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Contextualizing Calculus with Everyday Examples to enhance conceptual learning

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Abstract

Many engineering students in Sweden have difficulty passing the initial mathematics courses. Teachers complain that students are ill- prepared in pre-calculus and even the best students can only solve standard problems using standard procedures. The aim of teaching in mathematics at university is to develop deep understanding of the subject and to produce conceptually trained students who can then creatively solve unforeseen problems. But how should we educate such students' We hypothesize that the root of the problem lies in decontextualized abstract teaching. The approach adopted in this study is to introduce contextualized teaching of mathematics through concrete examples and to focus students' attention on the role of definitions in mathematics in order to scaffold their development of conceptual understanding. The general idea is to start from the most concrete, everyday examples and work towards more abstract mathematics. By everyday examples, we mean those that can instantly be understood by the students with reference solely to their life experience. Each new concept in the course is introduced verbally, numerically, graphically, and algebraically when applicable. Everyday examples are introduced in the verbal phase of the sequence. Application examples are also used to boost the students' motivation. Quantitative data were gathered from both a diagnostic test and the exam, and qualitative data come from a student questionnaire. The exam results show that the students in the intervention group succeeded better than the control groups. They also became more motivated and had a better grasp of abstract thinking in mathematics.

Introduction

Many students entering engineering education in Sweden do not show proficiency in abstract conceptual understanding in mathematics.[1] Diagnostic tests administered at the beginning of engineering programs at a mid-sized Swedish University have shown declining results during the last decade, a trend shared with other western countries. This substantially lowers the pass rates and overall grades in mathematical courses and creates problems in the subsequent courses in science and technology. Furthermore, it causes attrition and considerably delays graduation for many students with consequences for the individual and society at large.

Students may be unprepared for abstract thinking in mathematics for several reasons. One of them could be the simplified procedural teaching from earlier education. Many mathematics books from senior high school are designed in a way that stresses procedural solving of mathematical problems without paying enough attention to conceptual understanding. Students are thus taught to follow algorithms to solve similar problems. When the procedures learned are not suitable for solving the problem at hand, students are lost.

Students' understanding of mathematical definitions is also often deficient. The role of mathematical definition in the acquisition of conceptual understanding is not given sufficient focus. Definitions are mainly presented as something separate from both procedural and conceptual understanding.

For the conceptually oriented student, each new aspect of a mathematical concept – definition, procedure, graphical representation, examples – adds a new layer to a deeper understanding. This is not the case for the procedurally oriented student, who memorizes procedures to pass the exams. Each new aspect introduced in mathematics is thus perceived as a further burden on the memory.

A closer look into senior high school books shows that, although the students have been exposed to most of what is taught in the first calculus course at university, they still show little understanding of the abstract conceptual mathematics and find it problematic.

First-year students in engineering education have had a winning learning strategy in earlier mathematical courses in high school, mimicking algebraic examples, often without realizing the limitations of the procedures.[2] This winning learning strategy makes many students initially reluctant to adopt the offered approach to conceptual learning, for example, understanding definitions in mathematics, their role in the construction of mathematical theories, and how they can be used in concrete engineering applications. The literature as well as our experience tells us that most students are less focused when definitions are presented initially in the lectures.[3]

The major aim of university mathematics is to enable the students to develop abstract conceptual understanding that allows them to solve unforeseen problems in science and engineering and to create a meta-knowledge of how to develop as a self-learner. Students' initial unpreparedness in calculus means that a substantial portion of the first calculus course is devoted to focusing students' attention on how abstract concepts are developed.

The design of the course

The first calculus course, Analysis in One Variable, is designed as a general course given to all approximately 1500 engineering students in the first year. It consists of 21 lectures, 14 tutorials and four seminars. Students are additionally offered optional supplemental instruction and workshops in mathematics four days a week. The content of the course consists of limits, continuity, differentiation, integrals, series, differential equations, and Taylor expansions.

During the course, students are recommended not to use any devices or aids, like calculators, computers, or formula collections which they are not allowed to use during exams. Apart from the three interventions: everyday examples, the joint construction of definitions, and motivating application examples, the lectures are given in traditional fashion. The examination itself is centrally designed and administered to all first-year students. The course literature consists of a

Swedish book, *Analys i en variable*, by Persson and Böiers and *Calculus: A complete course* by Adams and Essex.

Aim of the study

The overarching aim of the study is to scaffold engineering students' development of conceptual understanding in the first calculus course. This is done by offering a secure and friendly learning environment where students are invited to actively contribute to the knowledge-building process. To achieve this aim, students have to be able to connect abstract mathematical concepts to their experience, understand the role of mathematical definitions in preparing them for solving unforeseen problems, and develop insight into the limits of a definition. Additionally, application examples, from computer science in this case, will be used to strengthen the ties between mathematics and their future professional field, thus making learning of mathematics an enjoyable and motivating experience and at the same time boosting the students' self-confidence.

The interventions

During the academic year 2012-2013, an intervention was introduced in the first calculus courses for 160 engineering students in Computer Science. The main hypothesis in the study is that many of the problems that students encounter in learning abstract conceptual mathematics are due to the fact that it has mainly been presented to them in an abstract, context-free manner. Abstract conceptual thinking is the product of learning sought for, but it is not necessarily the way to bring about conceptual understanding. To develop deep abstract conceptual thinking in mathematics, students have to experience contextualized mathematics by relating it to their own experience and connecting it to the concrete instances from which it has been abstracted.

To achieve this end, concrete *everyday examples* have been constructed for each concept in the calculus course. Everyday examples are examples that do not require more than the student's life experience to be understood. These everyday examples were introduced at the beginning of each lecture and all the numerical, graphical, and algebraic explanations used during the lectures were related to them, as were the definitions. During the lectures, students were invited, after a while, to suggest definitions that captured the general aspects of the examples. This was followed by open deliberations where the teacher made the consequences of students' suggestions clear to them.

Students' suggested definitions often introduced unnecessary restrictions, like derivations for defining a local maximum. The discussion ended when the scientific definition was reached. The role of definitions in mathematics was thus made clear by experience, and their importance for solving problems was appreciated.

When appropriate, examples of how mathematical concepts may be used in software design applications were demonstrated. It was made clear to the students that these were only demonstrations of the connection between mathematics and programming, their future profession, without any obligation for them to instantly understand them. Not all of the students were expected to grasp these examples, bearing in mind the initial differences in their programming proficiency. Their use was still found stimulating.

Rationale for the study – The context of mathematics in engineering education

Designing engineering education with mathematics and physics in the first two years of the program is not a law of nature.[4] It can hardly be argued that abstract mathematics is taught in the initial stages of programs for pedagogical reasons. Rather, such a design reflects a Tayloristic view of industrial production transferred to education where context-free bits and pieces are dispensed by specialists to be assembled to a coherent whole in the end.[5] Most engineering teachers claim that they need to build on a ”solid” mathematics and science base. Pedagogically motivated design would have integrated mathematics with applications subjects to partly out-design motivation and contextualization problems.

Much of the research in mathematics in engineering education takes for granted the traditional design of engineering education with mathematics courses in the first year. The aim of the research is then to alleviate learning problems created by the traditional design itself and the alienation of mathematics from the application fields. There is a substantial difference between teaching mathematics to future mathematicians and to engineering students.[6] Alternative designs like problem-based and some project-based learning integrate mathematics with engineering subjects to provide a context for mathematics.[7]

The quality of teaching has little correlation with students’ learning outcome in mathematics. It is amazing that almost no difference can be noted in the passing rates of the students regardless of the teaching proficiency of the teacher. There is a case where almost eighty percent of the students left the lecture hall after the third lecture of eighteen, never to return. No difference in their passing rate was observed here either. They took responsibility for their own learning by joining other lectures on site and online. There is, therefore, a need to study the different informal strategies that students adopt to manage their knowledge building in mathematics, like peer-learning, online lectures and graphics software available for free on the Web today.

De-contextualized teaching in mathematics

The strength of mathematics is its abstraction and its ability to be applied to different situations in science and engineering. However, the intended learning outcome in mathematics is mastering abstract concepts, not necessarily the learning process leading to this objective. Making abstractions is easier to understand when students know which concrete instances abstractions come from. They have thus seen the induction process.

Constructing concrete everyday examples is a way of providing the students with the instances that frame the process of understanding the definitions.

It takes quite an effort to come up with adequate everyday examples. Application examples are much easier to give but harder to understand. They raise the demands on the students to make the leap from one unknown mathematical concept to a yet unknown abstract concept in the application subject. In contrast, everyday examples frame the abstract mathematical concept in terms of everyday experience, thus making it easier for the student to use a known concept to understand the unknown one.

However, similar views of de-contextualized procedural teaching based on different kinds of DTP format (definition, theorem, proof) as an instruction model for developing conceptual understanding, as expressed by Wu[8] and Baker, Czarocha and Prabhu[9], have been criticized with reference to interpretations of research such as those expressed in the ”National Council of Teachers of Mathematics. USA (NCTM) Standards” by Brown, Seidelmann, and Zimmermann.[10] Pragmatic researchers like Lave have also found no evidence that such a transfer from procedural learning to conceptual understanding occurs.[11]

In traditional teaching, it is often seen as necessary to first acquire procedural knowledge in order to develop conceptual understanding.[12] ”Teaching for” procedural knowledge would simply mean presenting to the students readymade definitions, notations and procedures without at the same time presenting concrete examples to frame a ”deeper” understanding of the concepts involved.

Contextualized learning

”Teaching for” conceptual understanding, on the other hand, would connect the theory to concrete problems, thus scaffolding students in extracting the abstract structures from the problem at hand. Involving students in reasoning requires connections to their prior knowledge and experience.[9]

The use of the word ”example” requires some clarification. What do we really mean by ”example”? There are at least three kinds of examples:

a) Context-free examples

b) Application examples

c) Everyday examples.

The first category is often abstract examples using mathematical symbols to show how a definition could be used. These kinds of examples have no connection to experience, application, or the real world.

The second category, application examples, contains examples taken from future technical applications that the students are hopefully going to meet in subsequent courses. They are very good examples for motivating the student for coming uses of mathematical knowledge. The problem with examples of this kind is that the students, by definition, knows nothing about the subject they have not yet encountered. Introducing such examples means introducing an extra cognitive burden. Instead of only trying to understand the mathematical concept, students have to struggle to understand the application example, whether it be programming, chemistry, or physics. The usual reaction of the students here is, ”Do we have to learn this application example for the exam too?”

What is focused on in this paper is the third category, the use of everyday examples. They differ from the abstract examples of the first category because they are concrete and from the application examples of the second category because they do not introduce any cognitively new material. Their objective is to use something already known to understand something new.

Many studies show that the transfer of mathematics acquired in procedural settings to application subjects is not achieved to the extent to which it is expected.[13,14] Engineering faculty report that students to a great extent fail to recognize and apply the mathematics they are supposed to know in subsequent engineering subjects. ”The students would hardly recognize a vector if it flew through the room”. Vectors do not, of course, fly through rooms, but it was a drastic way for one engineering teacher to express his frustration.

In order to become proficient solvers of non-routine problems, students must be exposed to, and practice, non-routine problem solving. There is no automatic transfer from extensive drills of routine algorithms alone to developing proficiency in solving unforeseen problems in the world.[15]

Most students can follow procedures and algorithms, based on memory, to solve known problems. However, if translation between different mathematical expressions is required or if they are asked whether a given number is the correct answer to an equation, they hardly know what to answer. Engelbrecht et al. conclude that such behavior indicates a poor understanding of the abstract concepts involved.[16]

Procedural and conceptual understanding

The debate between conceptual and procedural knowledge among mathematics educators has been ongoing for at least three decades now.[17] The definition of these constructs given by Hiebert and Lefevre 1986 is still widely used.[18] ”Conceptual knowledge is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are prominent as the discrete pieces of information”. ”Procedural knowledge is constituted by two components, one made up by step-by-step procedures for mathematical problems solving, and one related to the symbolic representations used in such procedures, including a familiarity with the symbols used to represent ideas and an awareness of the syntactic rules for writing symbols in an acceptable form”.

Both procedural and conceptual knowledge may be deep or superficial and each of them may support the other.[19] A student with developed conceptual knowledge has the ability to understand mathematical concepts and apply them correctly to a variety of situations. She can also translate these concepts between verbal statements and their equivalent mathematical expressions and ”see” mathematical representations with her ”inner eye”.[15]

Although attempts have been made to develop conceptual understanding among university students, the traditional procedure-oriented teaching to solve standard problems by fostering procedural learning widely prevails.[20] Faculty are, nonetheless, still looking for students’ conceptual understanding. There is, therefore, a need for more research on efficient teaching methods to develop students’ conceptual understanding.

Due to the fact that research findings have difficulty reaching teaching situations, research methods involving a joint commitment from both faculty and educators should be considered. These joint efforts are also needed to understand the discrepancies between adopted teaching approaches and students' learning.

Studies conducted in Sweden and South Africa show that when teaching is designed to promote conceptual learning, students perform well even in the procedural domain.[16] The argument that we have to sacrifice something in the traditional teaching to focus on conceptual issues has little support in current research.

Chappell and Killpatrick have investigated 305 college-level calculus students in two groups.[21] The larger group had the "traditional" DTP format and in the smaller group conceptual learning was sought by providing verbal explanation, graphical representation, and extension to new situations, all of which were illustrated by examples.

The group in the conceptual teaching setting performed significantly better than the procedural group on procedural tasks although the instructors only demonstrated the basic use of each procedure.

Chappell and Killpatrick's conclusion is best summarized in their own words: "[The] results challenge the belief that focusing instruction on the development and understanding of calculus concepts requires sacrificing, to some extent, the development of the student's ability to perform computations".[17]

Similar studies by Simpson and Zakaria found that conceptually-oriented students use linking words, like "then" and "because" to explain their solutions.[22] This indicates that they are confident about the solution and, unlike procedurally oriented students, do not resort to external authority to justify their solution.[23] Conceptually oriented students make different use of diagrams and numerical reasoning even if they sometimes struggle to use algebraic reasoning. Qualitative analysis shows that concept-based learning can more easily be extended to new situations.[24]

It is, however, important to remember that the distinction between conceptual and procedural learning is problematic since they are not always easily separated from each other.[16] It has been found in empirical studies that the concept orientation in solving problems is part of the solution rather than of the problem itself. Conceptual and procedural knowledge in mathematics may sometimes be problematic and "highly complex" both in regard to the meaning of the constructs themselves and their use in teaching and learning outcomes.

However, if both teaching and examination of university mathematics is geared towards procedural knowledge, then the students cannot be blamed for not developing the conceptual knowledge hoped for.[2]

The pragmatic approach to building abstract concepts in mathematics

De-contextualized mathematics teaching that does not connect mathematical concepts to students' experience forces the majority of them to use memory to store readymade solution algorithms, thus adopting a surface approach in order to cope with examinations.[25] In line with these findings, Mitchelmore and White conclude that learning abstract concepts in a context-free setting usually yields ”concepts [that] are poorly understood, easily forgotten and rarely applicable”.[26]

In the pragmatic view of meaning, understanding language propositions is only achieved in the light of the learners’ experience (see Wittgenstein §317).[27] Giving concrete examples before constructing a definition and performing some calculations provides a link between students’ experience and the abstract concepts at hand. Consequently, not providing concrete examples to contextualize concepts denies students the opportunity of making connections between the abstract mathematical concepts and their own experience.

There is no evidence that those students who have not experienced at least a partial induction and contextualization with concrete instances anchored in their prior experience will be able to connect abstract concepts and problems in the world easily. The divide between the context-free teaching of mathematics and the proposed approach of contextualization through everyday examples can be captured in the dichotomy of abstract-isolated and abstract-connected.

In the abstract-isolated setting, the teacher relies on giving the definitions and proceeding by giving examples containing context-free symbols and equations. On the other hand, the pragmatic approach inherent in the abstract-connected concept starts from contextualized concrete examples. These examples connect the abstract mathematical concepts to the learners’ experiences, thus turning inert mathematical abstractions into living concepts and giving them faces that are easier to understand and remember.[28]

A further aspect central to the pragmatic approach is that everyday examples are introduced at the beginning of the lecture before anything else. A problem is presented and the mathematical concept is introduced as an efficient way of solving or describing the problem. Concrete everyday examples in this perspective are viewed as constitutive of the mathematical concept. It is in this light that the student understands the implications of the mathematical definition. Any kind of examples listed above can also be given at the end of the lecture as *illustrative* examples. Assigning the latter role to examples is proper to other kinds of pedagogical approaches.

Abstract concepts and concrete instances

The context-free setting in mathematics relies mainly on a double pre-supposition: that abstract concepts are best learned in abstract settings and that the use of mathematical concepts in applications is based on a deductive process that connects isolated abstract concepts to phenomena in the world. There is little evidence in the research that this is how the students go about learning abstract mathematical concepts. Nor are abstract mathematical concepts and phenomena in the world connected deductively.[29]

If the connection between abstract mathematical concepts and phenomena in the world is not made deductively, how then is it achieved?

The pragmatic answer is by analogy to family resemblance. This concept itself is best explained by an analogy. Not all members of a family are completely alike. Children resemble their parents in some respects but not others. Some of the children have the same eyes, a second the same mouth, and a third the same body posture. The family resemblance is made by discerning these particularities.[24]

According to the same logic, knowing that the shortest way to fly from Stockholm, Sweden, to New York is to fly over Greenland (near the North Pole), as an example of Lagrange multipliers (maximum and minimum), makes it easier to see connections to technical applications by discerning the similarities in the analogous examples.

The NCTM principles for enhancing conceptual learning are in line with the pragmatic view of building mathematical knowledge. Teaching for conceptual learning follows a sequence, when appropriate, from concrete to abstract: verbally, numerically, graphically, and algebraically. The initial verbal phase is an opportunity to connect to the learners' experience, which makes it easier to refer back to it at later phases. The whole sequence retraces the concept from its more specific to its most general phase: concrete to abstract.

What numerically, graphically, and algebraically mean is clear to mathematicians; what is meant by verbally is not always obvious. One way of clarifying it is by connecting to the students' experience in the verbal phase by using everyday examples.

The following are some examples of everyday examples used in the intervention:

1. *Derivative*: How fast does Usain Bolt run at a specific point in his 100-meter race?
2. *The number* e: Interest rate on interest. How much money do you get at the end of the year if you withdraw and deposit an amount of money, including its interest, more and more often during a year? Knowing this leads to a definition of the number e.
3. *Local and global maximum*: Marie waters her pumpkin with x liters a day. The pumpkin's growth is dependent on x (no water or infinite water will kill the pumpkin). There should be an optimal value. How can we formulate this optimal value?
4. *Limits*: A car starts from zero speed at time zero and accelerates at time 20 seconds to 50 kilometers/hour and drives thereafter at a constant speed. What happens to the average speed as time goes on? Does it have a limit? Will the average ever be 50 kilometers/hour?

Implications for mathematics teaching and learning

Most text books in calculus do not systematically use concrete examples to initiate abstract concepts, thus providing little scaffolding for the self-learner. Many teachers report that not even the most advanced students in mathematics are able to execute computations solely based on the definitions in the book, if they pay any attention to them at all.[3] A definition as a rule can be interpreted and used in many different ways. The rules of the definition do not include descriptions of how they should be used. As a rule, a definition is of little use to the novice because she does not understand its implications. The way in which mathematical rules should be used is part of what is shown in the traditional teaching of mathematics (see Wittgenstein §139).[27]

If the reader agrees with the above argumentation, then the most important part of teaching mathematics will be to contextualize the abstract mathematical concepts, for instance, by providing experience-based everyday examples. Everyday examples serve, therefore, as a preliminary bridge upon which the initial understanding of the abstract concepts rests. They are also concrete instances that provide meaning to the abstract definitions. Abstraction, according to the Web encyclopedia, is ”the process of formulating generalized ideas or concepts by extracting common qualities from specific examples” (see http://www.thefreedictionary.com/abstraction).

”Abstracting is an activity by which we become aware of similarities ... among our experiences. Classifying means collecting together our experiences on the basis of these similarities. An abstraction is some kind of lasting change, the result of abstracting, which enables us to recognize new experiences as having the similarities of an already formed class ...”.[30]

Thus, teaching mathematics without contextualizing from time to time, by providing concrete examples from which abstractions have been made, for example, compels students to adopt a surface approach to learning and to use memory to store definitions and procedural algorithms. There is no evidence that this approach to learning automatically leads to the conceptual understanding hoped for.[21]

Not only do the concrete examples scaffold the students' development of conceptual understanding, they also provide a concrete context for the abstract mathematical concepts. This contextual understanding together with the examples is, hypothetically, also an image that helps recognize future situations where the concept can be used. In a study by Chappell and Killpatrick, a student expresses it in the following way: ”Knowing where the formula comes from not only helps you understand what it accomplishes, but also helps you remember it.[21] I know what to do when the problem comes up and recognize what steps have to be taken”.

Characteristics of everyday examples

Introducing new mathematical concepts in a sequence that proceeds from concrete to abstract (verbally, numerically, visually, and algebraically) helps the student grasp the abstract meaning of the concept.

However, creating these everyday examples is not a trivial endeavor. Part of this project has, therefore, been devoted to developing a database with everyday examples for every concept in a One Variable Calculus course and making it available to the teachers.

According to the same pragmatic logic argued for in this paper, it is expected that, if a teacher sees ten everyday examples of mathematics, she is more likely to be prepared to come up with one of her own.

Everyday examples cannot logically be deduced from definitions: the family relationship is seen at once, analogically. They cannot be right or wrong, or only good or bad examples, which is far from the deductive traditions of mathematics and makes them risky.

If the process of building abstract concepts in mathematics is similar to the construction of abstract concepts in other domains, like native language, then a clear conclusion is that students' difficulties in grasping mathematical abstractions and being able to apply them properly are mainly due to the abstract teaching they have been exposed to. Defining abstract concepts before they have any experience to build on leaves the student with no other alternative than to rely on memory. Deep conceptual understanding of abstract propositions, definitions, on the other hand, is only possible through experience.[24]

Methods

The study was conducted in the first major course in mathematics, Calculus in One Variable, offered during the first semester of the computer science program. There were 160 students initially in the program. The method is both quantitative and qualitative. The quantitative data are from the results of both an initial diagnostic test administered at the beginning of the program and the exam results at the end of the course. It should be noted that the centrally designed and administered examination itself is not specially designed to test conceptual knowledge per se.

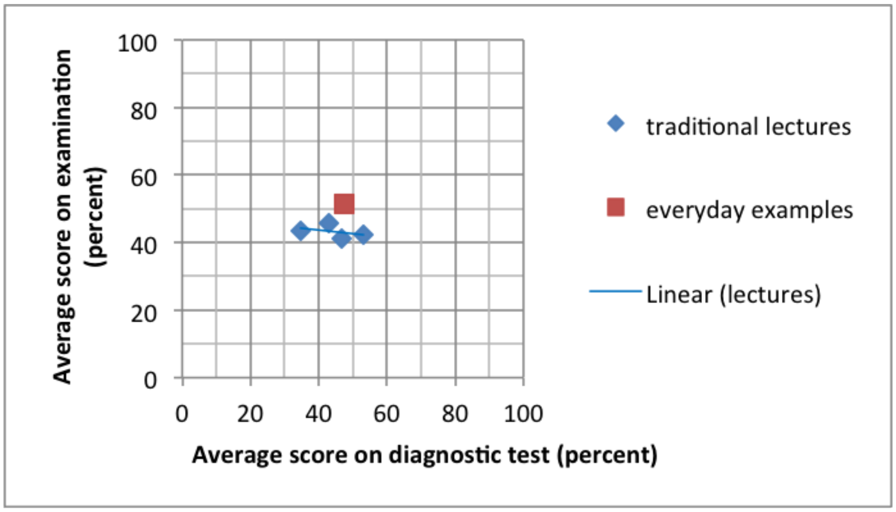
The exam is divided into three different parts: a) basic knowledge of the course, for those who did not succeed in the continuous examination in seminars b) some more basic knowledge at the intermediate level, c) which in some way corresponds to conceptual understanding for those who can (i) deduce some particularly important theorems and formulas, (ii) generalize and adapt the methods to fit in new situations, and (iii) solve problems that require complex computations in several steps.

The qualitative part of the study was conducted by letting the students answer a questionnaire presented at the end of the last lecture. 148 students present out of 160 in the program filled out the form.

To make it possible to compare the program with the intervention, using everyday examples, with the other four control programs, everything in the course has been left unchanged: the design of the course, the course book and examination are the same for all programs involved. The only difference is the use of everyday examples, discussions around definitions and application examples that have been introduced in the computer science program where the intervention took place.

Results

Quantitative results



In the table, the results of four programs in the diagnostic test and the examination are compared. The program where the intervention took place (red square in the graph) had a significantly better result in the examination although the number of points in the diagnostic test was lower than, or equal to, those in at least two of the other programs.

The quantitative results in the study show that the intervention program with everyday examples set in the calculus course had better examination results that the four control groups in this attempt although slightly more than half the group regularly participated in the lectures. The proportion of students who successfully passed the examination was 68 percent.

The fact that the difference between the intervention group and the control groups was larger can be related to the fact that the change in the design of the course was also minimal. The threshold for introducing changes in the mathematics course was deliberately kept as low as possible to appeal to traditional teachers as a first step in a change process.

Qualitative results

Of the total 160 students in the program, 148 have filled out the form. They answered 14 questions with five alternatives each on the last lecture in the course.

The majority of the students' participated in the lectures as well as in the other scheduled course activities. Participation in the optional supplemental instruction and mathematics workshop, on the other hand, was very low. More than three-quarters of the group used electronic facilities on the Web, like Khan-Academy or lectures from MIT.

Very few had systematically been exposed to the use of everyday examples in mathematics during their earlier senior high school studies. Almost all found the use of everyday examples stimulating, and agreed that they have gained a completely new understanding of mathematical definitions because they were drawn from the examples together with the teacher during lectures. The following are some citations of students' appreciations of the interventions in the course.

Everyday examples:

- *I have acquired a thorough knowledge in mathematics in such a way that theorems and methods can quickly be deduced. Therefore there is no need any longer to remember the methods.*
- *Everyday examples direct the focus to abstractions in mathematics*
- *The use of everyday examples in mathematics has made it more challenging and increased my motivation.*

Application examples:

- *I have gained an understanding for why abstraction is important in mathematics and in working life*
- *The teacher shows how computer scientists can use mathematics and that motivates me.*
- *Abstraction and application examples in programming were great.*

Construction of definitions:

- *My view of abstractions in mathematics has changed. I have picked up a new approach to problem solving.*
- *Mathematical definitions open a new field of understanding.*
- *It was an enriching learning experience to let the students construct mathematical definitions.*

Discussion and conclusions

The use of everyday examples has made it easier for many students to understand abstract concepts. Students appreciated them greatly and stated that they would appreciate the use of everyday examples even in the subsequent mathematical courses. Most students found that the use of everyday examples has also motivated them to work with the course material more than they did in the past.

The application examples, in programming in this case, were very much appreciated by those who could write computer code. They could see the connection between mathematics and their future profession. For those without coding experience, the application examples were presented to be taken at face value, simply demonstrating the connection without any demands for deep understanding. It was made clear to them that no instant understanding was required and that it would in no way affect their grades in the course.

Discussions about concept definitions were also appreciated and, together with everyday examples, they were central to the students' development of abstract understanding. Constructing concept definitions together presupposes that the students have seen the instances from which the definitions are drawn: everyday examples. There is also good reason to believe that the students, like those in the Chappell and Killpatrick study above, will be able to reconstruct the concepts at will and apply them in the future.[17]

Ultimately, using everyday examples is a conscious action where affordances for conceptual understanding are designed into the course and not only left to the student to attain in future once they have reached mathematical maturity through procedural struggle.

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