AC 2009-500: DEMONSTRATIONS THAT WORK IN THE MATHEMATICS CLASSROOM

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Elton Graves is a member of the Mathematics Department at Rose-Hulman Institute of Technology. He was the co-PI, along with Dr. Robert Lopez of the first $100,000 ILI grant to bring the use of CAS into the classroom. Professor Graves has continued his work on using CAS as a tool to improve the teaching of mathematics at the undergraduate level. He has also been instrumental in the use of physical models to promote the understanding of mathematical concepts.

Demonstrations That Work in the Mathematics Classroom

Abstract

Over the years we have developed several "hands on" demonstrations which help our students to visualize the mathematics they are learning. This paper will present several of these demonstrations including the cycloid curve and brachristochrone problem, Newton's Law of Cooling, directional derivatives, Lagrange multipliers, centers of mass, spring mass systems, and others. By seeing actual demonstrations, students see the relevance of mathematics to the real world situations, and thus gain a sense that the mathematics they are learning is important in their lives as engineers.

Introduction

In calculus courses, differential equations courses, and some upper division mathematics courses students are often presented with concepts that can be demonstrated with "hands on" demonstrations similar to those done in the chemistry, physics, or engineering class. Unfortunately, with the improvement of computer technology and the internet, some of these demonstrations have been relegated to a "show and tell" time for students to watch computer animation or downloaded videos. Still others believe such demonstrations are too time consuming or do not "add value" to the course. They may also believe that the apparatus used in these demonstrations is expensive. In this paper we will show several demonstrations that have been successfully used to help reinforce the mathematical concepts that the students are supposed to be learning.

While some of the equipment used does take some skill to build, none of the equipment used in these demonstrations is expensive. As the reader will see most of the equipment is made from "junk" that is lying around ones house, office, or can be borrowed from another department at your institution.

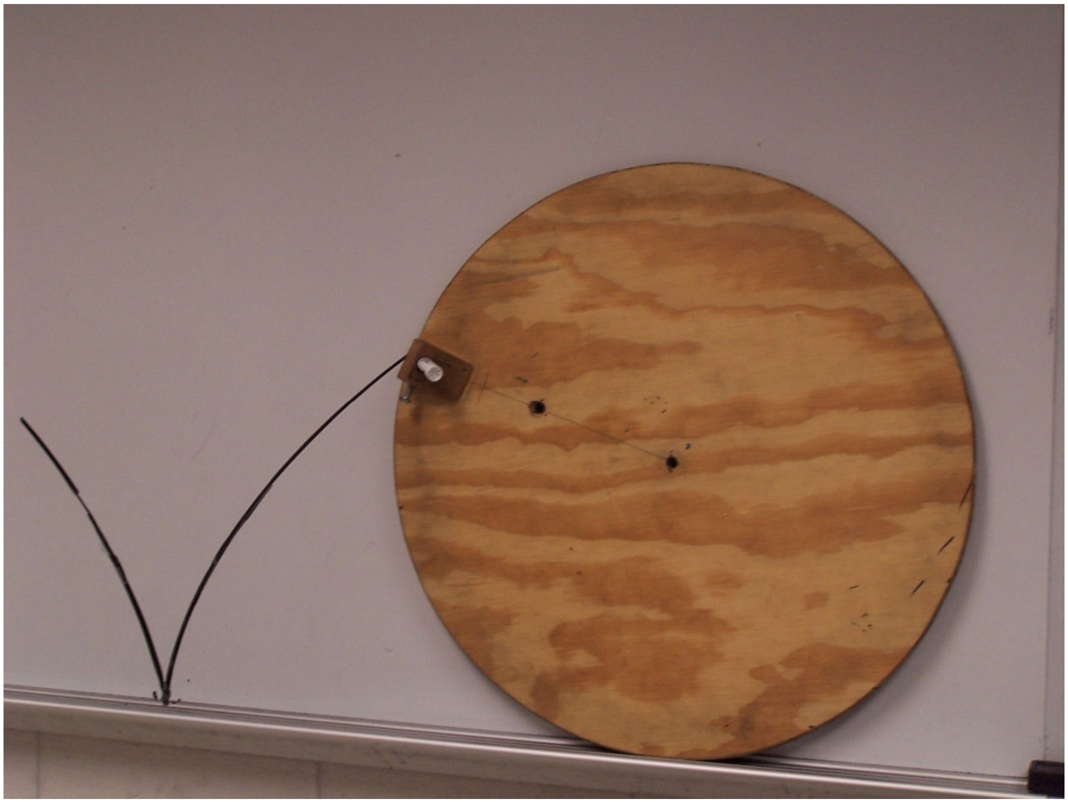
We will try to organize the demonstrations in an order that a student might encounter the topics in a standard mathematics curriculum at an institution where engineering is taught.

Demonstrations

A. The cycloid curve

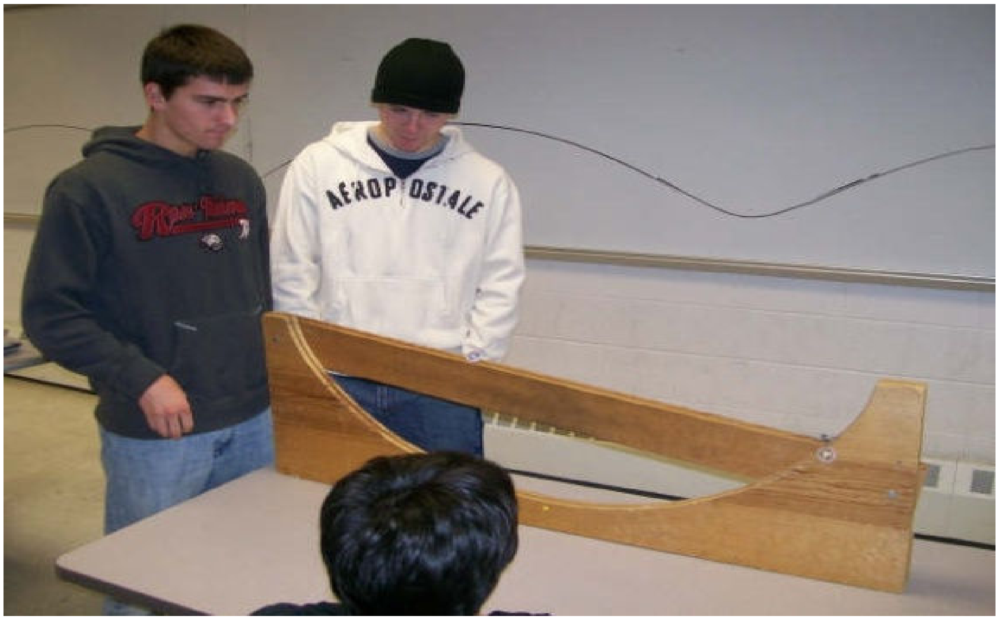
The first demonstration we will consider can be used in any calculus class where parametric equations are taught. This is a classical cycloid curve. To generate the curve we use a circular piece of wood in which a marker can be inserted on the circumference of the circle. (See the picture below.) Our apparatus also has a hole midway to the circumference which will demonstrate the path generated by a light or reflector which is placed beween the axis of a circle and its circumference.

Once the cycloid curve is drawn on the board, we derive the parametric equations for the cycloid curve. The derived parametric equations are then drawn by computer on the computer projection screen. The purpose is not only to derive parametric equations, but also to show that the curves derived in theory are in fact the curves drawn on the board by our apparatus. Our goal is always to show that the theory matches the actual expermimentation.



Picture of Cycloid Generator

Once we have derived and drawn the parametric equations for the cycloid we bring out our second piece of apparatus. This is an actual cycloid curved track made out of wood. We also have marbles that fit the track. We now have a couple of students "race" marbles. One student is allowed to pick the track on which he/she wishes to race his/her marble. The choices are the cycloid or "brachristochrone" track, or a straight line track. We then allow the students in the class to predict which marble will "win" the race. See the picture below. After about two races the students are convinced that the marble traveling along the cycloid will in fact get from point A to point B faster than a straight line. We can then explain to the students that the cycloid curve is the "optimal" curve for getting from point A to point B in the least amount of time.



Students Racing Marbles on Cycloid Track.

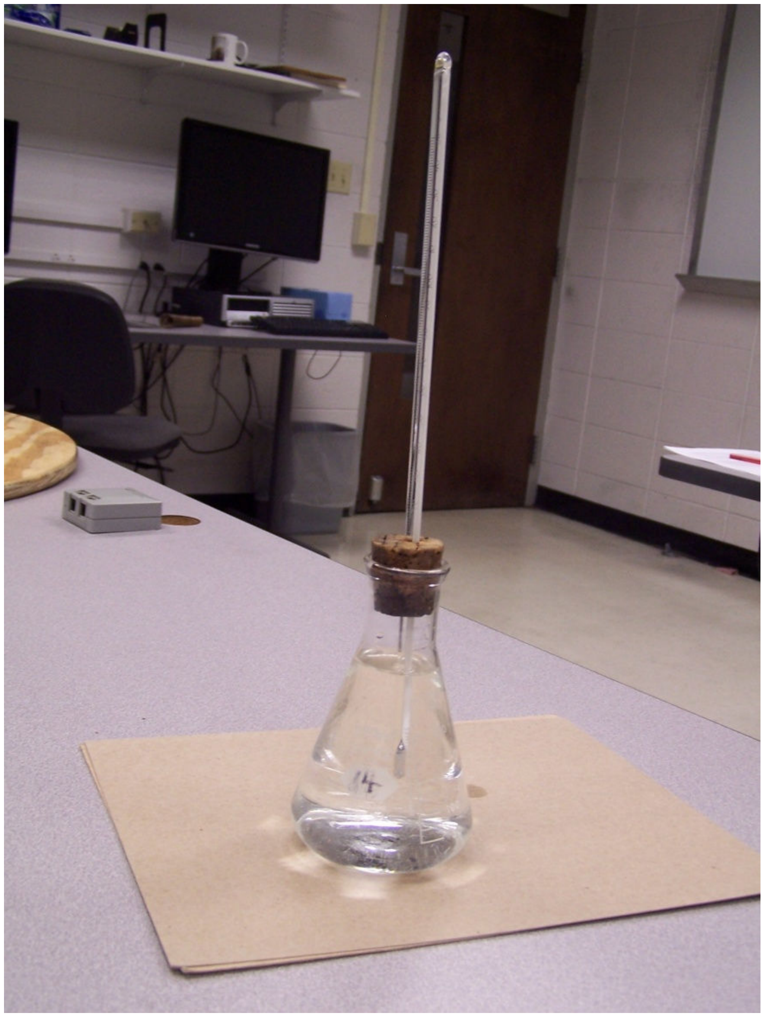
A second demonstration is to start two marbles at different points on the cycloid track and have the students guess which marble will reach the bottom of the curve the fastest. After several trials the students see that both marbles will reach the bottom of the track at exactly the same time. Again we can talk about the geometry and physical properties of the cycloid curve.

These classroom demonstrations, while they take time, do reinforce the idea of the value of parametric equations and how they might relate to the engineering student's studies and life as an engineer.

B) Newton's Law of Cooling

In this demonstration, which can be done in an integral calculus course or a first term differential equations course, we simply demonstrate Newton's Law of Cooling by using a flask, a stopper, a thermometer, and hot water. All the equipment can be borrowed from a Chemistry Department. See picture below.

We now have a student measure the room temperature and then have the student insert the thermometer into the hot water. At time intervals, time $t = 0$; and usually at five minutes; ten minutes; and fifteen minutes; we have the student announce the temperature of the water in the flask. While the class is waiting for the temperature readings, we derive the differential equation for Newton's Law of Cooling. Using the temperature data provided by our student we now solve the differential equation and check the theoretical results. It is not uncommon for the theoretical results at ten minutes to have a relative error of 0.1% of the actual temperature the student measured. We have never had a relative error of more than 1%, and once we actually had exact agreement between the theoretical and actual temperature of the water. Again our purpose is to not only show the derivation of Newton's Law of Cooling, but to show that mathematical models are, in fact, reasonable representations of real world situations.



Apparatus for Newton's Law of Cooling

C) Vectors, straight lines and planes.

In a multivariate calculus course vectors and straight lines can easily be demonstrated with rulers, yard sticks, and, in the author's case, a two meter stick. Two meter sticks can be obtained from the Physics Department. The author has a colleague who uses radio antennas from cars. These are nice since they can be made to be different lengths.

Cardboard is very useful in demonstrating planes and markers can be used to show perpendicular or parallel vectors to planes and lines respectively.

By using these "props" students are able to see the geometry, see when they need to find perpendicular vectors, and thus use the concept of the cross product of two vectors. Again the demonstrations are used to help motivate the students' ideas of the conceptual material being presented.

D) Gradient, directional derivatives, and Lagrange multipliers.

We are lucky there is a small hill on campus which helps us demonstrate the concepts of the gradient, directional derivative, and Lagrange multipliers.

One demonstration is to take the students outside on a "field trip" to demonstrate that the gradient vector is perpendicular to contour lines. To do this we simply ask a student (or instructor) to walk along the hill so that his/her elevation does not change. While the student is doing this we have the student hold out one of his or her arms (the appropriate one) perpendicular to the student's path (see picture below). Thus, for explicit functions where $z = f(x, y)$ the students see that the gradient vector is perpendicular to the contour lines of the function, in our case a hill.

The author was once asked what happens when you teach the class in the winter. The answer, as you can see in the pictures, is we go outside as usual. Snow actually adds to the demonstration as the path the student makes in the snow is visible and is in fact a contour line. Now students can actually see that the gradient vector is perpendicular to the contour line; i.e. the path in the snow.

Showing the Concept of the Gradient Vector for an Explicit Surface

We now ask the student to consider the hill as an implicit function. That is, consider the Earth as a ball. We ask the student to use the two meter stick to demonstrate the geometrical relationship of the gradient vector to the hill when the hill is considered an implicit function. The student simply places the two meter stick in a position so that it is perpendicular to the hill. See picture below.

Showing Concept of the Gradient Vector for an Implicit Surface

At the same time we talk about the gradient vector, we demonstrate the concept of the directional derivative. By standing on the hill and looking in different directions students see that the next step our student will take will be either uphill, or downhill, or possibly there will be no change in elevation as when you walk along a contour line. Once the students have physically seen the directional derivative we return to the classroom and do the mathematics. In this way we feel the students get a better grasp of what the directional derivative is.

Below is a picture which shows the case when the directional derivative is negative. While the student may be looking at a tree at some distance away, his very next step will be downhill.

Case Showing Negative Directional Derivative

When we begin talking about the concept of Lagrange multipliers, we again take a field trip out to the hill. The idea this time is to have a student or the instructor walk in an elliptical path on the hill. With one arm the student points in the direction of the gradient to his/her path, while the other arm points towards the highest point on the hill, the gradient for the hill. When the student arrives at the highest point of his/her path the class tells the student to stop. We now ask the students to discuss the geometric relationship of the student's arms (the gradient vector to the path and the gradient vector for the hill). The students usually will say they are the same, but we correct them to say that the vectors (arms) are in fact parallel. See pictures below. We do the same thing when the student reaches the bottom of the hill. Again the two gradient vectors are parallel. See pictures below. Now we tell the students that when you walk along a path that the maximum value the student obtains (highest point on the hill the student reaches) occurs when the gradient vectors are parallel. We also note that when the student reaches his/her lowest point on the hill the gradient vectors are again parallel. The students have now physically seen the concept of Lagrange multipliers and are ready for the theoretical mathematical discussion of the topic.

Point Where Student Reaches the Maximum Elevation

Point Where the Student Reaches the Minimum Elevation

E) Center of mass

In our multivariate calculus class, when we start talking about centers of mass, we start with center of mass along a line. To demonstrate this we use weights put on a yard stick. Now the weights we use are rather unconventional. As you can see from the picture we use pliers, chemistry clamps and paper clips as our weights. Again the idea is to use common items to demonstrate the mathematics. We can easily move the weights around to show how this will alter the location of the center of mass. The yard stick comes in handy as it has numbers on it which the students can actually read to show exactly where the center of mass is.

Center of Mass on a Straight Line

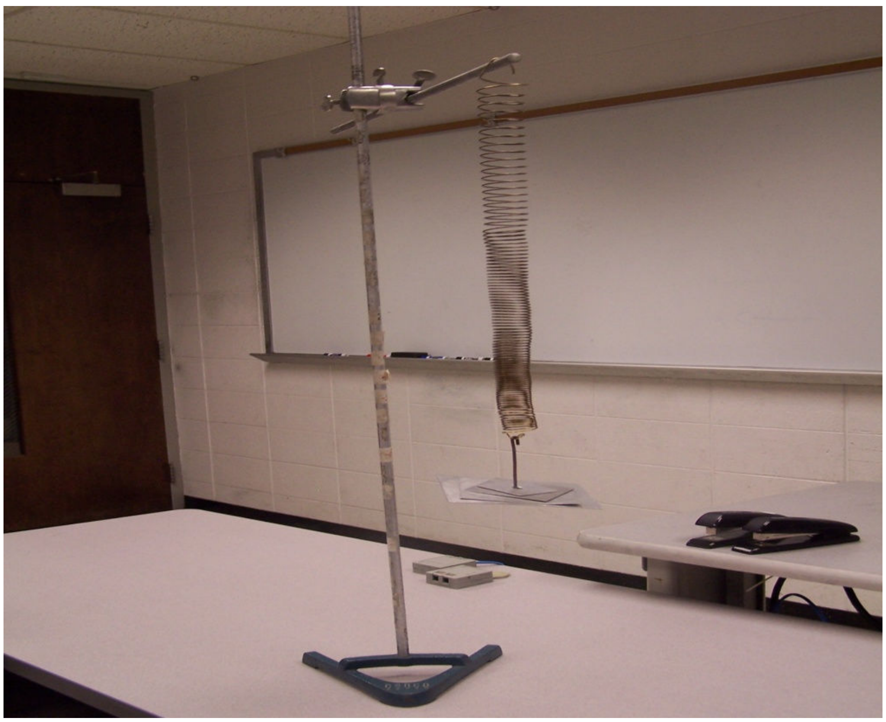
Having done the center of mass on a line we now want to introduce the concept of the center of mass for a plane region. To do this we have three planar surfaces that are basically the same shape. Each surface is made out of two pieces of manila folder paper that are taped together. See the picture below. In two of the "surfaces" we have placed pennies between the two sheets of folder paper. The third "surface" has no pennies in it. We now have three student volunteers balance each surface on the end of a finger. It is clear to the class that each surface balances at a different point. We now go into our discussion of center of mass for surfaces in two-space and the difference between the center of mass and the centroid of a surface.

The Three Similar Surfaces

Students Balancing the Surfaces

F) Spring Mass Systems

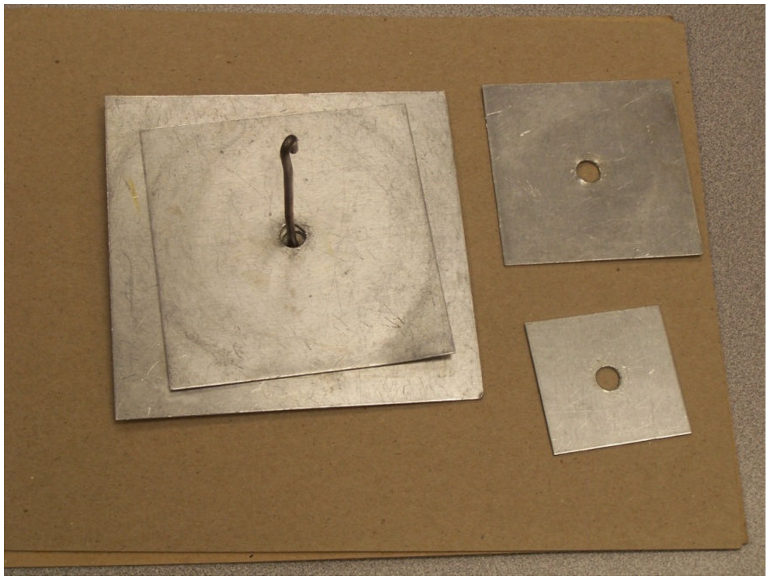
In our differential equations courses one of our standard models is the spring mass system. By bringing in a spring and a mass it is easy to demonstrate how different initial conditions can affect the motion of the mass. We then, of course, do the mathematics and show that mathematical model "agrees" with the physical model we demonstrated in class. See picture below.



A Simple Spring Mass System

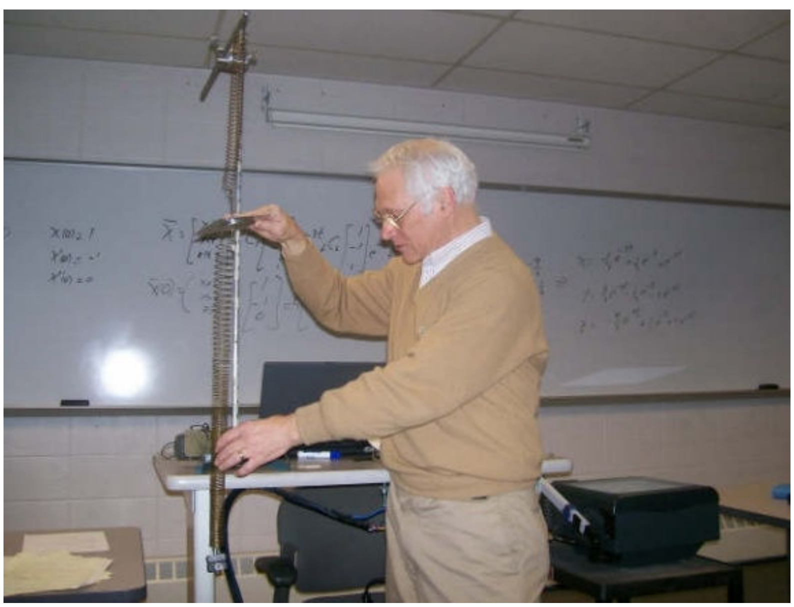
We also have a mass that consist of several squares of aluminum. With this particular mass we can show the effects of viscous damping by using squares of different sizes. When the larger squares are used it is obvious that there is more viscous damping as the mass stops vibrating a lot sooner than when only smaller squares are used. As one can see from the picture below we can easily interchange the metal squares. This allows us to do several different cases quite quickly.

This setup also allows us to set up a project where the students try to actually calculate the damping coefficient for the spring mass system. The students are given the apparatus shown above and a two meter stick. Their job is to find the spring constant for the spring, weight or mass of the squares, and also the damping coefficient for the spring mass system.



Masses used to Demonstrate Damping

When we are talking about systems of differential equations, we can bring in a second spring and a second mass. We can then demonstrate a coupled spring mass system. This demonstration also helps student see how the two second order differential equations are related and can consequently be changed into four first order linear ordinary differential equations.



A Coupled Spring Mass System

Student Response

The student response to using physical models to demonstrate the mathematical concepts discussed in the previous section has been good. The following are student comments from a recent course evaluation from a multivariate calculus course where many of the above demonstrations were used:

"The instructor's use of visual aids help make the 3d concepts easier to understand."

"I loved all of the visual aids (toys) that he used to help explain some of the 3D concepts!"

"The material was covered in full, with much time allotted to explaining the different principles, and solidifying them through visual displays."

"…while teaching with good methods (demonstrating gradients on a surface of grass outside of Crapo, using his 2m stick to show vectors, etc.)."

"He provided very good visual examples of a wide range of variety to help us visualize concepts that students have a hard time visualizing."

"He often brought in props or took students outside to show us how something worked. This helped me remember the material better and helped me visualize problems in three-space."

"…made class interesting by providing real life examples of surfaces and gradients and such"

"…brought props in order to demonstrate concepts to us,"

"He was great at using props to teach us the 3-space concepts."

As you can see from their comments, the students seem to enjoy seeing and participating in the demonstrations. The students also seem to think that the demonstrations help them visualize and remember the concepts better. Thus, while we don't have "proof" that using demonstrations help students learn the material better, it is our belief that the students do better simply because the demonstrations make the course more interesting.

Conclusion

With the advent of more sophisticated computer programs, it is tempting for mathematicians to use the latest and greatest graphics programs to illustrate the mathematical concepts that are trying to be taught in the classroom. While this does help the students graphically see what is going on, students sometime lose the sense that a lot of mathematical concepts and ideas came from real world experiences. In addition, a lot of the students at our school are visual and "hands on" learners. As engineers they actually like to "get their hands dirty" with an object to see how it works.

In this paper we have shown several demonstrations using objects that can easily be found at ones home, hardware store, or local Physics or Chemistry Department. Our idea is to show students a physical situation which when modeled mathematically can lead to an important mathematical concept that the student will use in their later engineering courses.

By actually seeing how the cycloid curve can be built and modeled, the students gain a better appreciation for not only the mathematics of parametric equations, but also the physical properties of the cycloid.

By actually seeing how the gradient vector and contours lines are related, or experiencing the directional derivative, or seeing Lagrange multipliers in action, the students are better able to grasp the concept of the gradient and its importance to engineering and mathematics.

Students come away with a better sense of why differential equations are important when they can actually see that the differential equations accurately model experimental situations such as Newton's Law of Cooling or the motion of a spring mass system.

While we should embrace the use of computers and technology in the classroom, we should not forget that a lot of the mathematics we teach our students was invented to solve real, sometimes very simple, physical situations. Going back to these simple situations is sometimes the best way to help our students get involved in the mathematics they will use throughout their careers.