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Coordinate Transforms and Dual Bases: a Teaching Aid for Undergraduate Engineering Students

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Coordinate transforms and dual bases: a teaching aid for undergraduate engineering students

When engineering students are introduced to subjects like classical mechanics, elasticity, electricity and magnetism they encounter – occasionally for the first time – tensors in practical applications. Tensors have an innate structure irrespective of the coordinates employed; the coordinate systems can be chosen as a matter of taste and convenience to make solving a problem as simple as possible. Coordinates are tools that allow the mathematical analysis of engineering problems, but they are not an intrinsic part of these problems. The freedom to choose coordinates for convenience without being dependent on that choice rests in the center of the understanding of tensors.

A complicating issue in students' understanding is the emergence of co- and contravariance in tensor algebra and calculus. Rectilinear but oblique angled coordinates capture the essence of this duality without the necessity of making use of derivatives. While Cartesian coordinates involve a metric that remains invisible, i.e. the unit matrix, oblique angled coordinates inherently produce non-diagonal matrices representing their metric tensor. Relations between the basis vectors are no longer simple due to the loss of orthonormality. Remedy to this loss of convenience can be found by a second set of basis vectors that is reciprocally orthonormal to the original basis, the dual basis. The representation of a tensorial quantity in the dual bases paves the way for the students' comprehension of co- and contravariance.

In this paper a computer program is presented that visualizes user-defined and, in general, oblique angled basis vectors, and plots the associated dual basis vectors. The basis vectors can be manipulated interactively by mouse drag and the associated dual basis vectors are modified simultaneously by the program. Position vectors can be inserted and their co- and contravariant components are computed and displayed. In addition, general rotations of a Cartesian coordinate frame, described by Euler angles, can be performed.

The visual output of the program is expected to increase and enhance understanding of dual bases and frame rotations and should therefore be well suited as a teaching aid. The program is freely available and can be downloaded from our institution's home page.

Introduction

In traditional elementary introductions in vector algebra, vectors are commonly represented by arrows and vector operations are defined in terms of algebraic operations on Cartesian components. This approach is in general perfectly sufficient for the purpose of preparing undergraduate students for the engineering courses in the first and second year of study. Nevertheless, within this treatment little or no insight into the nature of vectorial and tensorial quantities is provided.

A definition of vectors frequently used in introductory courses is that they are something possessing both direction and magnitude. Taken literally, this would include a broad variety of objects of our intuition or of our thinking. Another definition often encountered is that vectors are a set of numbers, called components. This definition lacks the fundamental property of invariance under coordinate transformations.

There are of course quantities that are usefully represented by arrows, like e.g. the velocity of a moving particle. The length of the arrow represents the particle's speed and the direction of the arrow corresponds to the direction in which the particle is moving. Both quantities do not depend on the choice of the coordinate system. The correspondence between the physical quantity and the mathematical object is preserved under coordinate transformations. These coordinate transformations do not only include rotations of the frame of reference, they also involve a change of the numerical scale of the coordinate axes.

For this second type of coordinate transformation – the scale change – not all vectorial quantities behave equally. As a popular example for this other type of vector the electric field within a parallel-plate capacitor is consulted (see, e.g. G. Weinreich[1]). The vector **d**, which is perpendicular to the plates and reaches from the negative to the positive plate, and the electric field vector **E** exhibit a similar behavior under frame rotations. The correspondence between the quantities “distance” and “electric field” and the vectors that represent them is preserved. When, however, the coordinate scale is altered this is no longer the case. The relation of length of arrow to quantity will be maintained for **d** but not for **E**. This is because the electric field **E** is the gradient of a potential, and a change of the length scale affects the denominator of the differential quotient.

Weinreich uses the term “arrow vector” for displacements and introduces the concept of a “stack” for quantities that behave like the electric field **E**. In a geometrical analogy similar to the arrow vector a stack consists of a number of parallel sheets, plus a loose arrowhead to indicate the sense. The direction of the stack is defined by the orientation of the sheets, whereas its numerical magnitude is given by their density[1]. The contrast between arrows and stacks becomes apparent if the space is compressed in the direction of the stack and the (parallel) arrow. This scale change will cause the arrow to become smaller while the magnitude of the stack gets larger.

Weinreich completes his pictorial menagerie of vectorial quantities by introducing “sheaves”, which are the results of cross products of arrow vectors, and “thumbtacks”, arising from the cross products of stacks. The formal names for these geometrically inspired quantities are “contravariant vectors” for arrows, “covariant vectors” (or, in the context of differential forms, “1-forms”[2]) for stacks, “contravariant vector densities” for sheaves, and “covariant vector capacities” for thumbtacks.

It cannot be the objective of introductory courses to teach that full menagerie. Nevertheless, the concept of co- and contravariance and dual bases strikes the authors as essential enough to be embedded into the course content of undergraduate engineering mathematics. Dual bases emerge in a variety of contexts, reaching from solid state physics over continuum mechanics to multiresolutional analysis.

In solid state physics, for instance, one takes advantage of the fact that the atoms are arranged in crystalline lattices. When considering waves propagating through such a lattice (x-ray diffraction, phonons, electrons), it is convenient to define the reciprocal or dual lattice (see, e.g. Ch. Kittel[3]) because each vector in that reciprocal lattice is orthogonal to a plane in the original crystal lattice and vice versa. Thus, the basis vectors of the reciprocal lattice form the dual basis of the crystal lattice vectors.

In continuum mechanics, the linear elastic behavior of a material is modeled by the relation between a second-order stress tensor and a second-order strain tensor. In the general theory of elastic deformation, the strain tensor is defined as the difference between the covariant metric tensors g_{ij} and G_{ij} for the coordinate systems in the unstrained and strained states of the material. Consequently, a covariant strain tensor is related to a contravariant stress tensor by a fourth-order contravariant elastic tensor[4].

In addition, there exist more abstract concepts of dual bases and biorthogonality. In the framework of multiresolutional analysis, for instance, orthonormal bases of compactly supported wavelets correspond to subband coding schemes with exact reconstruction in which the analysis and synthesis filters coincide (orthogonal wavelets). It proved to be possible to design reconstruction schemes with synthesis filters different to analysis filters, which give rise to two dual Riesz bases of compactly supported wavelets, i.e. biorthogonal wavelets[5]. Such biorthogonal wavelets provide a better compression performance compared with orthogonal wavelets and show a linear phase property.

These are just a few examples among many in science and engineering where dual bases naturally emerge. Undoubtedly, undergraduate engineering students will come in contact with such problems, if at all, only in higher semesters. Nevertheless, getting in touch with biorthogonal bases already in early semesters provides an opportunity to familiarize with this concept in a rather playful manner without being distracted by the entire mathematical background of the new subject area.

The curriculum of the Automotive Engineering undergraduate degree program at Joanneum University of Applied Sciences includes engineering mathematics courses in the first three semesters. The lectures follow typically the contents of textbooks like Kreyszig's Advanced Engineering Mathematics[6]. Elementary vector algebra is taught in the very beginning of the first semester; an advanced discussion of vector algebra takes place within the framework of linear algebra in the second semester of study. In this second semester course emphasis is laid on linear systems and ordinary differential equations. In the third semester the focus is on vector calculus, coordinate transformations, and partial differential equations, and in this course the above discussed consideration of dual bases takes place for the first time.

This paper is structured by first providing a brief outline of our approach to teaching dual bases to sophomore students. Subsequently, an application of dual bases is provided, which is based on a simple statics problem. And finally, a self-developed computer program for the visualization of dual bases is presented with which we provide our students so that they can "play around" with the concept of dual bases.

A brief approach to dual bases

In the following a derivation of dual bases with the aid of oblique-angled coordinate systems will be shown, which is embedded in the engineering mathematics course in the third semester of undergraduate study. In addition to the lectures an introductory guide to tensors calculus (Battaglia and George[7]) is suggested to our students for further reading.

As a common notation the Einstein summation convention is adopted. Whenever an index is repeated twice in a product the index is called dummy index and summation over it is implied. For instance, $c = a_i b_i$ means $c = \sum_{i=1}^{N} a_i b_i$. An index that is not a dummy index is called a free index. As an example, the free index *i* appears in the vector transformation $c_i = a_{ij} b_j = \sum_{j=1}^{N} a_{ij} b_j$ together with the dummy index *j*. This is a preliminary definition that needs to be extended in the course of this section.

A set of vectors $\{\mathbf{a}_i, i = 1, \ldots, N\}$ is said to be linearly independent if $\lambda_i \mathbf{a}_i = \mathbf{0}$ only when $\lambda_i = 0 \;\; \forall i$. The vector space is said to be *N*-dimensional if *N* is the maximum number of linearly independent vectors. In this case the vectors $\mathbf{a}_i$ are said to form a basis, and any other vector may be written as a linear combination that set of vectors.

Most important for engineering applications is the three-dimensional Euclidian space. In this space a basis may always be chosen to be orthonormal, i.e., the basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ are mutually perpendicular and normalized, which can be handily expressed by their mutual dot product as

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} , \tag{1}$$

where δ_{ij} is the Kronecker delta, a quantity that is equal to 0 if $i \neq j$ and 1 if $i = j$. Any vector **a** can be written as $\mathbf{a} = a_i \mathbf{e}_i$ where a_i is the *i*-th component of the vector **a** in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ that spans a Cartesian coordinate system. In this basis the dot product of two vectors becomes particularly simple, namely

$$\mathbf{a}_i \cdot \mathbf{b}_j = (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) = a_i b_j \delta_{ij} = a_i b_i , \tag{2}$$

and the components of vector **a** can be derived from dotting **a** with the associated basis vector

$$a_i = \mathbf{e}_i \cdot \mathbf{a} . \tag{3}$$

When the requirement of orthonormality of the basis vectors, however, is dropped, the dot product of the basis vectors can no longer be expressed by Equation (1). When we consider an arbitrary basis consisting of the three linearly independent vectors $\mathbf{g}_1, \mathbf{g}_2$, and $\mathbf{g}_3$ that are neither mutually perpendicular nor normalized, the dot products yield

$$\mathbf{g}_i \cdot \mathbf{g}_j \equiv g_{ij} \neq \delta_{ij} . \tag{4}$$

The dot products of the basis vectors g_{ij} form a symmetric matrix, which is generally denoted as the “metric”.

The new, arbitrary basis vectors $\mathbf{g}_i$ can of course be expressed as a linear combination of the orthonormal basis vectors $\mathbf{e}_j$, i.e., $\mathbf{g}_i = C_{ij}\mathbf{e}_j$. Thus, the metric g_{ij} can be derived from the transformation matrix $[C_{ij}]$ by the relation

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = (C_{ik}\mathbf{e}_k) \cdot (C_{jl}\mathbf{e}_l) = C_{ik}C_{jl}\mathbf{e}_k \cdot \mathbf{e}_l = C_{ik}C_{jl}\delta_{kl} = C_{il}C_{jl}\,, \tag{5}$$

that is, $[g_{ij}] = [C_{ij}][C_{ij}]^{\mathrm{T}}$.

Any arbitrary vector **a** can now be expressed as a linear combination of the new basis $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ so that

$$\mathbf{a} = a^i \mathbf{g}_i\,. \tag{6}$$

The notation in Equation (6) uses, in contrast to the previous notation, an upper index for the labelling of the vector components. The necessity for such a change in notation will become apparent with the introduction of the dual basis. At this point the Einstein summation convention needs to be specified so that a summation over a repeated index is implied only if it appears once in a lower and once in an upper position.

A first glimpse at the modifications necessitated by dropping the postulation of orthonormality of the basis vectors is provided by the determination of the vector components a^i. The dot product with the associated basis vectors $\mathbf{g}_i$, as suggested by Equation (3)

$$\mathbf{g}_i \cdot \mathbf{a} = \mathbf{g}_i \cdot (a^j \mathbf{g}_j) = a^j \mathbf{g}_i \cdot \mathbf{g}_j = a^j g_{ij} \neq a^j \delta^i_j \tag{7}$$

is not successful and the dot product of two vectors,

$$\mathbf{a} \cdot \mathbf{b} = (a^i \mathbf{g}_i) \cdot (b^j \mathbf{g}_j) = a^i b^j \mathbf{g}_i \cdot \mathbf{g}_j = a^i b^j g_{ij} \neq a^i b^j \delta_{ij} = a^i b^i\,, \tag{8}$$

also turns out to be not as simple as Equation (2).

According to the rules of Einstein summation free indices must be in the same position, i.e. subscript or superscript, on both sides of an equation. This implies the question if there exists another basis, labelled by upper indices, such that

$$a^i = \mathbf{g}^i \cdot \mathbf{a} = \mathbf{g}^i \cdot (a^j \mathbf{g}_j) = a^j \mathbf{g}^i \cdot \mathbf{g}_j = a^j \delta^i_j = a^i\,. \tag{9}$$

This other set of basis vectors $\{\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3\}$, the so called “dual basis”, obviously must fulfil the equation $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$. Due to the fact that the set of vectors $\{\mathbf{g}^i\}$ can be expressed by a linear combination of the basis vectors $\mathbf{g}_i$, a matrix $[C^{ij}]$ exists that relates the dual basis with the original basis vectors, $\mathbf{g}^i = C^{ij}\mathbf{g}_j$, and thus

$$\mathbf{g}^i \cdot \mathbf{g}_j = C^{ik}\mathbf{g}_k \cdot \mathbf{g}_j = C^{ik} g_{kj} = \delta^i_j\,. \tag{10}$$

So the required matrix $[C^{ij}]$ is obviously the inverse of the matrix $[g_{ij}]$. Due to the fact that

$$\mathbf{g}^i \cdot \mathbf{g}^j = C^{ik}\mathbf{g}_k \cdot \mathbf{g}^j = C^{ik}\delta_k^j = C^{ij} \tag{11}$$

the matrix $[C^{ij}]$, which has been identified as $[g_{ij}]^{-1}$, is simply the metric of the dual basis $[g^{ij}]$. Thus, the relation $\mathbf{g}^i = C^{ij}\mathbf{g}_j = g^{ij}\mathbf{g}_j$ that produces the dual basis from the "original" basis can be reversed by

$$g_{ij}\mathbf{g}^j = g_{ij}(g^{jk}\mathbf{g}_k) = g_{ij}g^{jk}\mathbf{g}_k = \delta_i^k\mathbf{g}_k = \mathbf{g}_i \, . \tag{12}$$

Hence, it can be concluded that if the dual basis of $\{\mathbf{g}_i\}$ is $\{\mathbf{g}^i\}$, then the dual basis to the latter is the original basis $\{\mathbf{g}_i\}$. In summary, it can be stated that in the case of a non-orthonormal basis another basis – its dual basis – naturally emerges. Every vector can arbitrarily be expressed either in the original basis $\{\mathbf{g}_i\}$ or in the dual basis $\{\mathbf{g}^i\}$, i.e.

$$\mathbf{a} = a^i\mathbf{g}_i = a_j\mathbf{g}^j \, , \tag{13}$$

where the components a_j can be found by dotting Equation (13) with $\mathbf{g}_j$ to get

$$\mathbf{a}\cdot\mathbf{g}_j = (a_k\mathbf{g}^k)\cdot\mathbf{g}_j = a_k\delta_j^k = (a^i\mathbf{g}_i)\cdot\mathbf{g}_j = a^i g_{ij} = a_j \, . \tag{14}$$

The components a^i, on the other hand, are obtained by dotting the same equation by $\mathbf{g}^i$:

$$\mathbf{a}\cdot\mathbf{g}^i = (a_k\mathbf{g}^k)\cdot\mathbf{g}^i = a_k g^{ki} = (a^j\mathbf{g}_j)\cdot\mathbf{g}^i = a^j\delta_j^i = a^i \, . \tag{15}$$

In the case of both the basis vectors and the vector components the metric g_{ij} as well as its inverse, g^{ij}, is applied to lower or raise the indices.

The vector components labeled by a lower index are called the "covariant" components, and those labeled by an upper index are called the "contravariant" components of **a**. A relation between the co- and the contravariant components, as well as of the original (covariant) and dual (contravariant) basis vectors is obtained by the metric g_{ij} and its inverse, i.e. g^{ij}. The dot product of two vectors can now be expressed as

$$\mathbf{a}\cdot\mathbf{b} = (a^i\mathbf{g}_i)\cdot(b^j\mathbf{g}_j) = a^ib^j\mathbf{g}_i\cdot\mathbf{g}_j = a^ib^jg_{ij} = a^ib_i = (a_i\mathbf{g}^i)\cdot(b_j\mathbf{g}^j) = a_ib_jg^{ij} = a_ib^i \, . \tag{16}$$

The distinction between co- and contravariant vectors is hidden when orthonormal bases are used because $g_{ij} = \delta_{ij}$; such bases are called self-dual – the co- and contravariant components coincide. The dot product of two vectors in orthonormal bases (Equation (2)) in fact do not reveal that such a product is actually performed between the vectors of a space and those of its dual space (as indicated in Equation (16)).

At this stage the covariant and contravariant components of a vector seem to be completely equivalent and do not comprise any physical meaning. They serve as a practical tool for calculations, as for example for the problem presented in the subsequent section, and provide a convenient orthonormality relation between dual spaces.

A deeper meaning of co- and contravariance with a physics context arises when coordinate transformations come into play. Displacements, for instance, transform like contravariant vectors, while gradients of scalar fields transform like covariant vectors. Although this would be essential for profound understanding it goes beyond the learning objective in the third semester of undergraduate studies. Tensor calculus is typically taught in senior or graduate-level courses when the students have a good grasp of linear algebra and calculus. Nevertheless, sophomore students have a sufficient background to become familiar with and understand the concept of biorthogonality and dual bases. It is the authors' opinion that exposing the students to these topics sooner in their academic career will ease, later on, their comprehension of tensor calculus.

An application of dual bases in an undergraduate statics problem

As an example for the application of dual bases a simple structural engineering problem is presented in the engineering mathematics lecture. For the rigid frame shown in Figure 1 the element forces shall be determined as a function of the arbitrary force **F** acting on the truss joint. **F** is given in Cartesian coordinates $\mathbf{F} = F_1'\mathbf{e}_1 + F_2'\mathbf{e}_2 + F_3'\mathbf{e}_3$ with the origin of the coordinate system in the joint of the truss. The force components in Cartesian coordinates are primed so that they can be distinguished from the covariant force components in the oblique-angled basis that will be introduced below.

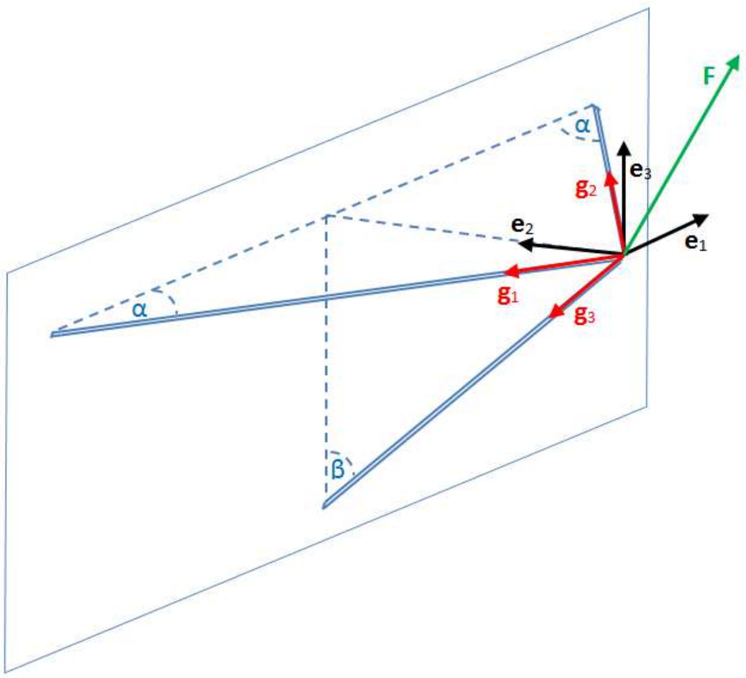

Figure 1: A rigid tripod frame attached to a wall. An arbitrary force **F** is acting on the truss joint.

The truss members at the joint provide the direction of the basis vectors of an oblique angled basis $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$, which can be expressed by the angles α and β as depicted in Figure 1 as

$$\mathbf{g}_1 = \begin{pmatrix} -\cos(\alpha) \\ \sin(\alpha) \\ 0 \end{pmatrix}, \ \mathbf{g}_2 = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \\ 0 \end{pmatrix}, \ \mathbf{g}_3 = \begin{pmatrix} 0 \\ \sin(\beta) \\ -\cos(\beta) \end{pmatrix} \tag{17}$$

with the metric

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \begin{pmatrix} 1 & -\cos(2\alpha) & \sin(\alpha)\sin(\beta) \\ -\cos(2\alpha) & 1 & \sin(\alpha)\sin(\beta) \\ \sin(\alpha)\sin(\beta) & \sin(\alpha)\sin(\beta) & 1 \end{pmatrix}. \tag{18}$$

These basis vectors are normalized and therefore the magnitudes of the truss forces $\mathbf{F}_1 = F^1\mathbf{g}_1$, $\mathbf{F}_2 = F^2\mathbf{g}_2$, and $\mathbf{F}_3 = F^3\mathbf{g}_3$ are simply the (contravariant) components F^1, F^2, and F^3. Static equilibrium in the joint of the truss is expressed by

$$F^1\mathbf{g}_1 + F^2\mathbf{g}_2 + F^3\mathbf{g}_3 + \mathbf{F} = \mathbf{0}. \tag{19}$$

The introduction of dual (contravariant) basis vectors $\{\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3\}$ allows the determination of the truss force F^i by simply dotting the force $\mathbf{F}$ with the respective dual basis vector $\mathbf{g}^i$:

$$-\mathbf{F} \cdot \mathbf{g}^i = F^j\mathbf{g}_j \cdot \mathbf{g}^i = F^j\delta^i_j = F^i. \tag{20}$$

The dual vectors can be determined with the help of the contravariant metric components g^{ij} by

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j. \tag{21}$$

The calculation of the components g^{ij}, i.e. the inversion of $[g_{ij}]$ ($[g^{ij}] = [g_{ij}]^{-1}$) can be avoided by using the relation

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j = \frac{\mathbf{g}_j \times \mathbf{g}_k}{[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]}, \tag{22}$$

with the scalar triple product

$$[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] = \mathbf{g}_i \cdot (\mathbf{g}_j \times \mathbf{g}_k) = \begin{vmatrix} -\cos(\alpha) & \sin(\alpha) & 0 \\ \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & \sin(\beta) & -\cos(\beta) \end{vmatrix} = 2\cos(\alpha)\sin(\alpha)\cos(\beta). \tag{23}$$

The dual basis vectors determined by (22) are

$$\mathbf{g}^1 = \frac{1}{2}\begin{pmatrix} -\sec(\alpha) \\ \csc(\alpha) \\ \csc(\alpha)\tan(\beta) \end{pmatrix}, \ \mathbf{g}^2 = \begin{pmatrix} \sec(\alpha) \\ \csc(\alpha) \\ \csc(\alpha)\tan(\beta) \end{pmatrix}, \ \mathbf{g}^3 = \begin{pmatrix} 0 \\ 0 \\ -\sec(\beta) \end{pmatrix} \tag{24}$$

and can be used for the calculation of the truss forces according to equation (20) to derive the required truss forces as functions of the geometry (α, β) and the (arbitrary) force components F_1', F_2', F_3' (in Cartesian coordinates) as

$$\begin{aligned} F^1 &= \tfrac{1}{2}F_1'\sec(\alpha) - \tfrac{1}{2}F_2'\csc(\alpha) - \tfrac{1}{2}F_3'\csc(\alpha)\tan(\beta) \\ F^2 &= -\tfrac{1}{2}F_1'\sec(\alpha) - \tfrac{1}{2}F_2'\csc(\alpha) - \tfrac{1}{2}F_3'\csc(\alpha)\tan(\beta) \\ F^3 &= F_3'\sec(\beta). \end{aligned} \tag{25}$$

Computer program for the visualization of dual bases

A computer program that visualizes the relation between dual bases in a three-dimensional vector space and, in addition, the rotation of an orthonormal basis by Euler angles was written and implemented in C#. The purpose of this program is to facilitate our students' understanding of duality in the case of oblique-angled bases and of frame rotations by a 3D visualization.

The program provides a graphical user interface consisting of an entry field (right hand side of Figure 2) and a visualization field (left hand side of Figure 2). In the visualization field a Cartesian coordinate system, represented by the (orange) basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ is illustrated. In the entry field within the program option "Custom Basis" the transformation matrix C_{ij} can be defined by entering the Cartesian coordinates of the new and in general oblique-angled basis vectors $\mathbf{g}_1, \mathbf{g}_2$, and $\mathbf{g}_3$. In addition, an arbitrary vector **a** is defined in the Cartesian basis. Then the button "Calculate" can be activated by mouse click and both the basis $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ (red) and the dual basis vectors $\mathbf{g}^1, \mathbf{g}^2$, and $\mathbf{g}^3$ (blue) are added to the orthonormal basis. The vector **a** is depicted in green and the components of this vector are printed in in the "Results" field for all three bases (Figure 3). In addition, the transformation matrix $[C_{ij}]$, its inverse $[C_{ij}]^{-1}$, and the co- and contravariant metric g_{ij} and g^{ij}, respectively, are calculated and displayed. The transformation matrices are used for the calculation of the vector components in the oblique-angled basis. From the representation of a vector **a** in Cartesian and oblique-angled bases,

$$\mathbf{a} = a'_i \mathbf{e}_i = a^j \mathbf{g}_j = a^j C_{ji} \mathbf{e}_i \ , \tag{26}$$

follows that the relation between a'_i and a^j is also provided by the transformation matrix of the bases, i.e. $a'_i = a^j C_{ji}$. Once again, the components are primed in the Cartesian basis in order to distinguish them from covariant components in oblique-angled bases.

The display of the different bases in the visualization field can be rotated by left-click and mouse drag or, alternatively, by the arrow keys on the keypad. Zooming in and out of the field can be carried out either by both left- and right clicking and mouse drag, or by the plus and minus keys on the number pad.

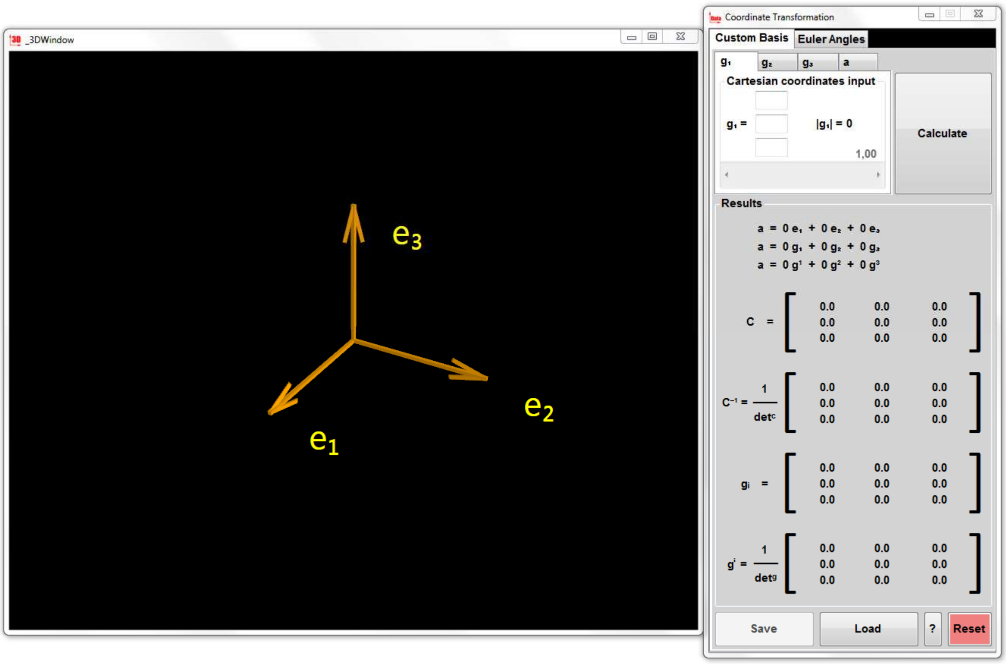

Figure 2: Graphical user interface with orthonormal basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ and empty entry field on the right.

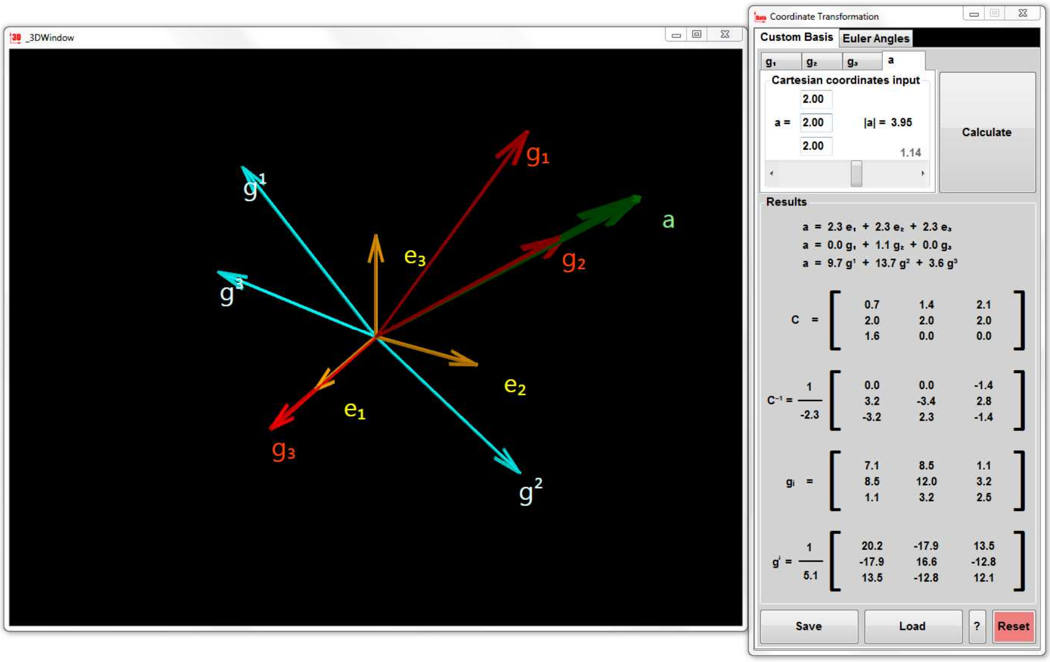

Figure 3: The covariant basis vectors $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ and the vector **a** are defined in the entry field; the metric and the dual basis $\mathbf{g}^1$, $\mathbf{g}^2$, and $\mathbf{g}^3$ are calculated and displayed. The components of **a** in all three bases are printed in the "Results" field.

As an example, the situation in the case of the rigid tripod frame of the previous section is reproduced with the program (Figure 4). With the truss joint in the origin of the coordinate system the red vectors $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ represent the direction of the trusses and the green vector the external force **F**. The Cartesian coordinate system spanned by $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ is displayed as reference frame for better orientation (compare with Figure 1). The dual basis vectors $\mathbf{g}^1$, $\mathbf{g}^2$, and $\mathbf{g}^3$ used for the calculation of the truss forces F^1, F^2, and F^3 are displayed in blue.

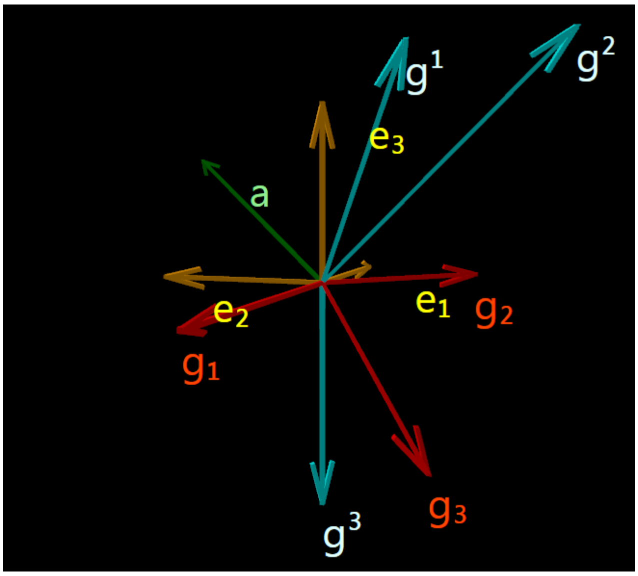

Figure 4: The oblique-angled basis of the rigid tripod frame (red) and an arbitrary force vector (green) are reproduced. The dual basis vectors used for the calculation of the truss forces are displayed in blue.

The conditional equation $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$ for the definition of a dual basis is readily identifiable in the figures. The vector $\mathbf{g}^3$ in Figure 4, for instance, is parallel and opposite to $\mathbf{e}_3$ because it must be perpendicular to the $\mathbf{g}_1 - \mathbf{g}_2$ plane and must point in the negative $\mathbf{e}_3$ direction to form an acute angle with $\mathbf{g}_3$. The magnitudes of associated vectors in dual bases behave in a reciprocal way. The horizontal slider in the “Custom Basis” tab of the user interface (see Figure 3) allows the continuous variation of the magnitude of the selected basis vector $\mathbf{g}_i$. When the vector is stretched the associated dual vector $\mathbf{g}^i$ is compressed, and vice versa. This is exemplified in Figures 5 and 6 with the aid of the vectors $\mathbf{g}_1$ and $\mathbf{g}^1$.

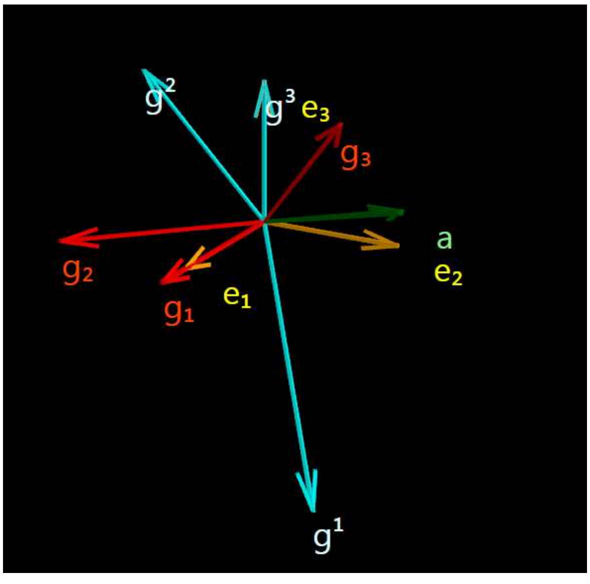

Figure 5: An oblique-angled basis (red) and an arbitrary vector **a** are defined. The dual basis is displayed in blue.

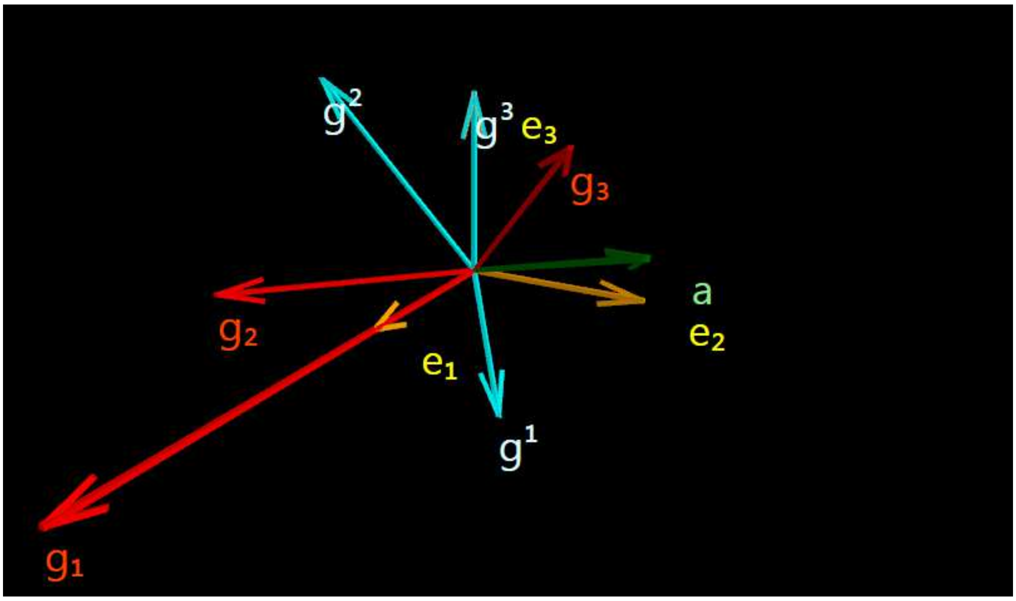

Figure 6: The magnitude of vector $\mathbf{g}_1$ in Figure 5 has been increased by the slider bar and instantaneously the vector $\mathbf{g}^1$ has been accordingly shortened.

An additional feature of the program is the visualization of three-dimensional rigid rotations of an orthonormal coordinate frame. The axes of the original frame are represented by the basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ and the axes of the rotated frame by $\mathbf{h}_1, \mathbf{h}_2$, and $\mathbf{h}_3$ (see Figures 7 and 8). Once again an arbitrary vector – here denoted as **b** – can be read in in the original frame. Then that frame can be rotated by the three Euler angles ψ, ϑ, and φ in the sequence $z - x' - z''$, i.e., a rotation about the *z* axis by ψ, about the new *x* axis by ϑ, and a rotation about the new *z* axis by φ.

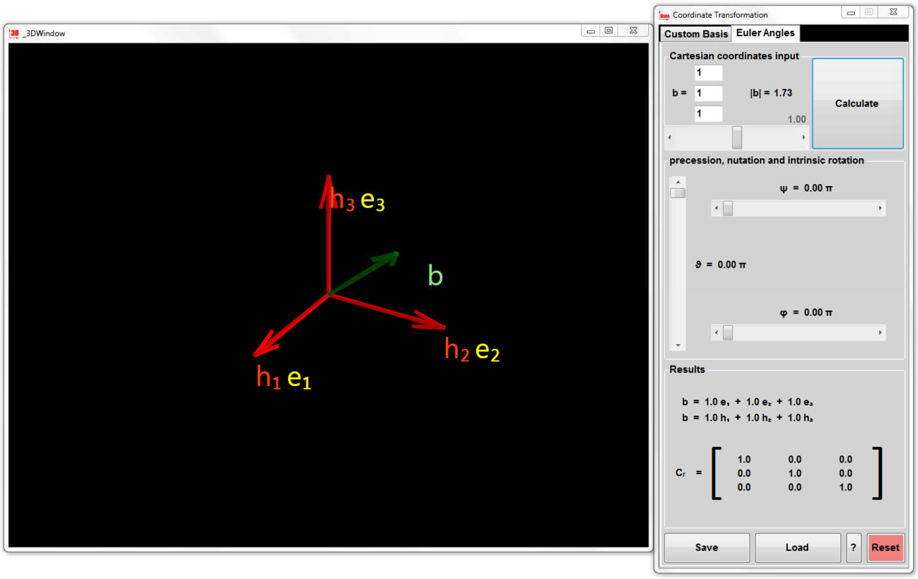

Figure 7: The orthonormal bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3\}$ coincide; all rotation angles are zero. An arbitrary vector **b** (green) is defined in those bases.

The rotation angles ψ, ϑ, and φ can be continuously varied by the sliders in the "Euler angles" tab. These sliders are arranged in an intuitive way and labelled "precession", "nutation" and "intrinsic rotation". In Figure 8 precession and nutation angles are selected and the associated rotation matrix C_r is calculated and printed in the "results" field. In the same field the components of the vector **b** are presented in the original and the rotated frame.

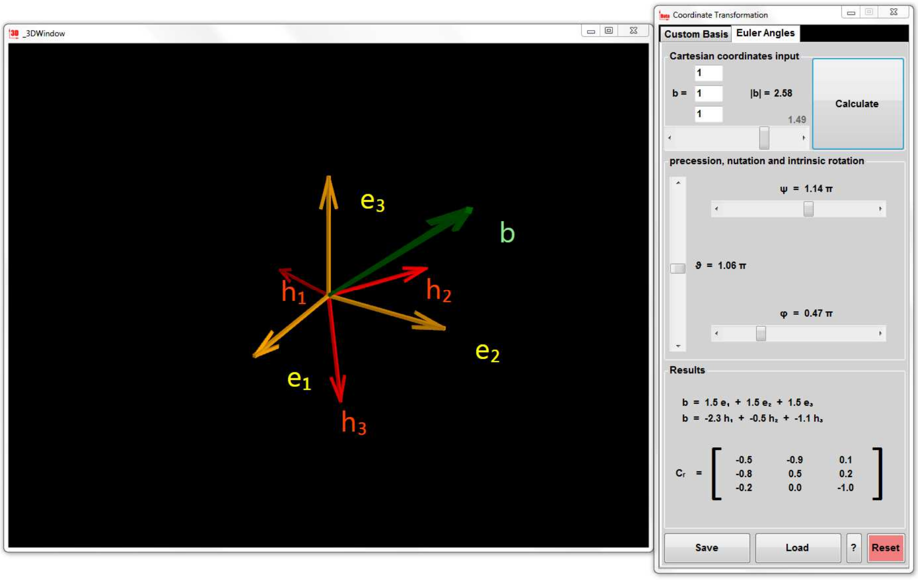

Figure 8: A rotation about the Euler angles ψ and ϑ has been performed. The resulting rotation matrix C_r and the components of **b** (for both bases) are printed in the "Results" field.

In the visualization field the basis vectors of the original frame are depicted in orange, the basis vectors of the rotated frame in red and the vector **b** in green. A change of the rotation angles by the slider bars leads to an immediate frame rotation in the visualization field and both the modified rotation matrix and the new vector components are simultaneously calculated and displayed.

The program can be downloaded, free of charge, from the website https://fahrzeugtechnik.fh-joanneum.at/software/Coordinate-transformation/, and can be installed and executed on computers running the Windows operating systems. The authors would like to thank Wolfgang Dautermann for establishing this link.

Summary and Conclusions

In this paper, a teaching approach for familiarizing sophomore students with the concept of co- and contravariance and dual bases has been presented. This approach comprises the derivation of dual bases with the aid of oblique-angled rectilinear coordinate systems, and the application of biorthogonality to an intuitively accessible engineering problem. Both the derivation and the application are embedded in the regular engineering mathematics course in the third semester of undergraduate study.

In addition, a computer program has been presented that visualizes user-defined and, in general, oblique-angled basis vectors, and plots the associated dual basis vectors. These basis vectors can be manipulated interactively by mouse drag and the associated dual basis vectors are modified simultaneously by the program. Position vectors can be inserted and their coand contravariant components are computed and displayed. Furthermore, general rotations of a Cartesian coordinate frame, described by Euler angles, can be performed.

This approach provides an early and playful access to higher semester topics without being impaired by the additional theoretical structures associated with these topics.

The efficacy and acceptance of the computer program by our students, however, has not been tested so far because it is available quite recently and its use optional. The program can be downloaded, free of charge, from our institution's website and the authors would appreciate feedback on its applicability and suggestions for improvement.

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