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Design Equations Developed by Geometric Programming

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He was born and raised in Pittsburgh, PA, Graduated from Penn State in Industrial Engineering in 1963, Graduated from Berkeley in 1964 with a MS in IEOR, worked for US Steel from 1964-66, returned as a full time Instructor in the Department of Industrial Engineering and was a PhD Student in Metallurgy and graduated in1972. He taught Metallurgy at Grove City College and started a Management Engineering Program from 1972-1976 and returned to Penn State in IE from 1976 to 1979. He went to West Virginia University in1979 and taught in Industrial Engineering until retirement in 2014. He has published books on Manufacturing Processes, Geometric Programming and Strategic Cost Fundamentals. He has been a member of ASEE since 1968.

Design Equations Developed by Geometric Programming

History of Geometric Programming

Dr. Clarence Zener is credited for the first paper related to geometric programming and is considered to be "the father of geometric programming" and is also known for the development of the Zener Diode. His publication in 1961, "A Mathematical Aid in Optimizing Engineering Designs[1] in 1961 is considered as the start of geometric programming. Dr. Zener was director of the Westinghouse Research Laboratories in Pittsburgh, PA. Dr. Zener also co-authored, with Professor Richard J. Duffin and Graduate Student Elmor Peterson of Carnegie Institute of Technology in 1967, the first book focused entirely on geometric programming- *Geometric Programming – Theory and Applications*[2]. Geometric Programming was a popular research area with over 60 dissertations published on the topic during 1965 – 1995. These dissertations primarily focused on specific problem solutions, but they did not involve the development of design equations. This paper illustrates the development of design equations for metal casting riser design. Design equations allow the determination of process variables without needing to resolve the original problem. Geometric programming is used when the project objective and/or the constraint equations are non-linear. Previously the project objective and constraint equations had to be linear equations and the problem was solved using linear programming. Geometric programming is used to control the process when the problem objective function and/or the constraints are non-linear. Engineers develop process models and develop design equations to keep the process under control.

In 1967 I was a Ph.D. Candidate in the Metallurgy Department and working as an Instructor in the Industrial Engineering Department at The Pennsylvania State University. I applied and was accepted to attend an NSF Optimization Short Course at the University of Texas taught by Dr. Douglas Wilde and Dr. Charles Beightler using their book, *Foundations of Optimization*[3]. Professor Doug Wilde presented the chapter on geometric programming and during his lecture I immediately recognized that geometric programming as a useful tool for the design of risers(feeders) in metal casting and the development of design equations. I was excited, but my research in metallurgy was focused on the Imperial Smelting Process for lead-zinc blast furnaces, taking various metallurgy courses and since I also had a fulltime Instructor teaching position in Industrial Engineering, I was unable to focus on applying geometric programming to the metal casting riser design problem for two years.

Design Equations Developed Using Geometric Programming for The Metal Casting Riser Design Problem

I had some time available to work on the riser problem in 1969-70 and the problem was to minimize the riser volume for rapid freezing alloys such as cast iron and steel with the constraint that the riser would solidify after the casting. The riser was designed to prevent internal shrinkage of the cast by feeding liquid metal to the casting as it solidifies. The riser size is often large and will be removed from the casting and remelted to supply liquid metal for another casting, but it represents a significant part of the casting cost. My first geometric programming publication was "Optimal Riser Design by Geometric Programming"[4] in the *Cast Metals Research Journal*, in 1971 while I was still an Industrial Engineering Instructor and a metallurgy graduate student at Penn State. The paper developed design equations for minimizing the riser volume for the basic riser shapes of the cylindrical side riser, the cylindrical top riser, and a cylindrical riser with hemispheric top. The example illustrated is the cylindrical side riser. These risers were "blind risers", which were surrounded by sand except for the connection to the casting and for metals that had a short freezing range. The theoretical basis was to assure that the riser(also called feeder) did not solidify completely until after the casting solidified to prevent shrinkage voids in the casting and is based upon Chvorinov's Rule.

Chvorinov's Rule for solidification is:

$$t = q(V/SA)^2 \qquad (1)$$

The solidification time constraint becomes:

$$t_r \geq t_c \qquad (2)$$

which becomes:

$$q_r(V_r/SA_r)^2 \geq q_c(V_c/SA_c)^2$$

and can be reduced to:

$$(V_r/SA_r) \geq (V_c/SA_c) \qquad (3)$$

where:

- t_r = solidification time of the riser
- t_c = solidification time of the metal casting
- $q_r = q_c$ = solidification constants for the molding material are equal as both the riser and the casting surfaces are in the same material – sand.
- D = diameter of cylindrical side riser(Primal Variable)
- H = height of cylindrical side riser(Primal Variable)
- V_r = riser cylindrical volume (Primal Objective Function) = $\pi D^2H/4$ (4)
- SA_r= cooling surface area of riser = $\pi DH + 2\pi D^2/4$ (5)
- Y_c = casting modulus =(V_c/SA_c) (6)
- V_c = casting volume
- SA_c= casting surface area

The objective function is to minimize the riser volume and is:

$$\text{Minimize} \quad V_r = \pi D^2H/4 \qquad (7)$$

The constraint $t_r \geq t_c$ becomes:

$$V_r/SA_r \geq V_c/SA_c = Y_c$$

$$((\pi D^2H/4)/(\pi DH + 2\pi D^2/4)) \geq Y_c \qquad (8)$$

The constraint must be written in the form of ≤ 1; thus Equation (8) becomes

$$4Y_c D^{-1} + 2Y_c H^{-1} \leq 1 \qquad (9)$$

The Primal Form of the cylindrical side riser design problem is:

$$\text{Minimize:} \quad V_r = \pi D^2 H/4 \qquad (7)$$

Subject to: $4Y_c D^{-1} + 2Y_c H^{-1} \leq 1$ (9)

The primal form formatted for starting the dual form sets up as:

$$\text{Minimize} \quad \pi D^2 H/4 \qquad (7)$$

$$\text{Subject to:} \quad 4Y_c D^{-1} + 2Y_c H^{-1} \leq 1 \qquad (9)$$

The sigma values obtained from the primal form are signs of the terms which are all positive and thus 1; they are:

Objective function $\sigma_{01} = 1$

Constraint Terms $\sigma_{11,}\ \sigma_{12,}$ and $\sigma_{10} = 1$

The coefficients of sigma are the signs(+ is positive 1(+1) and – is negative 1(-1) and the subscripts indicate the row and the column; the objective function(only one term) term has the subscript $\sigma_{01.}$ The dual formulation(ω values) is based on Equations 7 and 9 and is:

Objective Function:	$1*\omega_{01}$			$= 1$
D terms (exponents)	$2*\omega_{01}$ -	$1*\omega_{11}$		$= 0$
H terms (exponents)	$1*\omega_{01}$		$- 1*\omega_{12}$	$= 0$

Solving these equations, the values are: $\omega_{01} = 1$, $\omega_{11} = 2$, and $\omega_{12} = 1$ (10)

The linearity inequality equation is:

$$\omega_{10} = \omega_{mt} = \sigma_m \sum \omega_{mt}\, \sigma_{mt}$$

$$\omega_{10} \quad = (1)\,[(2*1) + 1*1)] = 3 \qquad (11)$$

The dual objective function is:

$$d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{Tm} (C_{mt} * \omega_{mo} / \omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma}$$

$$d(\omega) = 1[[\{(\pi/4)*1/1\}^{(1*1)}] * [\{(4Y_c * 3/2)\}^{(1*2)}] * [\{(2Y_c * 3/1)^{(1*1)}]]^1$$

$$= 1 *[[\ (\pi/4) \ ] * [\ (6Y_c)^2 \ ] * [\ (6Y_c) \ ]]^1$$

$$d(\omega) = (\pi/4)*(6Y_c)^3 \qquad (12)$$

This is the volume of the riser as determined by the dual function.

Using the primal-dual relationships for the first and second terms of the constraint:

$$4Y_cD^{-1} = \omega_{11}/\omega_{10} = 2/3$$
$$\text{and } D = 6Y_c \qquad (13)$$

$$2Y_cH^{-1} = \omega_{12}/\omega_{10} = 1/3$$
$$\text{and } H = 6Y_c \qquad (14)$$

Thus, for any casting which solidifies by following Chvorinov's Rule, the design equations for the cylindrical riser diameter(D), the cylindrical riser height(H) and cylindrical riser volume(V_r) are:

$$D = 6(V_c / SA_c) \qquad (15)$$
$$H = 6(V_c / SA_c) \qquad (16)$$
$$V_r = (\pi/4)*(6 * V_c / SA_c)^3 \qquad (17)$$

If a casting with is a plate shape of dimensions of 2 x 4 x 4, the volume would be 32 cubic units and the surface area would be 64 square units. The casting modulus (volume/surface area) is 0.50 units and the riser diameter and riser height would be 3 units and the riser volume would be 21.2 These design equations, made when the solidification constants were equal, resulted in a simple relationship and requiring only the volume and surface area of the particular casting to be produced. The solidification constants would be different if insulating sleeves or other exothermic materials are used. If the constraint is controlled by the shrinkage rate, another solution would be needed. This model was developed in the 1969-71 era - before personal computers and calculators were available. Later publications considered top risers, hemispherical top risers, modified hemispherical risers, tapered risers, and insulated risers. Design equations were developed for these riser types during 1970's into the 1990's.

The Cobb-Douglas Cost Minimization Model for the Civil Engineering Construction Sector of Turkey

Ibrahim Guney and Ersoy Oz published the paper "An Application of Geometric Programming"[5] in 2012**.** Their example concerned the minimization of production costs for a fixed production level in the civil engineering construction sector in Turkey using the Cobb-Douglas production function. The authors presented the objective function, the Cobb-Douglas production function, the input data, and the solution obtained. They used geometric programming and presented their solution results in detail for that specific problem, but they did not develop design equations or present any details on their procedure for determining the two output variables. . After reading their paper, I wanted to determine the design equations for the variables in terms of the input constants. The plan was to solve the dual and primal problems and then use the primal-dual relationships to determine the design equations for the primal variables. The design equations results were initially published in a paper in 2015[6] and later in a book[7].

The primal objective function, $Y(x)$, is to minimize production costs which are:

$$Y(x) = r_1x_1 + r_2x_2 \tag{18}$$

Subject to the Cobb-Douglas production constraint, which is:

$$q = A\, x_1^{\alpha}\, x_2^{\beta} \tag{19}$$

The variables are:

- x_1 = labor amount
- x_2 = capital amount

The input constants are:

- r_1 = labor rate = C_{01}
- r_2 = capital rate = C_{02}
- q = desired output level
- A = total productivity factor
- $(q/A) = C_{03}$
- α = labor elasticity
- β = capital elasticity
- $(\alpha + \beta) = 1$

The Cobb-Douglas production constraint must be written in the less than or equal unity form, that is:

$$q/\left(A\, x_1^{\alpha}\, x_2^{\beta}\right) \le 1 \quad \text{or as}$$

$$(q/A)\, x_1^{-\alpha}\, x^{-\beta} \le 1 \tag{20}$$

In Linear Programming, the primal $Y(x)$ and dual $d(\omega)$ objective functions must be equal and this also applies in Geometric Programming. The dual formulation appears more complex, but it results in linear equations which are easier to solve. The dual objective function is NOT linear and is solved after the dual variables have been determined. The dual objective function is:

$$d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{Tm} \left(C_{mt}\ \omega_{mo} / \omega_{mt}\right)^{\sigma_{mt}\omega_{mt}} \right]^{\sigma} \tag{21}$$

where

- σ = signum function for objective function =1 (1 for minimization and -1 for maximization)
- σ_{mt} = signum function for dual constraints ($\pm$ 1)
- $C_{mt} > 0$ positive constant coefficients are required
- ω_{m0} = dual variables from the linear inequality constraints
- ω_{mt} = dual variables of dual constraints
- σ_{mt} = signum function for dual constraints
- $\omega_{00} = 1$ (by definition = the sum of the components of the objective function

Since all the coefficients in equations have positive signs. All the signum values for the dual will be positive, that is:

σ_{00} =1 (objective function is minimization)

σ_{01} =1

σ_{02} =1

σ_{10} =1 (right hand side of constraint is positive)

The dual can be formulated with signum values and equations 18 and 20 determining the dual variables. The objective equation is initially:

Objective Function	$\sigma_{01}*\omega_{01} + \sigma_{02}*\omega_{02}$			$= \sigma_{00} * \omega_{00}$	(22)
X_1 terms	$\sigma_{01}*\omega_{01}$		$-\alpha*\omega_{11}$	$=0$	(23)
X_2 terms		$\sigma_{02}*\omega_{02}$	$-\beta*\omega_{11}$	$=0$	(24)

Inserting the values of σ and ω the dual objective function and variable constraints become:

Objective function	$1*\omega_{01} + 1*\omega_{02}$			$=1* 1$	(22)
X_1 terms	$1*\omega_{01}$		$-\alpha*\omega_{11}$	$=0$	(23)
X_2 terms		$1*\omega_{02}$	$-\beta*\omega_{11}$	$=0$	(24)

Using Equations 22-24, one obtains

$\omega_{00} = 1$

$\omega_{01} = \alpha / (\alpha + \beta)$ = fraction of total cost by first term of primal (25)

$\omega_{02} = \beta / (\alpha + \beta)$ = fraction of total cost by second term of primal (26)

$\omega_{11} = 1 / (\alpha + \beta)$ (27)

Now the value of ω_{m0} can be determined from inequality constraints being positive and m=1, that is:

$$\omega_{m0} = \omega_{10} = \sigma_m \sum_{t=1}^{T_m} \sigma_{mt}\ \omega_{mt} = \sigma_{10}*\sigma_{11}*\omega_{11} = 1 * 1 * (1/(\alpha + \beta)) = 1/(\alpha + \beta) \quad (28)$$

and note that $\omega_{10} = \omega_{11} = (1/(\alpha+\beta))$

The dual objective function(21) can now be formulated as:

$$d(\omega) = 1*[\{r_1*1/(\alpha/(\alpha+\beta))\}^{(1*(\alpha/(\alpha+\beta))} *\{r_2*1/(\beta/(\alpha+\beta))\}^{(1*(\beta/(\alpha+\beta))} *\{q/A\}^{(1*/(\alpha+\beta))}\ ]^1 \quad (29)$$

Since the primal and dual terms must be equal, they can be used to determine the primal variables.

$$C_{0t} \prod_{N=1}^{N} (x_n)^{mtn} = \omega_{0t}\ \sigma_{0t}\ d(\omega) \quad (30)$$

This results in: $r_1 * x_1 = (\alpha/(\alpha+\beta)) * 1 * d(\omega)$ (31)

$r_2 * x_2 = (\beta/(\alpha+\beta)) * 1 * d(\omega)$ (32)

Dividing Eqn 31 by Eqn 32 one obtains:

$$(r_1*x_1)/(r_2*x_2) = (\alpha/\beta) \quad (33)$$

Solving for x_1 one obtains:

$$x_1 = (\alpha/\beta)(r_2/r_1)\, x_2 = [(\alpha * r_2) / (\beta * r_1)] * x_2 \quad (34)$$

For the constraint terms: $C_{mt} \prod_{N=1}^{N} (x_n)^{mtn} = \omega_{mt}/\omega_{mo}$ (35)

Results in $(q/A) = x_1^{\alpha} * x_2^{\beta}$ (36)

Using Equation 34 for x_1 and using Equation 36 to solve for x_2

$$(q/A) = x_1^{\alpha} * x_2^{\beta}$$

$$(q/A) = [(\alpha * r_2) / (\beta * r_1)\, x_2]^{\alpha} \; x_2^{\beta}$$

$$(q/A) = [(\alpha * r_2) / (\beta * r_1)]^{\alpha} * x_2^{(\alpha+\beta)}$$

$$x_2 = (q/A)^{1/(\alpha+\beta)} * [(\alpha * r_2) / (\beta * r_1)]^{((-\alpha)/(\alpha+\beta))} \quad (37)$$

and $x_1 = (\alpha * r_2)/(\beta * r_1)\, x_2 = [(\alpha * r_2)/(\beta * r_1) * (q/A)^{1/(\alpha+\beta)} * [(\alpha * r_2)/(\beta * r_1)]^{((-\alpha)/(\alpha+\beta))}$

$$x_1 = (q/A)^{(1/(\alpha+\beta)} [(\alpha * r_2)/(\beta * r_1)]^{((1-\alpha)/(\alpha+\beta))} \quad (38)$$

The primal objective function can be determined from the primal variables and Equation 18

$$Y(x) = (q/A)^{(1/(\alpha+\beta)} * [\, r_1 * (\alpha * r_2)/(\beta * r_1)^{(1-a)/(\alpha+\beta)} + r_2 * (\alpha * r_2)/(\beta * r_1)^{(-\alpha/(\alpha+\beta))} \,] \quad (39)$$

Now that the formulas for x_1(Eqn. 38), x_2 (Eqn. 37), Y(x) (Eqn. 39) and $d(\omega)$ (Eqn. 29) have been developed, the results are presented in Table 1.

Table 1. Output Data and Input Data for The Cobb-Douglas Cost Minimization Model for the Civil Engineering Construction Sector of Turkey

	Input	Input	Input	Output	Output	r1* x1	r2 * x2	Primal Y(x)	Dual d(ω)	Check Cost r1*x1 + r2*x2
	Production	Labor	Capital	Labor	Capital	Labor	Capital	Primal	Dual	Labor Cost +
Year	Index	Index	Index	Estimate	Estimate	Cost	Cost	Total	Total	Capital Cost=
	q	r1	r2	x1	x2	r1*x1	r2*x2	Y(x)	D(ω)	r1*x1 + r2*x2
2006	291.90	121.88	114.32	62.56	155.63	7625	17791	25416	25416	25416
2007	311.23	137.8	122.32	63.50	166.91	8750	20416	29166	29166	29166
2008	283.64	153.85	140.06	59.82	153.32	9203	21473	30676	30676	30676
2009	228.09	158.53	131.48	46.73	131.48	7408	17286	24695	24695	24695

The other input values for the study were: A=1, α=0.53 and β=0.47 and (α + β = 1)

The results from the design equations produced the exact same results as those obtained by Guney and Oz. Results include both the primal and dual results for the objective function and the total of the labor cost and the capital cost. The fixed input values for the study were: A=1, α=0.53 and β=0.47 and the annual input values were for the Production Index(q), the Labor Index (r_1) and the Capital Index (r_2). I decided to vary the three fixed input variables to investigate the effect upon the results. The values of A were varied and it appeared all positive values gave reasonable results. When the values of α and β were randomly changed, problems occurred. It was found that the values of α and β had to be positive and that the sum of α plus β had to equal 1. The reduced design equation formulas with α+β = 1 are presented in Table 2.

Table 2. Design Equations for Output Variables, Annual Costs, and Objective Functions
(These are the reduced equations using the requirement that α+β = 1)

Variable Name	Symbol	Design Equation
Labor Estimate	x_1	$x_1 = (q/A)* [(\alpha * r_2)/(\beta * r_1)]^{(\beta)}$
Capital Estimate	x_2	$x_2 = (q/A) * [(\alpha * r_2)/(\beta * r_1)]^{(-\alpha)}$
Primal Objective Function	Y(x)	$Y(x)=(q/A)*\{r_1*[(\alpha*r_2)/(\beta*r_1)]^{(\beta)}+r_2*[(\alpha*r_2)/(\beta*r_1)^{(-\alpha)}]\}$
Dual Objective Function	d(ω)	$d(\omega)= [\{r_1*1/(\alpha)\}^{(1*(\alpha)} *\{r_2*1/(\beta)\}^{(1*(\beta)}*\{q/A\}]$
Labor Cost		$= r_1*x_1$
Capital Cost		$= r_2*x_2$
Total Cost		$= r_1*x_1 + r_2*x_2$
Cobb-Douglas Production Index	q	q(input)
Labor Index	r_1	r_1(input)
Capital Index	r_2	r_2(input)

The output in the Geometric Programming Cobb-Douglas Cost Minimization Model produced the exact same results as reported in the paper by Guney and Oz. The primal, dual and Check Cost solutions obtained the same answers for all values of A tested, but they also indicated that the α and β values must be positive and the sum of α and β must be 1. This sum of unity is often required in most Cobb-Douglas cost models, but it may not be necessary in all models. The Design Equations permit an analysis of the impact of the values of the production index, labor index, capital index, A, α, β upon the labor estimate, capital estimate, labor cost, capital cost and the total cost determined by the primal cost, dual cost, and check cost. Additional example problems are presented the book Geometric Programming for Design Equation Development and Cost/Profit Optimization[7] which contains fifteen different detailed examples of design equations development.

Conclusions

Geometric Programming is a technique for optimizing non-linear problems. It can also be used to develop design equations for the variables in small problems, such as for the riser design problem for minimizing the riser volume. It was used in evaluating other riser design for different shapes and different factors such as casting shrinkage and the use of exothermic materials. The Cobb-Douglas cost minimization model was a complex constraint, but also provided three evaluations to the minimum cost and the design equations provided a rapid process for evaluating alternatives values for the input variables. The development of design equations gives a better understanding of the behavior of the process and the impact of the specific variables upon the results. More work needs to develop design equations for the more complex problems.