AC 2009-215: DAMPED BEAMS: A VERSATILE MATLAB SCRIPT FOR THE ANIMATION OF A VARIETY OF BEAM VIBRATION PROBLEMS

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DampedBeams: A Versitile MATLAB Script for Animation of a Variety of Beam Vibration Problems

Abstract

The authors report development of a single piece of software that animates responses of an elastic beam with external viscous damping and four different sets of support conditions. The motion can be unforced, stimulated by initial deflection, or when externally forced by several combinations of temporal and spatial forms for the forcing function. Several examples are presented.

Introduction

Beams are fundamental building blocks of structures and machine assemblies, hence understanding of their lateral vibration is key to understanding the dynamics of those systems. Free and forced vibration of beams may be analyzed by employing eigenfunction expansions and solving the problem in normal coordinates which are the time dependent coefficients of the generalized Fourier expansion in the eigenfunctions.[1,2] The purpose of this paper is to create an awareness of this and other available software for the animation of beam vibration. The software reported here is designed to animate both free and forced vibration of beams with a variety of boundary conditions and forcing functions, and is intended for a second course in vibrations.

There is a considerable body of literature dealing with the visualization of partial differential equation solutions. A detailed bibliography on visualization of partial differential solutions is available at **http://www.eng.uwyo.edu/classes/matlabanimate**. Reports of animation to enhance understanding of vibration principles have been in the papers of Gramoll and his colleagues, including axial vibration of an elastic bar.[3,4] The first report of animation employing MATLAB handle graphics was by Watkins et al.[5] Animation of the beam lateral vibration using MATLAB was first reported in 2001.[6] An alternative to writing scripts in MATLAB or some other programming language is to use a commercial finite element program as reported by Barker.[7] The application of the handle graphics to animate a wide variety of partial differential equation problems is illustrated in reference.[8]

Why MATLAB[TM]?

MATLAB is perhaps the most widely used general-purpose scientific and engineering software package used in engineering education and engineering practice. It is thus appropriate to develop software for the purpose given here in that computing environment. The reasons to choose MATLAB over other numeric and symbolic computing environments are
- MATLAB is a compact computing environment with simple syntax and array computing;
- It incorporates powerful 2-D and 3-D graphics;
- It is portable and readily available;
- There exist intrinsic robust ODE solution routines available for initial value problems,
- A student edition is available;
- Authors have positive past experiences with other applications.

Visualization Strategies

There are several ways to present dynamic beam deflection which will be denoted here as $y(x,t)$. They are:

- Plots of $y(x,t_i)$ as a function of x for selected values of t (freeze frame),
- Plots of $y(x_j,t)$ as a function of t for selected values of x (measured values of $y(x,t)$ at various locations x_j),
- A plot of $y(x,t)$ as a function of x and t in three dimensions,
- Using animation to plot and erase $y(x,t)$ versus x for a series of closely spaced values of t such that a movie of $y(x,t)$ is attained.

Problems Considered

Consider the Bernoulli-Euler elastic beam of length L, bending stiffness EI and mass per unit length μ with a load $A_0 f(x)\, g(t)$ as illustrated in Figure 1. The constant A_0 will have different units depending on the selections of $f(x)$ and $g(t)$.
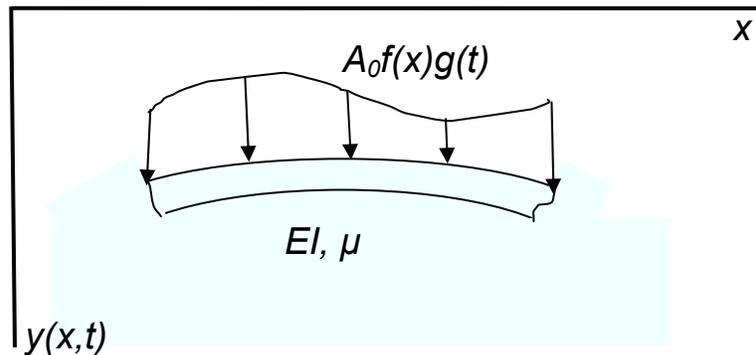
Figure 1. A Segment of a Bernoulli-Euler Beam

The developed software considers four commonly encountered sets of boundary conditions:

1. Pinned-Pinned (Simply Supported),
2. Clamped-Free (Cantilever),
3. Clamped-Pinned,
4. Clamped-Clamped.

These are the conditions which exclude rigid body motion of the beam. The software will solve both the forced and unforced problem. Either there will be no forcing function or the spatial distribution, when present will take on one of two possible forms:

1. Point load at $x = a$ ($f(x) = \delta(x - a)$),
2. Uniform load in space ($f(x) = 1$).

When the forcing function is present the temporal portion of the forcing function $g(t)$ will take on one of three possible forms:

1. No forcing function present, initial deflection alone ($g(t) = 0$),
2. A unit impulse at $t = 0$ ($g(t) = \delta(t)$),
3. A unit step at $t = 0$ ($g(t) = u(t)$),
4. A cosine function of radian frequency ω ($g(t) = cos\ \omega t$).

For the cosine forcing function the interest here is only in the steady-state sinusoidal response. When there is no external forcing function the initial deflection of the beam may be specified in two ways:

1. A concentrated load deflection curve,
2. Modal expansion coefficients.

The analysis for the general problem is carried out in Appendix A and in all cases examined five modes are used in calculating the responses. Graphical results are presented in four ways corresponding directly to the four visualization strategies discussed earlier:

1. A plot of the response of several points along the beam as a function of time,
2. A plot of the beam deflection as a function of x for several values of time,
3. A 3-D plot of the response as a function of time and location,
4. An animation of the beam response as a function of location for many closely spaced values of time.

Several examples should clarify what the software accomplishes.

Example 1.

In this case consider a pinned-pinned (simply supported) beam that is forced by a point load at $x = 0.3L$. The load has a temporal step form so this falls under Case 3 as discussed in Appendix A. The damping ratio in the first mode is $\zeta_1 = 0.2$. The response at the quarter points along the beam is shown in Figure 2. Note that irregularities in the response curves are due to the participation of higher modes.
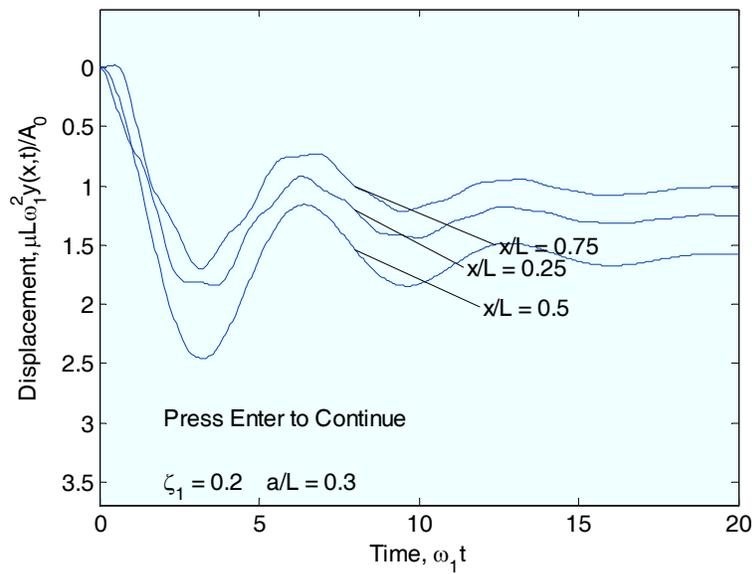
Figure 2. Temporal Responses of the Pinned-Pinned Beam Forced by a Point Load at $x = 0.3L$ that is a Step in Time.

For this problem eleven selected animation frames from the first quarter cycle (first mode) of motion are shown in Figure 3.
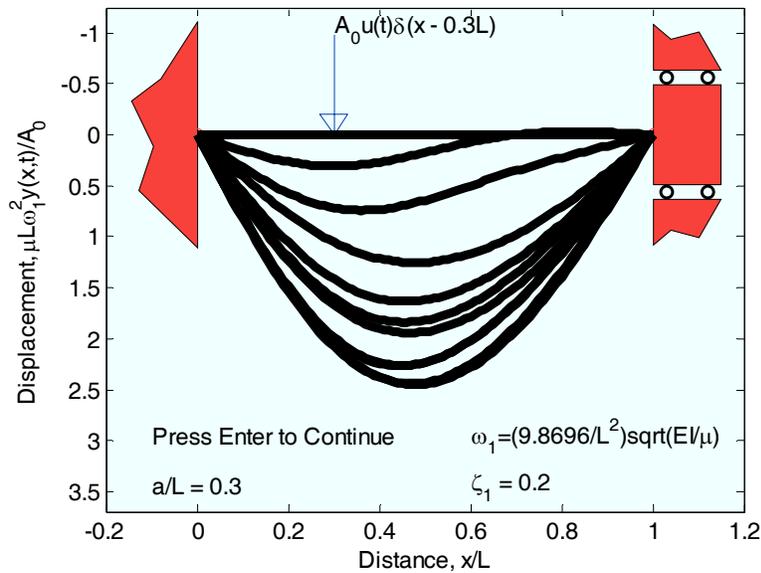
Figure 3. Eleven Selected Animation Frames for the Response of the Pinned-Pinned Beam Forced by a Point Load that is a Step in Time.

Illustrated in Figure 4 is a 3-D plot of the response. Note that as time gets large the deflection approaches the static deflection curve for a point load at $x = 0.3L$.
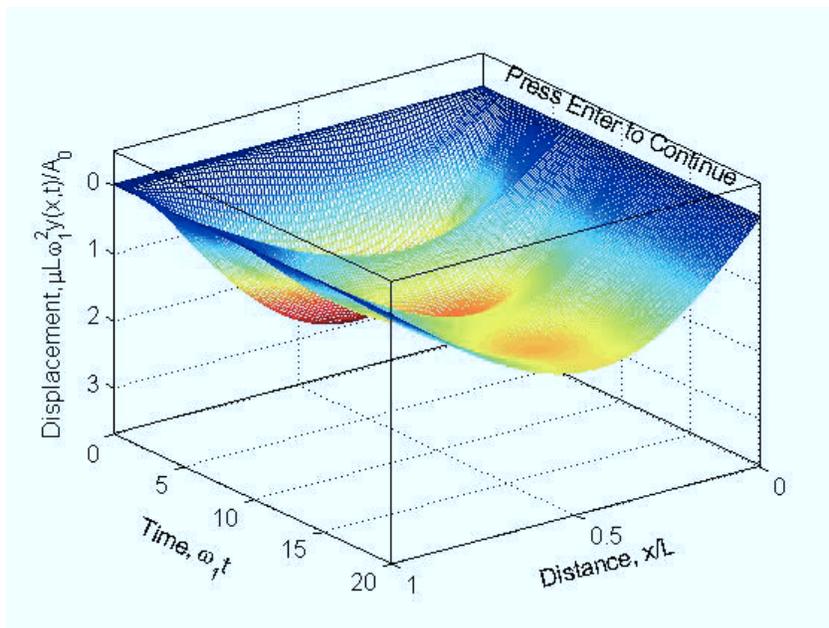
Figure 4. 3-D Plot of Response of the Pinned-pinned Beam Forced by a Point Load at *a = 0.3L* that is a Step in Time.

Example 2

Consider the case of a cantilever (clamped-free) beam forced by a uniform load that is an impulse in time so this is Case 2 as developed in Appendix A. This would be the case of a sudden blast from an acoustic wave. The damping ratio in the first mode is assumed to be $\zeta_1 = 0.01$. The response at the quarter points of the beam is shown in Figure 5. The irregularities in the response curves are, as before, due to participation of higher order modes.
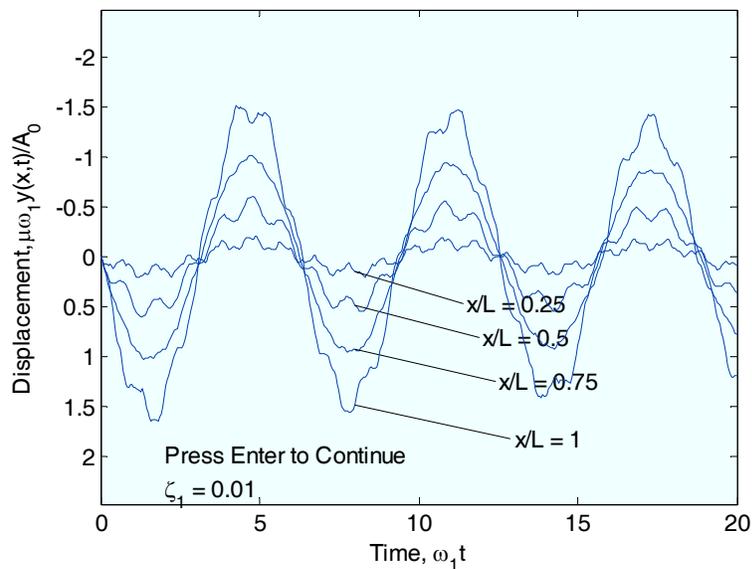
Figure 5. Cantilever Beam Responses for Uniform in Space and Impulsive in Time Applied Force. The First Mode Damping Ratio is $\zeta_1 = 0.01$.

Eleven frames from the first half of the first natural period from the beam animation are illustrated in Figure 6.
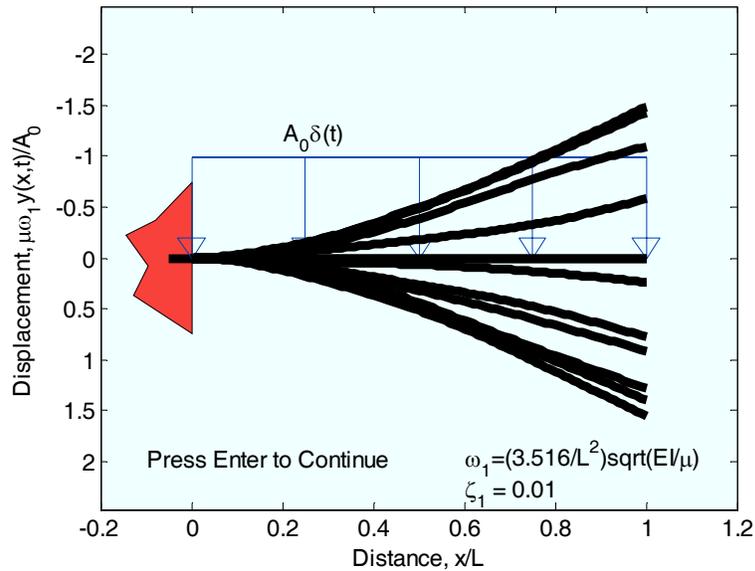
Figure 6. Eleven Selected Animation Frames for the Response of a Cantilever Beam Forced by a Uniform Load that is an Impulse in Time. The First Mode Damping Ratio is $\zeta_1 = 0.01$.
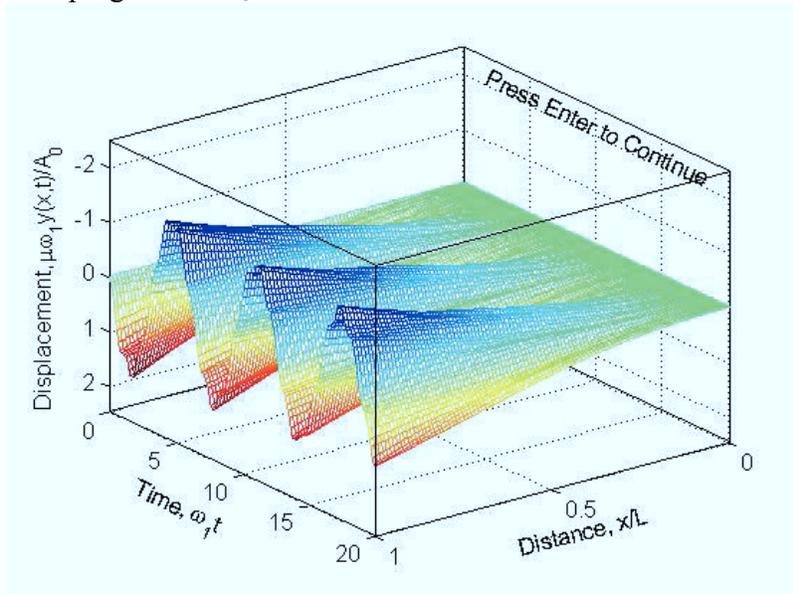
Figure 7. 3-D Plot of Response of a Cantilever Beam Forced by a Uniform Load that is an Impulse in Time. The First Mode Damping Ratio is $\zeta_1 = 0.01$.

Conclusion

The authors have presented an account of a MATLAB script which graphically illustrates dynamic beam problems with a variety of boundary conditions and stimuli to excite the motions.

This should be of use to those teaching a second course vibrations or one in structural dynamics. The script can be employed to illustrate points in the classroom but to be useful in homework exercises will probably require redesign of homework. Since the code length is in excess of 550 lines it is not included in the text of the paper but it and other MATLAB vibration codes for beams and strings complete with an index can be found, free of charge, at the website:

http://www.eng.uwyo.edu/classes/matlabanimate

The authors' attempt to animate the solution to problems with two spatial variables and time (membranes, plates and shells) revealed that time to render the 3-D images in MATLAB is excessive and these problems await a new generation of hardware and software. These problems can be animated by making a MATLAB movie off-line.

Acknowledgement

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Appendix A

With the usual assumptions of a prismatic beam, small angles, plane sections remaining plane and an elastic material the governing equation of motion is[1]

$$EI\frac{\partial^4 y}{\partial x^4} + c\frac{\partial y}{\partial t} + \mu\frac{\partial^2 y}{\partial t^2} = A_o f(x)g(t) \tag{1}$$

Here EI is the bending stiffness, μ the mass per unit length and c the viscous damping coefficient. Note that the forcing function is of the separable type or a product of a spatial function $f(x)$ and a temporal function $g(t)$. With various possibilities, the constant A_0 will have different units for different combinations of the spatial and temporal forcing functions. For a cosine forcing function the interest here is only in the steady-state sinusoidal response. For the unforced, undamped problem the eigenfunctions for the pinned-pinned support condition are

$$\varphi_i(x) = \sqrt{2}\sin\beta_i x \qquad i = 1,2,... \tag{2}$$

and for the other three sets of boundary conditions are

$$\varphi_i(x) = \cosh\beta_i x - \cos\beta_i x + \alpha_i(\sinh\beta_i x - \sin\beta_i x) \qquad i = 1,2,... \tag{3}$$

Eigenvalues $\beta_i L$ for all the boundary conditions have been tabulated by Young and Felgar[9] and by Bishop and Johnson[10] and are reproduced in Table 1.

Table 1
The First Five Eigenvalues $\beta_i L$ for Various Boundary Conditions

BCs i	Pinned- Pinned	Clamped- Free	Clamped- Pinned	Clamped- Clamped
1	π	1.87510	3.92660	4.73004
2	2π	4.69409	7.06858	7.85320
3	3π	7.85476	10.2102	10.9956
4	4π	10.9955	13.3518	14.1732
5	5π	14.1372	16.4934	17.2788

The constants α_i in the eigenfunctions are tabulated in Table 2.

Table 2
The constants α_i in the Eigenfunctions

BCs i	Clamped- Free	Clamped- Pinned	Clamped- Clamped
1	0.7340955	1.0007773	0.9825022
2	1.0184664	1.0000014	1.0007773
3	0.9992245	1.0000000	0.9999665
4	1.0000336	1.0000000	1.0000015
5	0.9999986	1.0000000	0.9999999

The relation between the eigenvalues and the radian undamped natural frequencies is

$$\omega_i = \frac{(\beta_i L)^2}{L^2}\sqrt{\frac{EI}{\mu}} \qquad i = 1,2,... \tag{4}$$

The eigenfunctions for a given set of boundary conditions form a complete orthogonal set and are normalized such that

$$\int_0^L \varphi_i^2(x)\,dx = L \tag{5}$$

The damping can be specified in terms of the first mode damping ratio ζ_1 defined as

$$\zeta_1 = \frac{c}{2\mu\omega_1} \tag{6}$$

and it can be shown that the damping ratios in the higher modes are given by

$$\zeta_i = \zeta_1 \frac{\omega_1}{\omega_i} \qquad i = 2,3,... \tag{7}$$

This implies that the damping ratios for the higher modes decrease with increasing i. This may not be the most realistic damping model but is one that allows the use of the well-known tabulated undamped eigenfunctions for expansions of the responses. An expansion of the spatial portion of the forcing function, $f(x)$, is given in terms of the orthogonal beam functions as

$$f(x) = \sum_{i=1}^{\infty} f_i \varphi_i(x) \tag{8}$$

where the expansion coefficients are given by

$$f_i = \frac{1}{L}\int_0^L f(x)\varphi_i(x)\,dx \quad i = 1,2,... \tag{9}$$

This integral may be evaluated numerically to great accuracy employing the trapezoidal rule. The solution to (1) can be given by a similar expansion

$$y(x,t) = \sum_{i=1}^{\infty} q_i(t)\varphi_i(x) \tag{10}$$

Since the $\varphi_i(x)$ are the eigenfunctions for the unforced and undamped problem we can show that

$$\frac{d^4\varphi_i(x)}{dx^4} = \beta_i^4 \varphi_i(x) \quad i = 1,2,... \tag{11}$$

If (9) and (10) are substituted into (1) using relation (11) the result is a set of uncoupled ordinary differential equations in the normal coordinates or

$$EI\beta_i^4 q_i + c\dot{q} + \mu\ddot{q}_i = A_0 f_i g(t) \qquad 1 = 1,2,... \tag{12}$$

Division through by μ and noting the definitions of (4), (6) and (7) the result is a set of ordinary differential equations in the modal coordinates or

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \frac{A_0 f_i}{\mu}g(t) \quad i = 1,2,... \tag{13}$$

For the forcing functions considered, this equation can be easily solved analytically and substituted into (10) to give the desired solution. In all cases five eigenfunctions were used in the calculation of the response.

Case 1-Initial Deflection, No Forcing

In this case the right hand side of (13) is zero and all that is needed is the initial deflection shape $y(x,0)$ so (10) becomes

$$y(x,0) = \sum_{i=1}^{\infty} q_i(0)\varphi_i(x) \tag{14}$$

where the $q_i(0)$ are given by

$$q_i(0) = \frac{1}{L} \int_0^L y(x,0)\varphi_i(x)dx \quad i = 1,2,... \tag{15}$$

This integral has been evaluated for many common forms for $y(x,0)$ or it may be evaluated to great accuracy by numerical integration using the trapezoidal rule. With the $q_i(0)$ known the solution to (13) is

$$y(x,t) = \sum_{i=1}^{\infty} q_i(0)\varphi_i(x)[\ e^{-\zeta_i\omega_i t}(\ \cos\omega_{di}t + \frac{\zeta_i}{\sqrt{1-\zeta_i^2}}\ \sin\ \omega_{di}t\ )] \tag{16}$$

where

$$\omega_{di} = \omega_i\sqrt{1-\zeta_i^2} \tag{17}$$

Case 2-Impulse in Time

In this case $g(t) = \delta(t)$ and the solution to equation (13) is

$$q_i(t) = \frac{A_0 f_i}{\mu} e^{-\zeta_i\omega_i t} \sin\omega_{di}t \quad i = 1,2,... \tag{18}$$

And the response is given by substitution of (18) into (10)

Case 3-Step in Time

In this case $g(t)$ is the unit step function $u(t)$ and the solution to equation (13) is

$$q_i(t) = \frac{A_0 f_i}{\mu\omega_i^2}\ [\ 1 - e^{-\zeta_i\omega_i t}(\ \cos\omega_{di}t + \frac{\zeta_i}{\sqrt{1-\zeta_i^2}}\ \sin\ \omega_{di}t\ )]\quad i = 1,2,... \tag{19}$$

and as before substitution into (10) yields the response of the beam.

Case 4-Cosine in Time (Steady-State Response)

In this case $g(t) = \cos\omega t$ and the steady-state solution to equation (13) given by the phasor method is

$$q_i(t) = \frac{A_0 f_i}{\mu\omega_i^2}\ |H_i(j\omega)|\cos(\omega t + \angle H_i(j\omega))\qquad i = 1,2,... \tag{20}$$

where

$$H_i(j\omega) = \frac{1}{1-(\omega/\omega_i)^2 + j2\zeta_i(\omega/\omega_i)}\qquad i = 1,2,... \tag{21}$$

and j is the imaginary unit. The solution to the complete problem is now given by expression (10) with substitution of (20) and (21).