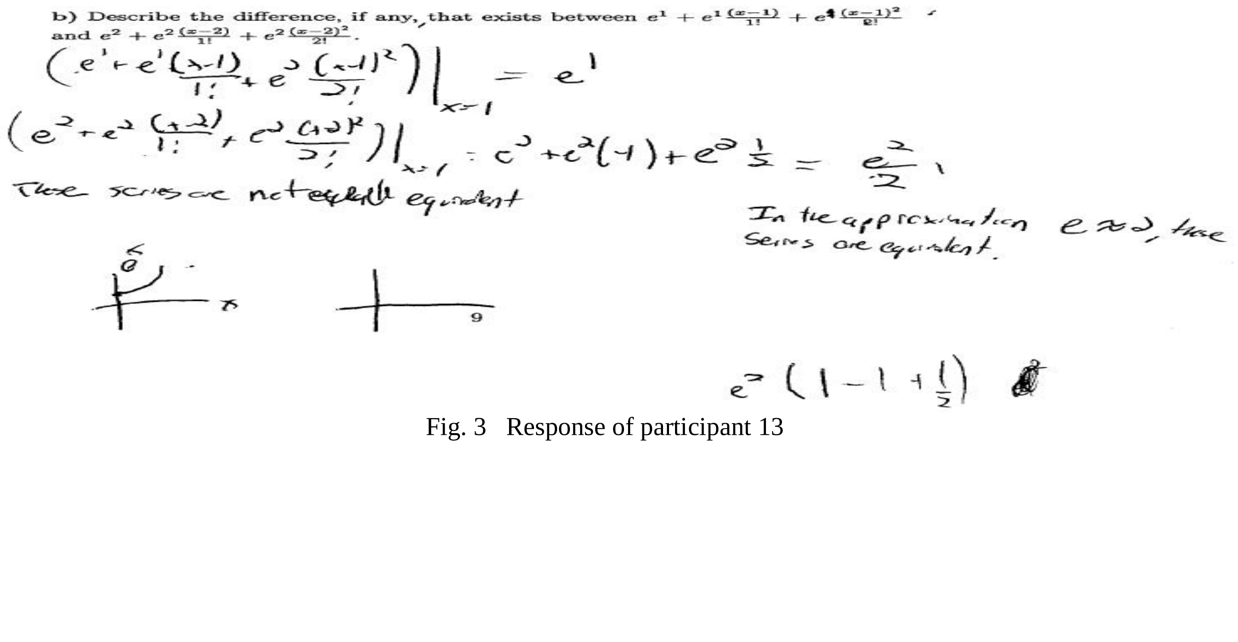


Fig. 3 Response of participant 13

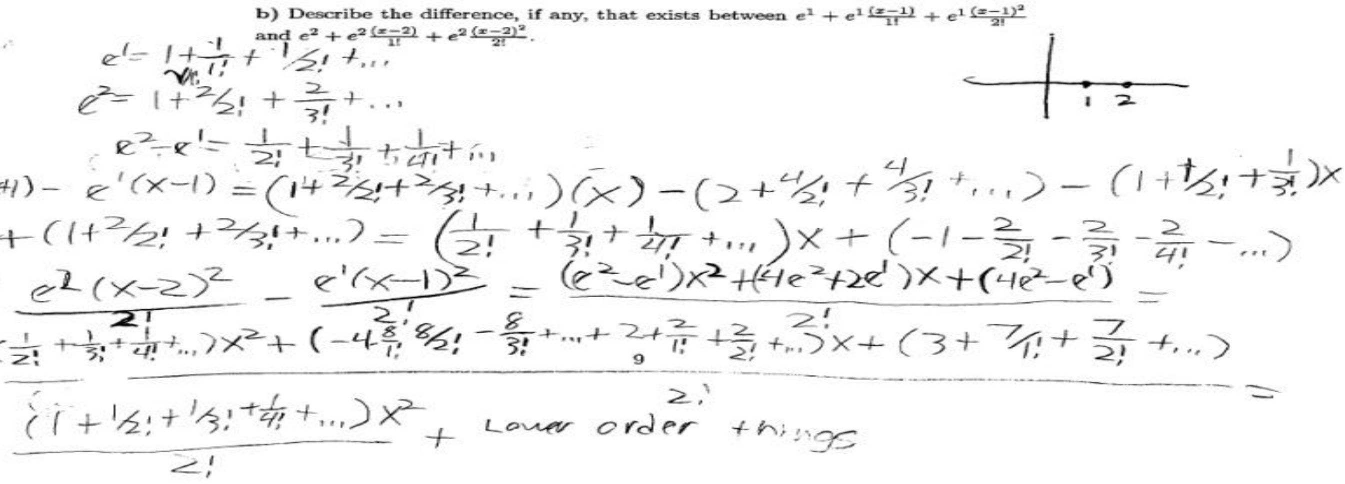


Fig. 4 Response of participant 17

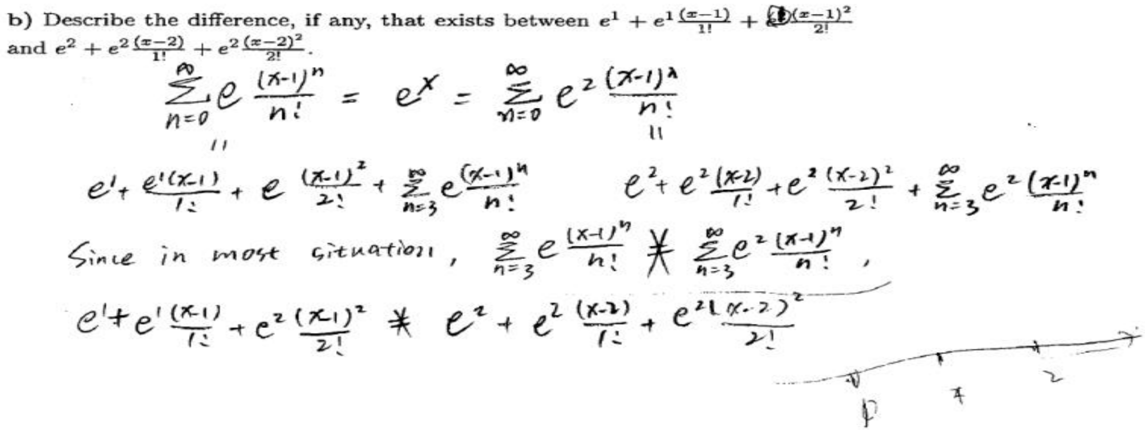


Fig. 5 Response of participant 14

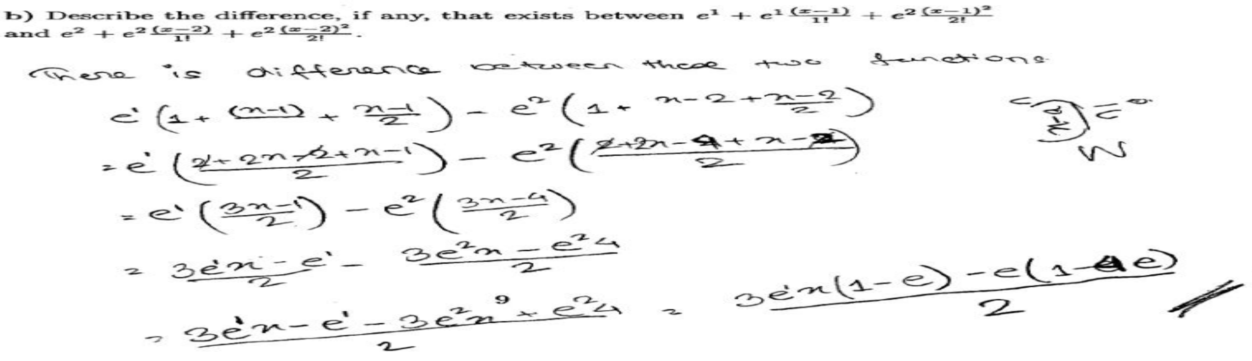


Fig. 6 Response of participant 8

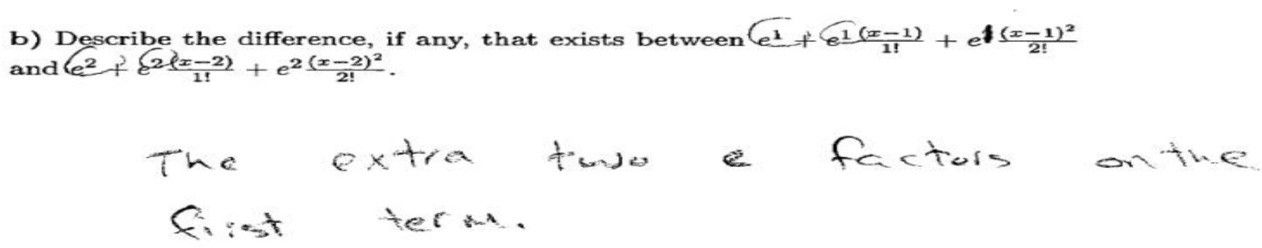


Fig. 7 Response of participant 16

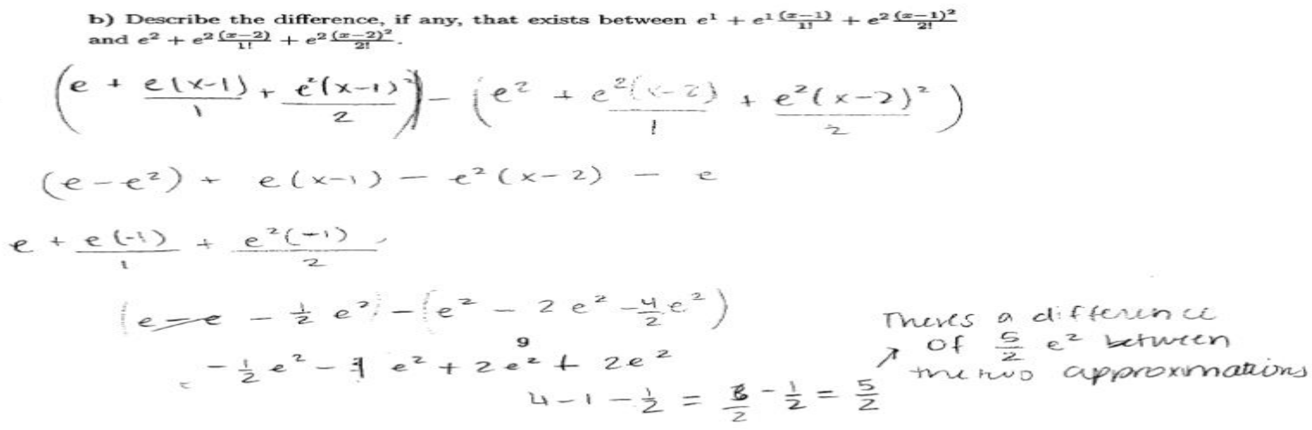


Fig. 8 Response of participant 1

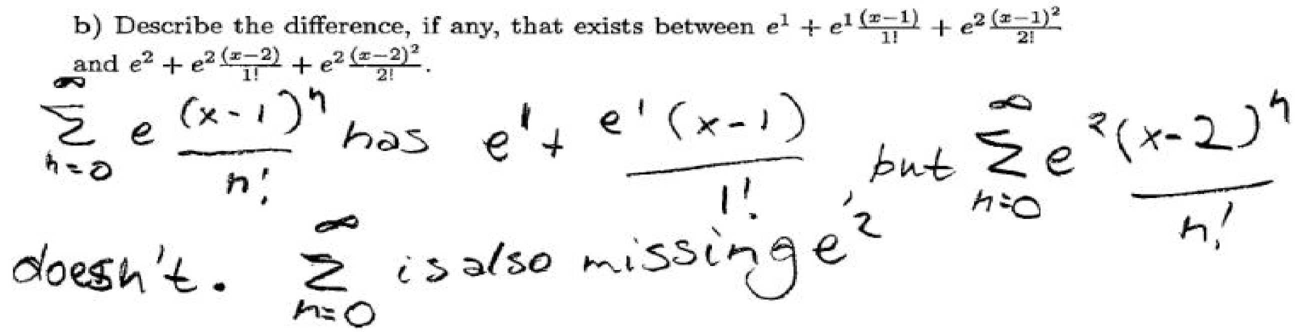


Fig. 9 Response of participant 4

The correct answer to the question is given by the participants 3, 5, 9, 10, 11, and 12 with participant 5 having the best written response among all the participants. Participant 6 was not sure about his/her response to the question neither before the interview nor after the interview.

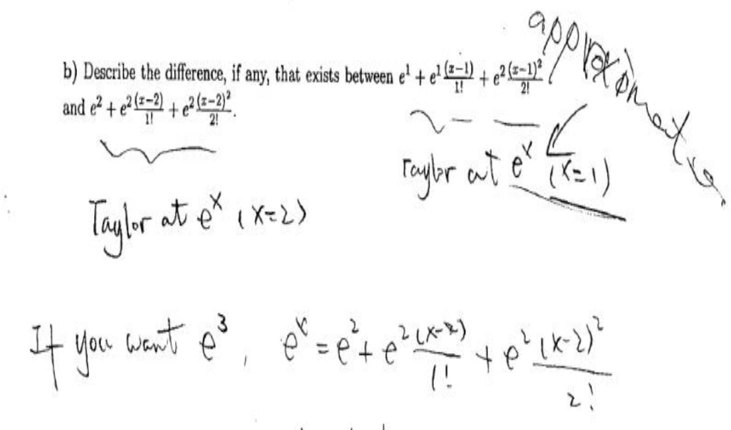


Fig. 10 Response of participant 5

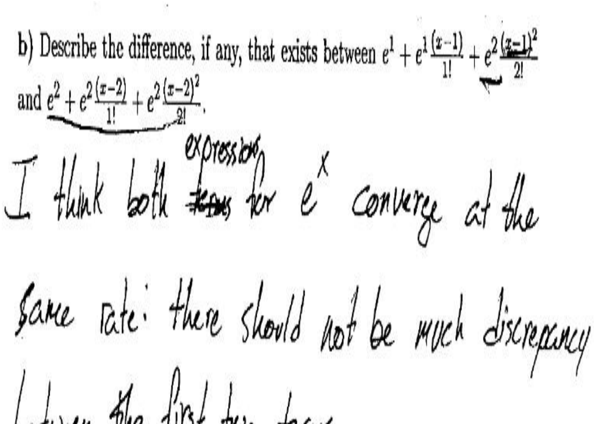


Fig. 11 Response of participant 6

Research participant 2 left the question blank and did not want to respond to the question during the interviews either. One of the Computer Science majors, RP 15, responded to the research question from computational difference perspective. The written response of the participant is given in Figure 12 below:

b) Describe the difference, if any, that exists between $e^1 + e^1\frac{(x-1)}{1!} + e^1\frac{(x-1)^2}{2!}$ $= a$ and $e^2 + e^2\frac{(x-2)}{1!} + e^2\frac{(x-2)^2}{2!}$. $= b$

$a < b$

a takes less computation

Fig. 12   Response of participant 15

Overall, 56.25% (9 out of 16 participants) had the right response to the research question (b) after the interviews. The way that the research question stated seemed to confuse some of the participants prior to the interviews which indicates the importance of how the question should be stated. The pedagogical evaluation of the question is only done respecting the after interview responses.

Maclaurin & Taylor Series' Approximation Differences

In this section participating students' written and interview responses to the research question

c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2\frac{(x-2)^n}{n!}$.

will be evaluated. Students were expected to realize that the given infinite series are Taylor series expansion of exponential function with centers x=0 and x=2 to solve the question.

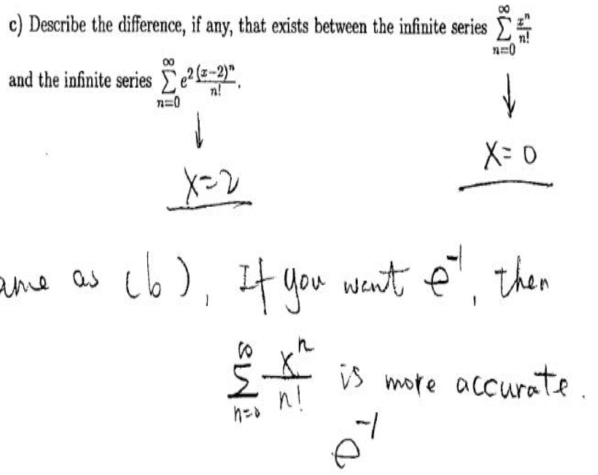
RP 1 was not able to explain the difference of Taylor series centers during the interviews. The given two series were compared from equality perspective. Participant 4 had conflicting view of the series given in Figure 13.

c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2\frac{(x-2)^n}{n!}$.

$\sum_{n=0}^{\infty} e^2\frac{(x-2)^n}{n!}$ will be larger than $\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty} e^2\frac{(x-2)^n}{n!}$$

Fig. 13   Response of participant 4

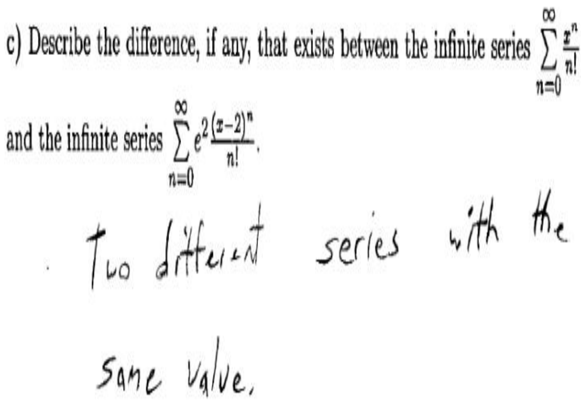


c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

$X=2$ $X=0$

Same as (b), If you want e^{-1}, then $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is more accurate. e^{-1}

Fig. 14 Response of participant 5



c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

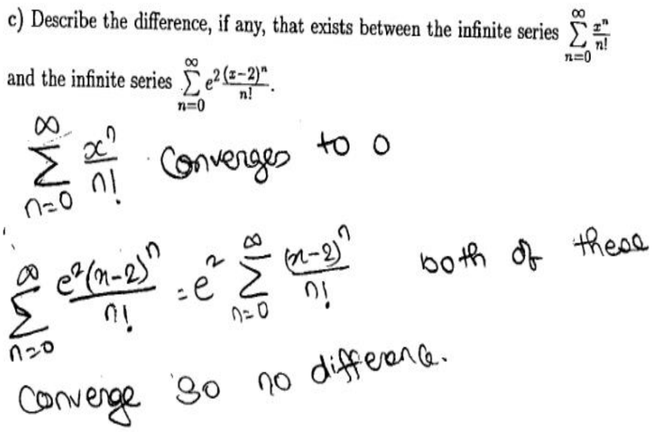
Two different series with the same value.

Fig. 15 Response of participant 7

Participant 7 had the correct location analysis of the question during the interview:

Interviewer: …describe the difference between these two series and you are saying two different series with the same value. And what is the difference in terms of location?
RP 7: The second one is around one, the second one is about two so. Just still end up being equal, shouldn't it, if you take the infinite. Because you are not having any error in either case.

Participants 8 and 9 both tried to answer the question from convergence perspective but did not succeed (given in Figures 16 and 18).

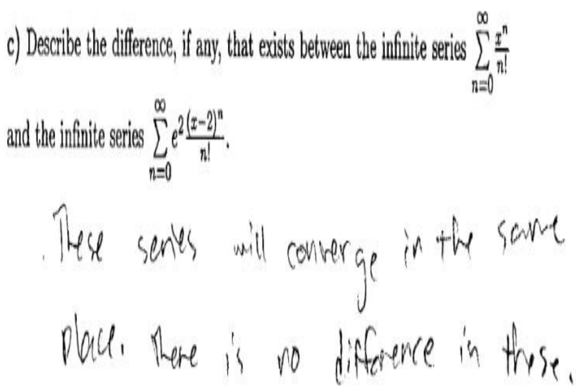


c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Converges to 0

$\sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!} = e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$ both of these converge so no difference.

Fig. 16 Response of participant 8

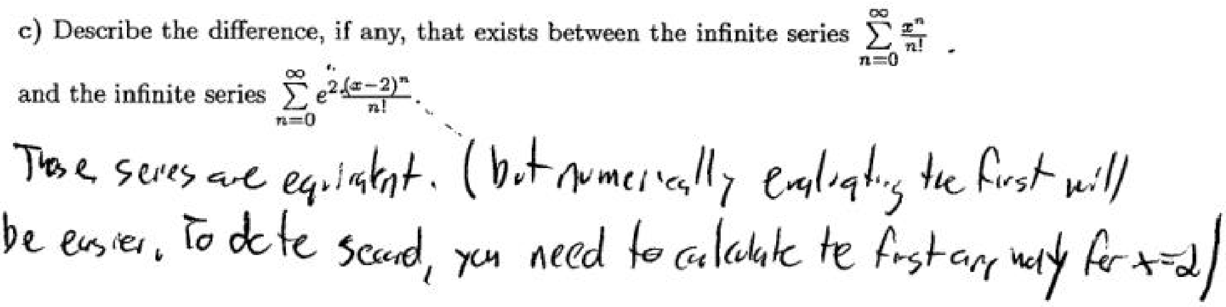


c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

These series will converge in the same place. There is no difference in these.

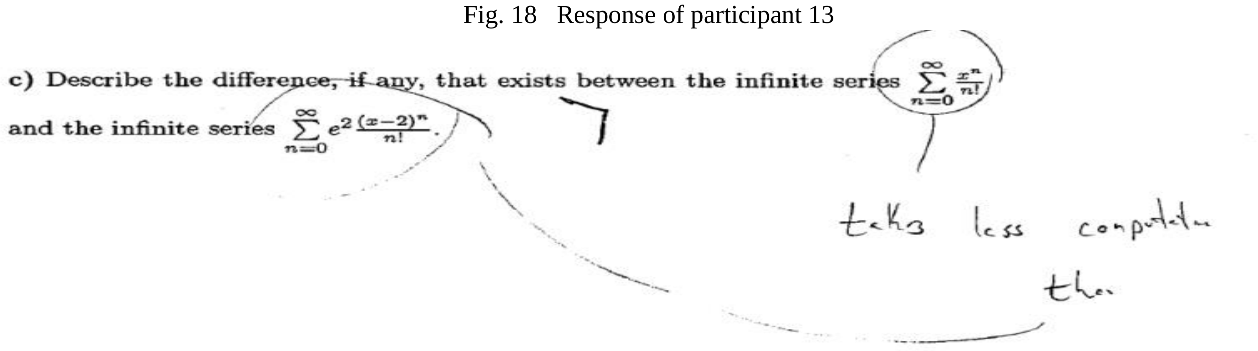
Fig. 17 Response of participant 7

Interviewer: …here describe the difference, if there is any, between the two infinite series
RP 8: One is located at zero the other one at two.


c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

These series are equivalent. (but numerically evaluating the first will be easier. To do the second, you need to calculate the first any way for x=2)

Fig. 18 Response of participant 13


c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

takes less computation than

Fig. 19 Response of participant 15

Participant 13 tried to explain his/her written further in detail during the interview:

RP 13: I mean there is the problem that I just had, I mean just noticed with this series. You can't get the value e^2 out of this. But these two series should be the same elsewhere. I mean I don't see why they shouldn't be. At all values except x=2, but these series also requires you already know the value e^1 or e^2, although could we look back here…This series (pointing the series centered at x=2) in the presetting requires you to know e^2. So evaluating this series (pointing the Maclaurin series) will be more indirect whereas, say we already know the value of e^1 or e^2 (pointing the series centered at x=2) to calculate this number here (pointing the series centered at x=2) we need to use this series (pointing the Maclaurin series) or the series before if we want to find this numerically.
Written response of participant 15 was from a computational perspective while his interview response was more comprehensive due to the set of questions raised by the interviewer:

Interviewer: …and here you are saying?
RP 15: Takes less computation.
Interviewer: …in terms of computational complexity of it you are approaching from or number of computations you can count here? …in terms of location, what is the difference?
RP 15: This is around zero (pointing the Maclaurin series), this (pointing the series centered at x=2) is 2.
Interviewer: ….is there a difference between them in terms of function?
RP 15: This is (pointing the series centered at x=2) bigger than this one (pointing the Maclaurin)

Overall only 16 out of 17 participants responded to question (c). Only 37.5 % of the participants had the correct response to question (b). Majority of the participants corrected or responded right to the question during the interviews. One of the participants preferred to not answer the question.

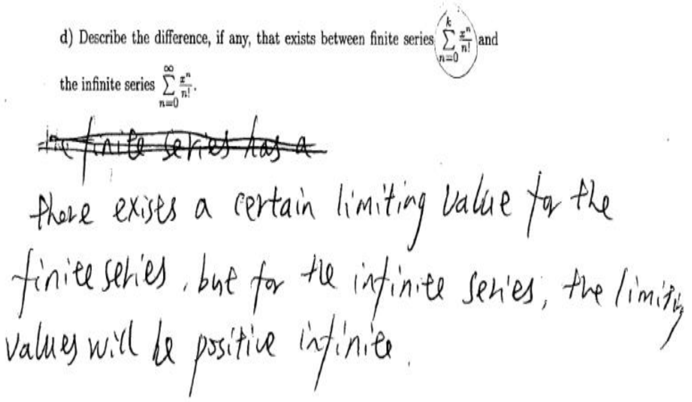
Finite & Infinite Maclaurin Series Difference

In this section participating students responses to the research question

d) Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

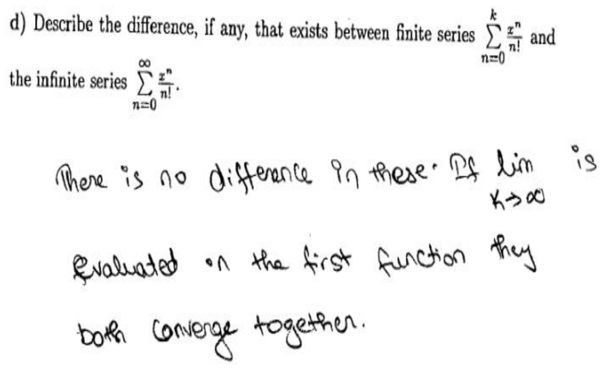
will be evaluated based on their written questionnaire and interview responses. The purpose of this question is to understand graduate and senior undergraduate engineering and mathematics students' basic knowledge of Maclaurin series and its' approximation. Participants' cognitive reasoning to describe the differences between e^x and $\sum_{n=0}^{k} \frac{x^n}{n!}$ is observed similar to the research question (a). The difference between the given two terms' is explained in various ways by the participants. Participants 2, 8, and 9 responded to question (d) from a "limiting value" perspective (given in Figures 20-22) with somehow improper written explanations. Participant 2 tried to explain his/her written response during the interview furthermore:

Interviewer: Okay, what about this one (part d)? Is there any difference between these two?
RP 2: Yeah, I think you can get actual value from this one (pointing the Maclaurin series), but this one goes to the positive infinity so maybe the value also goes to the positive infinity.
Interviewer: That's the difference?
RP 2: Yeah.


d) Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

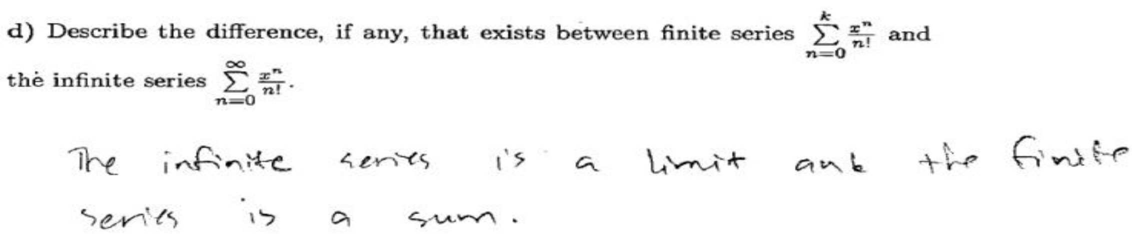
~~the finite series has a~~
There exists a certain limiting value for the finite series, but for the infinite series, the limiting values will be positive infinite.

Fig. 20 Response of participant 2


d) Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

There is no difference in these. If $\lim_{k\to\infty}$ is evaluated on the first function they both converge together.

Fig. 21 Response of participant 8


d) Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

The infinite series is a limit ank the finite series is a sum.

Fig. 22 Response of participant 9

The rest of the participants, 76.47% (13 out of 17 participants), had the correct approximation response to the question either during the interviews as the written response.

APOS Theory Classification & General Results

The research question investigated in this work is designed to study senior undergraduate and graduate mathematics and engineering students' strengths and weaknesses on Taylor and Maclaurin series. Exponential function is chosen for ease of Taylor series expansion representations; Maclaurin and Taylor series representations of the exponential function are provided to the participants. The finite or the infinite series representation of the exponential function centered at x = 0, 1, or 2 are the focus points of the research question. The APOS theory classification of the participants is determined by using all four parts (a)-(d) of the question. "Action" ability of the participants was measured in part (a) with their ability to respond this question correct. 88.23% of the participants showed their ability to relate the first three terms of the finite series approximation of e^x and the exponential function itself indicating their ability to act on their finite Maclaurin series knowledge. They are expected to transform this knowledge to "Process" by showing their ability to respond to question (d). 76.47% of the participants were able to carry their series knowledge to the "Process" level by showing relatively advanced knowledge of Maclaurin series. Correct responses to question (c) indicated participating students' ability to relate Maclaurin series to the Taylor series of the exponential function centered at x = 2 and determined the "Object" classification. 62.5% of the participants are qualified to be in this category. Two Taylor series approximations of the exponential function centered at x =1 and x=2 are given in question (b). Participants who determined the difference between these two approximations are assumed to have the conceptual understanding of series representation and classified in the "Schema" stage of the APOS theory. 31.25% of the participants are qualified to be in the Schema category of the APOS theory.

Conclusions & Future Work

In this work STEM graduate and senior undergraduate students' Taylor series knowledge is evaluated based on the written and interview responses to the following research questions:

- Describe the difference, if any, that exists between e^x and $1 + \frac{x}{1!} + \frac{x^2}{2!}$
- Describe the difference, if any, that exists between $e^1 + e^1\,\frac{(x-1)}{1!} + e^1\,\frac{(x-1)^2}{2!}$ and $e^2 + e^2\,\frac{(x-1)}{1!} + e^2\,\frac{(x-1)^2}{2!}$
- Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2\,\frac{(x-2)^n}{n!}$
- Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k}\frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

To the best of our knowledge there was no pedagogical research on engineering majors' Taylor series knowledge prior to this work. The four research questions developed in this study are designed to evaluate participants' Taylor series knowledge by using APOS theory. The following content is observed as a part of these questions:

1. The difference between the Maclaurin series of exponential function and its approximation.
2. The difference between the Taylor series approximation of the exponential function centered at x=1 and x=2.
3. The difference between Maclaurin and Taylor series expansion (when x=2) of the exponential function.
4. The difference between finite and infinite series: the Maclaurin series of the exponential function and its approximation up to a number of finite terms.

Determining engineering majors' Taylor series knowledge can be particularly important due to the use of finite series approximation for error term calculations in approximation theory. In particular, Calculus, Numerical Methods and Numerical Analysis instructors who are teaching STEM majors can benefit from the results of this study. Participating STEM majors written and oral interview results indicated

- well established approximation knowledge
- poor conceptualization of the Taylor series' center
- well established ability to deal with infinite series

The results indicated 88.23% of the participants are at the "Action" level; 76.47% of the participants are at the "Process" level; 62.5% of the participants are at the "Object" level; 31.25% of the participants are at the "Schema" level.

Furthermore investigation on engineering and mathematics students' conceptual knowledge of power series is needed by applying different pedagogical methodologies. For instance, concept image and concept definition mismatch of Taylor series knowledge of students appeared to be one of the main reasons of power series misconception of the participants in this article, therefore developing a pedagogical method centered at the conceptual power series definitions and the corresponding images can strengthen the conceptual understanding of students with the support of technology. Researchers and educators are invited to investigate pedagogical impact of combining the concept image and concept definition of mathematical series to teach power series of functions.

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