Convolution for Engineers, Technologists, Scientists, and Other Non-PhDs

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ABSTRACT

One of the more important and one of the least understood principles in electronic engineering technology is convolution. The convolution integral provides a convenient mathematical equation that expresses the output of an linear time invariant system based on an arbitrary signal, x(t), and the system's impulse response, h(t). Because the interpretation takes some effort, most instructors take advantage of the linear transformation into the frequency domain where convolution becomes simply multiplication, eg. Laplace and Fourier transforms. After performing the analysis in the frequency domain, the results are transformed back to the time domain. Regrettably, the students lose the sense that mathematical statements have meaning. This paper examines several MATLAB examples that can be used to vividly present the concepts involved with the convolution integral in less mathematically frightening, more normal terms to the engineering technology and other scientific students that prefer relatable applications rather than mathematical theory.

keywords: convolution and MATLAB

INTRODUCTION

For a continuous, linear, and time-invariant (LTI) system with an impulse response *h(t)*, the response function, *y(t)*, to an input function, *x(t)*, is determined by a convolution integral. This convolution relation, denoted as $y(t) = x(t) \otimes h(t)$, is

$$y = \int_{-\infty}^{\infty} x(\tau)h(\tau - t)d\tau \qquad (1)$$

where τ is a dummy integration variable. Convolution is commutative such that

$$y(t) = x(t) \otimes h(t) \qquad (2)$$

implying

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(\tau - t)d\tau = \int_{-\infty}^{\infty} x(\tau - t)h(\tau)d\tau. \qquad (3)$$

The interpretation of the convolution integral is not intuitive. Engineering students are taught to solve the convolution integral using semigraphical methods while engineering technology students use transform methods like the Laplace transform [1].

Transform methods like the Laplace and Fourier transforms convert signals in the time domain to signals in the frequency domain. Typically, the transformations have a one-to-one correspondence with the original function. Because of this one-to-one correspondence, the time domain functions and the frequency domain functions generated by the Laplace and Fourier transforms are simply tabulated in widely available math tables.

One of the properties associated with the transformation from the time domain into the frequency domain is that convolution in the time domain becomes multiplication in the frequency domain. Likewise convolution in the frequency domain becomes multiplication in the time domain. Many engineering technology students have used the Laplace transform to convert from the time domain to the frequency domain where they do their analysis and then convert their solution back into the time domain.

Semigraphical Methods

Semigraphical methods typically break the convolution integral into distinct graphical sections. For example, consider a system with and impulse response, $h(t)$, that is a 1 unit high and 1 unit wide and an input function, $x(t)$, that is 1 unit high and 1 unit wide as shown in Figure 1.

1. The first graphical step is to determine $h(-\tau)$ which simply involves folding $h(\tau)$ about the vertical axis as shown in Figure 2. Note t is replaced by a dummy variable, τ.

2. The function $h(t-\tau)$ slides from $t = -\infty$ to $t = 0$ without intersecting $x(\tau)$, Figure 3.

3. Letting $h(t-\tau)$ slide from $t = 0$ to $t = 1$, $h(t-\tau)$ and $x(t)$ will intersect at an increasing rate, Figure 4.

   The convolution integrals yields

$$\int_0^t 1\times 1\, d\tau = \tau\Big|_0^t = t \qquad (4)$$

4. Letting $h(t-\tau)$ slide from $t = 1$ to $t = 2$, $h(t-\tau)$ and $x(t)$ will intersect, but at a decreasing rate, Figure 5.

$$\int_{t-1}^1 1\times 1\, d\tau = \tau\Big|_{t-1}^1 = 1-(t-1) = 2-t \qquad (5)$$

5. Sliding $h(t-\tau)$ past $t = 2$ yields no more intersections, Figure 6.

The mathematical solution to the convolution of the two functions *h(t)* and *x(t)* for this particular example is:

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau)h(\tau - t)d\tau = \begin{cases} 0 & t < 0 \\ t & 0 < t < 1 \\ 2 - t & 1 < t < 2 \\ 0 & t > 2 \end{cases} \qquad (6)$$

The output, *y(t),* is shown graphically in Figure 7.

Figure 7 shows that *y(t)* is zero from $-\infty < t < 0$. That is, neither signal intersects the other. For $0 < t < 1$, there is more intersecting area until $t = 1$ is reached where the two signals overlay each other and the bounded area is maximum. From $1 < t < 2$, the limits of integration start decreasing, and the bounded area decreases.

The Joy of Convolution

There is little joy realized when doing the semigraphical method of convolution. The student has to carefully observe when the transitions in the signals cause the limits of integration to change. In the previous simple example, the integration changes four times ($-\infty < t < 0$, $0 < t < 1$, $1 < t < 2$, and $-\infty < t \leq 0$) with two actual changes in the limits of integration. As signals become more complex, there are more transitions and more possibilities for the students to lose their way.

Of course, students must do a few of the simple examples to realize the methodology for the more complex problems. Nevertheless, the mathematical process of convolution quite often overwhelms the technological realities. One method that pumps some joy into the convolution process is the use of convolution applets [2]. Applets are programs written in Java [3]. Since Java is a programming language, the applications are limitless. Nevertheless, applets are typically small embedded applications in a webpage. They are small because they are typically transferred over the internet.

Quick and Dirty

My classes have found that using MATLAB [4] to perform convolution is more informative than either the convolution integral or the convolution applets. The advantage of MATLAB is the ease with which students can program the application and the graphical capabilities that the programs allow. The MATLAB program that demonstrates the convolution of Figure 1 is shown in Program 1 and Figure 8. Program 2 and Program 3 show a couple of MATLAB-based convolution examples that are solved as homework problems. There MATLAB representations are shown Figures 9 and Figure 10, respectively. The convolutions for Program 2 and Program 3 are calculated in Appendix A.

Conclusions

The Java applet and MATLAB application programs each vividly represent the same technological realities as the convolution integral, however, in less mathematically frightening, more graphically acceptable terms for the average engineering, engineering technology, science, and other non-Ph.D students. Although many engineering technology curriculums mask convolution by transferring the solution of problems into the frequency domain, a working knowledge of convolution is important in several disciplines within engineering and engineering technology such as digital signal processing and medical imaging. Convolution can be learned and is really useful, and it doesn't require a Ph.D. in mathematics. Using MATLAB, it has never been easier to experiment with convolution.

References

[1] Goldberg, I., Block, M., and Rojas, R. *A Systematic Method for the Analytical Evaluation of Convolution Integrals*. IEEE Transactions on Education, vol. 45, no. 1, February 2002. pp 65-69.

[2] Crutchfield, Stephen. *The Joy of Convolution*. http://www.jhu.edu/~signals.

[3] *Java Applets: Introduction*. http://www.echoecho.com/applets.htm

[4] MathWorks, *MATLAB & SIMULINK, Student Version*. The MathWorks, Inc. Natick, MA. 2007.

Figures and Tables

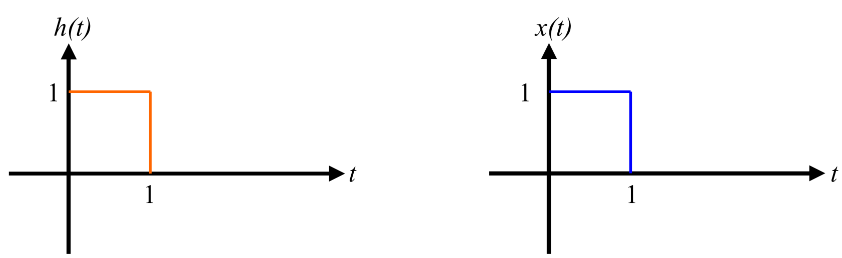

Figure 1.

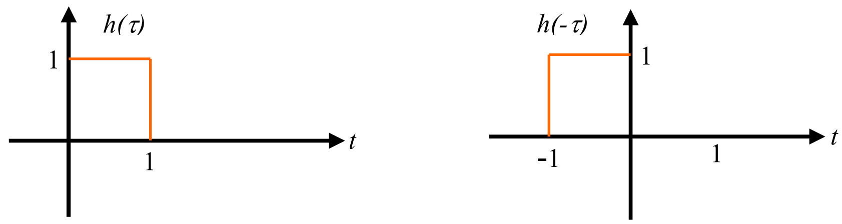

Figure 2.

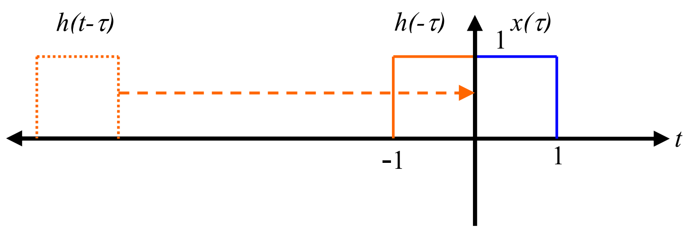

Figure 3.

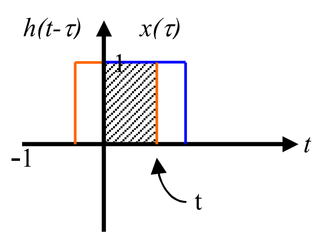

Figure 4.

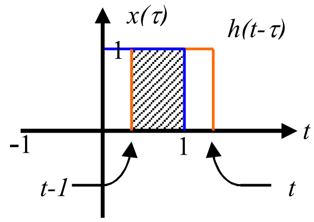

Figure 5.

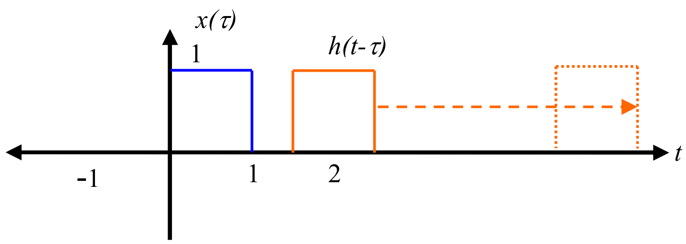

Figure 6

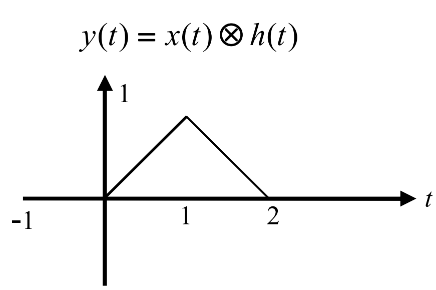

Figure 7

$y(t)=\int_{-\infty}^{\infty} x(\tau)h(\tau-t)d\tau$
 	
$y(t)=\begin{cases}0 & t<0\\ t & 0<t<1\\ 2-t & 1<t<2\\ 0 & t>2\end{cases}$
	`tint=0;`
	`tfinal=10;`
	`tstep=.01;`
	`t=tint:tstep:tfinal;`
	`x=1*((t>=0)&(t<=1));`
	`subplot(3,1,1), plot(t,x)`
	`axis([0 4 0 2])`
	`h=1*((t>=0)&(t<=1));`
	`subplot(3,1,2),plot(t,h)`
	`axis([0 4 0 2])`
	`t2=2*tint:tstep:2*tfinal;`
	`y=conv(x,h)*tstep;`
	`subplot(3,1,3),plot(t2,y)`
	`axis([0 4 0 2])`

Program 1

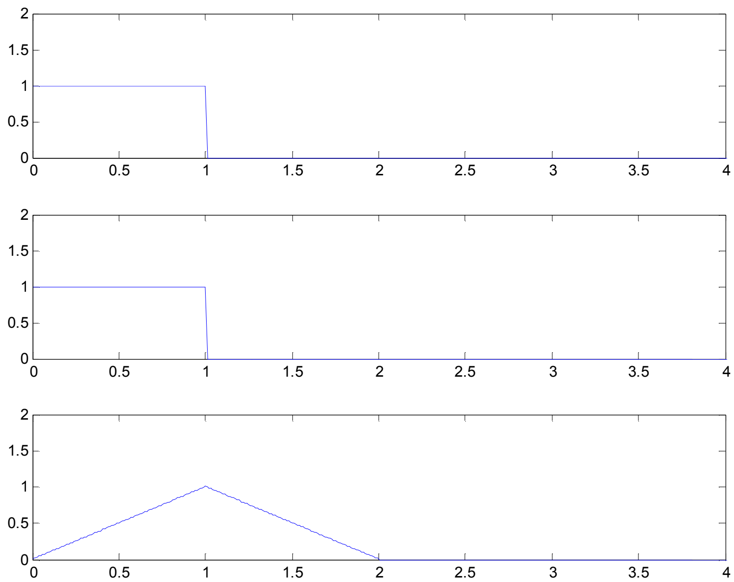

Figure 8

$$y(t)=\begin{cases}\dfrac{t^2}{2} & 0<t<1\\ \dfrac{1}{2} & 1<t<2\\ \dfrac{1-(t-2)^2}{2} & 2<t<3\\ 0 & \text{elsewhere}\end{cases}$$	`tint=0;`
	`tfinal=10;`
	`tstep=.01;`
	`t=tint:tstep:tfinal;`
	`x=t.*((t>0)&(t<1));`
	`subplot(3,1,1), plot(t,x)`
	`axis([0 5 0 1.5])`
	`grid on`
	`h= 1*((t>=0)&(t<=3));`
	`subplot(3,1,2), plot(t,h)`
	`axis([0 5 0 1.5])`
	`grid on`
	`t2=2*tint:tstep:2*tfinal;`
	`y=conv(x,h)*tstep;`
	`subplot(3,1,3),plot(t2,y)`
	`axis([0 5 0 1.5])`
	`grid on`

Program 2

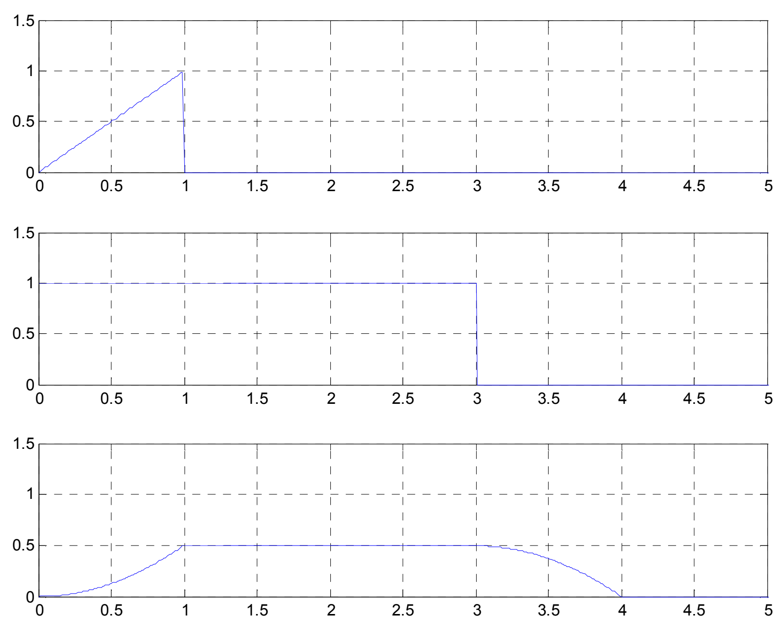

Figure 9

<table>
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$$y(t)=\begin{cases}2\left(e^{t}-e^{-1}\right) & -1<t<0\\ 4-2e^{-t}-2e^{-1} & 0<t<1\\ 4\left(1-e^{-1}\right) & 1<t<2\\ 4-2e^{-1}-2e^{(t-3)} & 2<t<3\\ 2\left(e^{(3-t)}-e^{-1}\right) & 3<t<4\\ 0 & \text{elsewhere}\end{cases}$$

</td>
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```
tint=-3;
tfinal=10;
tstep=.01;
t=tint:tstep:tfinal;
x=(exp(t).*((t>-1)&(t<0)) + exp(-t).*((t>0)&(t<1)));
subplot(3,1,1), plot(t,x)
axis([-3 5 0 2])
grid on
h=2*((t>=0)&(t<=3));
subplot(3,1,2), plot(t,h)
axis([-3 5 0 3])
grid on
t2=2*tint:tstep:2*tfinal;
y=conv(x,h)*tstep;
subplot(3,1,3),plot(t2,y)
axis([-3 5 0 3])
grid on
```

</td>
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Program 3

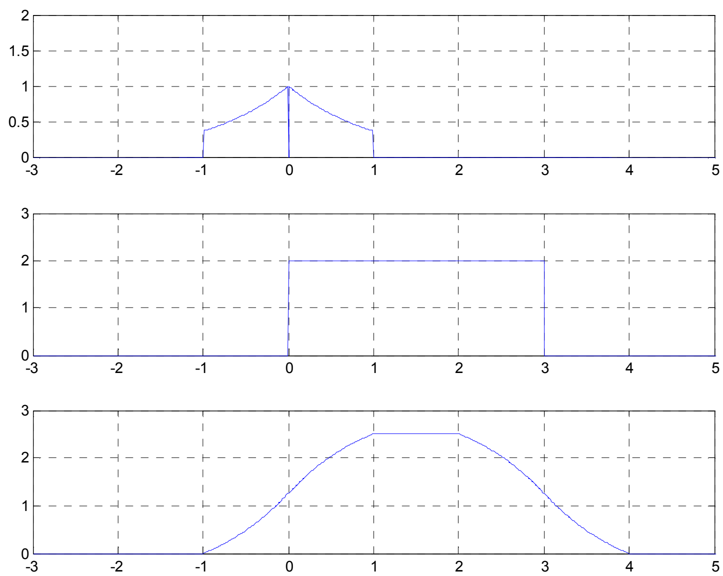
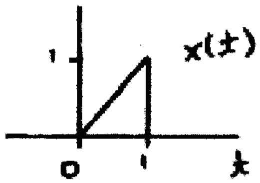
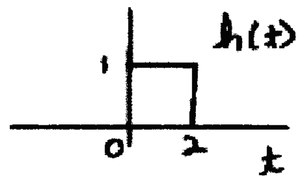


Figure 10

APPENDIX A
Convolution Examples

Program 2



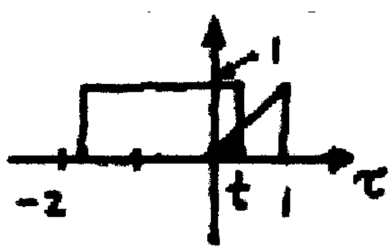


$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

For -∞ < t < 0,

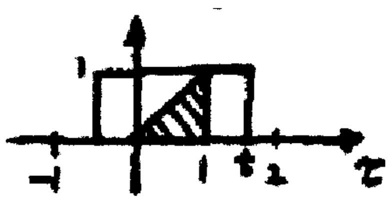
$$y(t) = \boldsymbol{0}$$

For 0 < t < 1



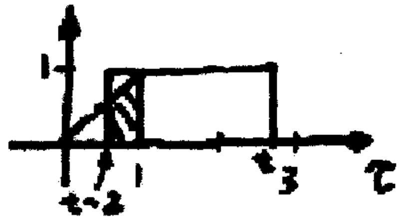
$$y(t) = \int_{0}^{t} \tau \, d\tau = \left.\frac{\tau^2}{2}\right|_0^t = \frac{t^2}{2}$$

For 1 < t < 2



$$y(t) = \int_{1}^{2} \tau \, d\tau = \left.\frac{\tau^2}{2}\right|_1^2 = \frac{(2-1)^2}{2} = \frac{1}{2}$$

For 2 < t < 3

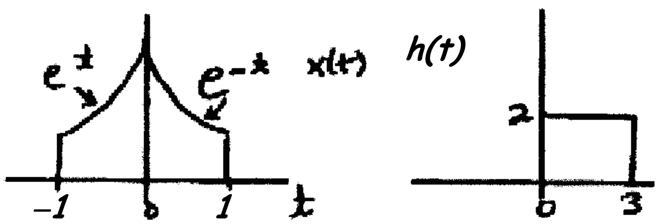


$$y(t) = \int_{t-2}^{1} \tau \, d\tau = \left. \frac{\tau^2}{2} \right|_{t-2}^{1} = \frac{1-(t-2)^2}{2}$$

For 3 < t,

$$y(t) = 0$$

Program 3.

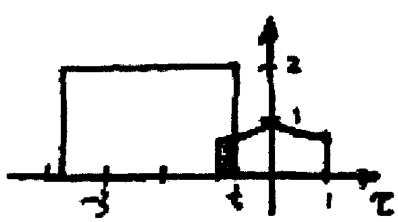


$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

For -∞ < t < -1,

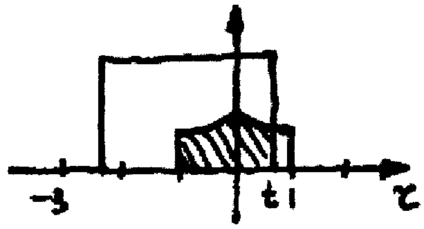
$$y(t) = 0$$

For -1 < t < 0



$$y(t) = \int_{-1}^{t} 2e^{\tau} d\tau = 2e^{\tau} \Big|_{-1}^{t} = 2\left(e^{t} - e^{-1}\right)$$

For 0 < t < 1

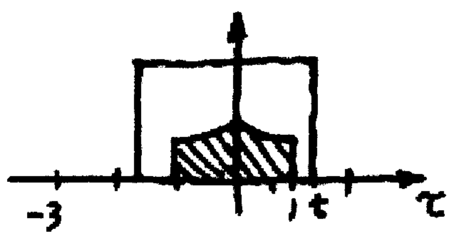


$$y(t) = \int_{-1}^{0} 2e^{\tau} d\tau + \int_{0}^{t} 2e^{-\tau} d\tau = 2\left(e^{\tau}\Big|_{-1}^{0} - e^{-\tau}\Big|_{0}^{t} \right)$$

$$= 2\left(e^{0} - e^{-1} - e^{-t} + e^{0}\right) = 2\left(2 - e^{-t} - e^{-1}\right)$$

$$= 4 - 2e^{-t} - 2e^{-1}$$

For 1 < t < 2

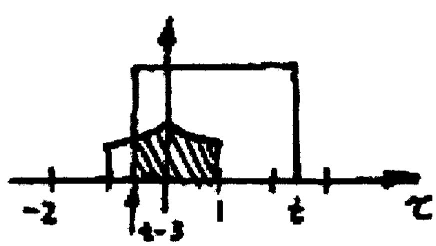


$$y(t) = \int_{-1}^{0} 2e^{\tau} d\tau + \int_{0}^{1} 2e^{-\tau} d\tau = 2\left(e^{\tau}\Big|_{-1}^{0} - e^{-\tau}\Big|_{0}^{1} \right)$$

$$= 2\left(e^{0} - e^{-1} - e^{-1} + e^{0}\right) = 2\left(2 - 2e^{-1}\right)$$

$$= 4 - 4e^{-1} = 4\left(1 - e^{-1}\right)$$

For 2 < t < 3

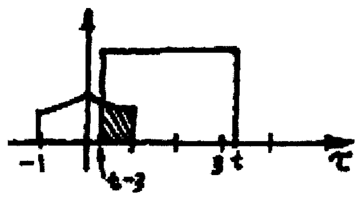


$$y(t) = \int_{t-3}^{0} 2e^{\tau} d\tau + \int_{0}^{1} 2e^{-\tau} d\tau = 2\left(e^{\tau}\Big|_{t-3}^{0} - e^{-\tau}\Big|_{0}^{1} \right)$$

$$= 2\left(e^{0} - e^{(t-3)} - e^{-1} + e^{0}\right) = 2\left(2 - e^{-1} - e^{(t-3)}\right)$$

$$= 4 - 2e^{-1} - 2e^{(t-3)}$$

For 3 < t < 4



$$y(t) = \int_{t-3}^{1} 2e^{-\tau} d\tau = 2\left(-e^{-\tau}\Big|_{t-3}^{1} \right)$$

$$= 2\left(-e^{-1} + e^{-(t-3)}\right) = 2\left(e^{-(t-3)} - e^{-1}\right) = 2\left(e^{(3-t)} - e^{-1}\right)$$

For 4 < t

$$y(t) = 0$$