Design of Shaft and Bearing system in Eccentric and Nonaligned Gears Mounted on Rotating Shafts

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Overview

In Machine Design courses, students usually learn how to design a system consisting of a shaft and its bearings under rotating, bending, transverse, axial, and torsional loads. Although most machine-design textbooks available today cover Rayleigh's and Holzer's methods, which are used in the classroom to find fundamental natural frequencies of the system in question, other important dynamic effects in shaft and bearing system design are not treated or discussed by them. Typically, considering fatigue loading effects, the diameter of the shaft is calculated, and then the deflection of the shaft is evaluated by using static deflection formulas. The static deflection assumption might be reasonable to make in ideal manufacturing situation; however, it will cause serious errors in shafts' deflection resulting from eccentric and nonaligned gears mounted on the shaft. This manufacturing defect induces loads that depend on the rotating speed of the shaft, which, in turn, causes dynamic deflections that are speed-sensitive and could fall beyond the allowable limits of deflection at the shaft's operating speed. The authors addressed this potential manufacturing defect issue in a Machine Design class as a term project, which also required students to transfer and apply content knowledge from their dynamics and vibrations' courses to come up with a viable design for the system in question. The authors and the students together believe that this project rendered important engineering education objectives in this design-oriented course. In this paper, we present all aspects of this successful experience of implementing ABET strategies in the engineering classroom to maximize its reach and potential impact.

To meet these objective and, thus, satisfy the ABET requirement to enhance the content knowledge of engineering design courses, the following project was assigned in our senior level machine design class. Students had four weeks to complete this project and turn in their final design. In the meantime, topics regarding dynamic load effects on the shaft and bearing designs were elaborated upon and discussed in the classroom to advance students knowledge and to help them develop an intuitive understanding of the relevance of these topics alongside assembly tolerances to principal problems in rotating machinery. It is to be mentioned that, although students were asked to design the bearings for their designed shaft, the aspects of that design are not elaborated upon in this paper. The main thrust of the work presented here is to show the dynamic effects of load in shaft design and compare it with the case where only static loads are considered.
Problem Statement

A shaft, transmitting 40.12 hp at 1200 rpm, and its bearings are to be designed to support two spur gears C and D (See Fig. 1). Both gears have pressure angle of 20°, and their radial loads are in the same plane. Gear C is mounted to the right of the left bearing, 8 inches from the left bearing, and has a pitch diameter of 8 in. Due to a manufacturing defect, the symmetry axis of the gear makes an angle of 20° with the axis of the shaft and, therefore, is not properly aligned with the shaft. The bearings are 24 inches apart. Gear D, with a pitch diameter of 6 in., is placed at 6 in. to the left of the right bearing. Its center of mass is off from the shaft’s center line by 0.02 in., but the gear is mounted at right angle to the axis of the shaft. Gears C and D weigh 60 lb and 45 lb, respectively. As a rough guideline for smooth operation of gears, the allowable transverse deflection and slope for spur gears used in this application are 0.01 in and 0.00045 rad, respectively. The angular deflection of the shaft in ball and roller bearings should not exceed 0.04°. In your design, you should be concerned about whirling of the shaft and should guard against excessive deflection of the shaft at the desired operating speed. It is desired that bearings have a life of 5000 hr at a combined reliability of 0.995. After getting thoroughly familiar with this apparatus’ purpose and its requirements or limitations, select the bearings and design the shaft. Make a sketch to scale of the shaft showing all fillet sizes, keyways, shoulders and diameters. Specify the material that you used for your design and justify your choice. Cite any and all references used on a separate page of your report titled References.

Make sure you analyze torsional vibration of the shaft and guard against resonance at the fundamental frequency of torsional vibration. Ignoring the mass of the shaft itself, calculate the critical speed of the shaft using Rayleigh’s method, Dunkerley’s approach and setting the determinant of $\mathbf{K} - \mathbf{M}\omega^2 = 0$. Neglect damping and flexibility of the supporting bearings. Ignore the effects of gyroscopic couples that act on the system.

![Figure 1](image-url)
Load and Shaft Diameter Consideration

To show the importance of dynamic load effects, calculations for shaft design are carried out considering the manufacturing and adjustment defects of the gears; the results are then compared with the usual treatments of static load considerations.

Assuming that gear C delivers the power to the shaft and gear D transfers the power from the shaft, tangential loads are readily evaluated as:

\[
T = \frac{hp \times 550}{2\pi N/60} = \frac{40.12 \times 550}{2\pi \times 1200} \times \frac{60}{2} = 180 \text{ ft. lbf} = 2160 \text{ lbf. in}
\]

\[
F_{1t} = \frac{2T}{d_{pC}} = \frac{2160 \times 2}{8} = 540 \text{ lbf}
\]

and \(F_{2t}\) will be:

\[
F_{2t} = F_{1t} \times \frac{d_{pC}}{d_{pD}} = 540 \times \frac{8}{6} = 720 \text{ lbf},
\]

where \(hp\) is the horse power delivered, \(T\) is the shaft torque, \(N\) is shaft's rpm, \(d_{pC}\) and \(d_{pD}\) respectively are gears C and D pitch diameters, and \(F_{1t}\) and \(F_{2t}\) are tangential gear forces on gears C and D, respectively.

Knowing the pressure angles, the radial forces on gears C and D are found, respectively, as:

\[
F_{1r} = F_{1t} \tan 20^\circ = 197 \text{ lbf} \quad \text{and} \quad F_{2r} = F_{2t} \tan 20^\circ = 262 \text{ lbf}.
\]

The dynamical effects are considered by reflecting first on gear C's tilt of its axis relative to the shaft axis (See Fig. 2).

Figure 2

As it is indicated in the problem statement of the project, the axis of gear C makes an angle of \(\beta = 2^\circ\) relative to the shaft’s axis (axis Z). It was assumed that the center of gravity of the gear C
lies on the shaft’s axis, and hence no dynamic force is exerted on the gear. However, because of the inclination of its axis relative to the shaft’s, rocking moment (alternating moment) is experienced by shaft and gear. To evaluate that rocking moment, three coordinate systems were adopted: An inertial coordinate system XYZ as it is seen in both Figure 1 and Figure 2, a coordinate system xyz attached to the shaft and obtained by rotating the XYZ coordinate about the Z axis through angle \( \theta \) such that \( \theta = \omega t \), in which \( \omega \) is the angular velocity of the shaft, \( t \) is the time, and \( \theta \) is the rotational angle of the shaft. The coordinate system x’y’z’, as it is seen in Figure 2, is aligned with the principal axes of the gear (assuming to be a disk) and is obtained by rotating the xyz-coordinate system through (y) by an angle of inclination \( \beta \). Applying Euler’s equation to the disk and writing the equations in xyz-coordinate, while recalling that \( \omega \) is along the fixed z axis (that is, \( \omega = - \omega_z = \text{constant} \)) and that xz is the plane of symmetry of the gear (\( I_{xz} = 0 \)), one obtains:

\[
\begin{align*}
M_x &= -I_{xx}\dot{\omega} + I_{yz}\omega^2 = 0 \\
M_y &= -I_{yz}\dot{\omega} - I_{xx}\omega^2 = -I_{xz}\omega^2 \\
M_z &= I_{zz}\dot{\omega} = 0,
\end{align*}
\]

where \( I_{xx} \) is the mass product of inertia, and \( M_y \) is the couple on the disk along the y axis. To obtain \( I_{xx} \) one can use the transformation equations for moment of inertia. Assuming \( I_{xy} \) is the mass moment of inertia of the disk about its principal axis x’, and \( I_{yz} \) is its mass moment of inertia about axis z’, then \( I_{xx} \) is given by:

\[
I_{xx} = \frac{I_{xz}z'z' - I_{x'z'}}{2} \sin 2\beta + I_{x'z'} \cos 2\beta = \frac{1}{8}mR^2 \sin 2\beta
\]

where \( m \) and \( R \) are the mass and the radius of the gear, respectively, and \( I_{z'z'} - I_{x'x'} = \frac{1}{2}mR^2 - \frac{1}{4}mR^2 = \frac{1}{4}mR^2 \) is being used in Equation (b). Consequently, Equation (a) renders \( M_y \) as

\[
M_y = -I_{xz}\omega^2 = -\frac{1}{8}mR^2 \omega^2 \sin 2\beta.
\]

Realizing that the disk exerts a moment - \( M_y \) on the shaft, and expressing this moment in terms of the inertial coordinate XYZ, one obtains:

\[
\begin{align*}
M_x &= -M_y \cos \theta = \frac{1}{8}mR^2 \omega^2 \sin 2\beta \cos \theta = \frac{1}{8}mR^2 \omega^2 \sin 2\beta \cos \omega t \\
M_y &= -M_y \sin \theta = \frac{1}{8}mR^2 \omega^2 \sin 2\beta \sin \theta = \frac{1}{8}mR^2 \omega^2 \sin 2\beta \sin \omega t
\end{align*}
\]

These are the rocking moments, which will cause a rocking force on the bearings. As can be seen from the equations, they are affected by rotation rate and have a harmonic nature as well.

The other dynamic load is due to the existence of eccentricity in the disk D. This will induce whirling of the shaft\(^6\), which, in turn, applies a dynamic load on the shaft, which depends on the rate of rotation. Figure 3 depicts this situation.
In this figure, the point O is the intersection of the line, which goes through both the bearings and the disk D. The point G is the center of mass of the disk, and S is the geometric center of the disk, which is also the intersection of the bowed shaft and the disk. Assuming steady synchronous whirl, that is: $r = \dot{r} = \dot{\theta} = 0$ and $\ddot{\theta} = \omega$, ignoring damping and assuming rigid bearings, the amplitude of the whirl is:

$$r = \frac{e \omega^2}{\omega_n^2 \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \quad (f)$$

where the eccentricity $e = SG$ (see Fig. 3), and $\omega_n$ is the natural frequency of the disk’s shaft system. To avoid resonance and excessive deflection of the shaft, it is recommended that the natural frequency of the system be:

$$\omega_n \approx 2 \times \omega \text{ or } 3 \times \omega . \quad (f1)$$

Then, at these operating speeds, the phase angle between $e$ and $r$ is 0. Consequently, the dynamic force exerted on the disk is:

$$F = m(r + e)\omega^2. \quad (f2)$$

Assuming $\frac{\omega}{\omega_n} = 1/2$, (which can be checked and corrected after the natural frequency is calculated), Equation (f) gives $r = \frac{e}{3}$; in addition, components of the dynamic force on the shaft are opposite to the force on the gear and are equal to:

$$F_{SX} = 4m \frac{e}{3} \omega^2 \sin \omega t \quad \text{and} \quad F_{SY} = 4m \frac{e}{3} \omega^2 \cos \omega t ,$$

where $F_{SX}$ and $F_{SY}$ are the X and Y-components of the force exerted on the shaft by the gear D, respectively. The amplitude of this force is:
The amplitude of the rocking moments on the shaft at gear C is also calculated as:

\[
\frac{1}{8} m R^2 \omega^2 \sin 2\beta = \frac{1}{8} \times \frac{60}{32.2} \times \left( \frac{4}{12} \right)^2 (40\pi)^2 \times 12 \times \sin 4^\circ \approx 342 \text{ lbf in} \quad (f3)
\]

Knowing the loading condition in both planes XZ and YZ on the shaft, one can find the maximum alternating bending moment, the maximum mean bending moment, and the mean torque. The Distortion Energy Goodman\(^1\) approach then is utilized to evaluate the shaft’s diameter. Figure 4 and Figure 5 on the following page depict the alternating loading state on the shaft’s axis in planes XZ and YZ, respectively.

Since \(\sin \omega t\) and \(\cos \omega t\) are alternating between \(\pm 1\), the rocking moments at C and dynamic loads at D are going to be in the same or opposite direction of gear loads. Applying equations of
equilibrium, the reaction loads at bearings are evaluated and shear diagrams and bending moment diagrams are readily obtained. Due to the weight of the gears, one should also acquire the mean bending moment diagram as well. Also, note that, when $\cos \omega t$ is at its maximum or minimum, that is ±1, then $\sin \omega t = 0$ and vice versa. Therefore, when a dynamic load in a plane (either XZ or YZ) is at its maximum value (for example +49 lb or +342 lbf.in), its counterpart in the other plane is zero. With this in mind, the alternating bending moment diagrams in planes XZ and YZ, for the worst case scenario, together with the mean torsional diagram are obtained and depicted in Figure 6-Figure 8 in the next page. The mean bending moment diagram is shown in Figure 9.

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**Figure 6 - The torsional diagram.**

**Figure 7 - The alternating bending moment diagram in the XZ plane.**

**Figure 8 - The alternating bending moment diagram in the YZ plane.**
From these diagrams it is evident that point D, which is at the position of the gear D, is the position of the critical bending moment diagram. The total alternating moment at that point is:

$$M_a = (M_x^2 + M_y^2)^{\frac{1}{2}} = (2160^2 + 1882^2)^{\frac{1}{2}} \approx 2865 \text{ lbf. in}$$

With the mean bending moment at D, $M_m = 323 \text{ lbf. in}$, one can now calculate the shaft’s diameter using Distortion Energy Goodman\(^1\) Criterion:

$$d = \left[ \frac{32 \cdot N_f}{\pi} \cdot \sqrt{\left(K_f \cdot M_a \right)^2 \frac{2}{S_e}} + \sqrt{\left(K_{fm} \cdot M_m \right)^2 + \frac{3}{4} \left(K_{fsm} \cdot T_m \right)^2 \frac{2}{S_{ut}}} \right]^{\frac{1}{3}}$$

where $d$ is the shaft’s diameter, $N_f$ is the factor of safety, $K_f$ is fatigue stress concentration factor for alternating bending $M_a$, $K_{fm}$ and $K_{fsm}$ are fatigue stress concentration factors for mean bending $M_m$ and mean torsion $T_m$, respectively, $S_e$ and $S_{ut}$ are modified endurance limit of steel and ultimate tensile strength of steel, respectively. Assuming machined surface 1050 cold-drawn steel and a stepped shaft at location D, the shaft’s diameter is calculated. The detail of such calculations is given in the attached MathCad file. As it is observed from that file, the shaft’s diameter is evaluated to be

$$d = 1.5 \text{ in.}$$

With the shaft’s diameter determined, the next step is to evaluate its deflection under the loads.

Shaft Deflection

The nature of dynamic loads requires vibration consideration of the shaft. Neglecting damping and modeling both the shaft and its loads as lumped parameters, one has:

$$M \ddot{X} + KX = F(t) \quad (g)$$

where $M$ is the mass matrix, $X$ the deflection vector, $K$ the stiffness matrix, and $F(t)$ the applied load to the shaft. Since loadings are both in XZ and YZ planes, Equation (g) is written in both of
these planes, and the total deflection is assumed of the vector sum of the deflection in each plane. The deflection vector in each plane is then defined as:

\[ X = \begin{bmatrix} \delta_1 \\ \theta \\ \delta_2 \end{bmatrix}, \]

where \( \delta_1, \theta, \) and \( \delta_2 \) are deflection of the shaft under gear C, rotation of the shaft under gear C, and deflection of the shaft under gear D, respectively. The mass matrix \( M \) is defined as:

\[ M = \begin{bmatrix} m_c & 0 & 0 \\ 0 & I_c & 0 \\ 0 & 0 & m_d \end{bmatrix} \]

where \( m_c \) is the mass of gear C, \( I_c = m_c k^2 \) is the mass moment of inertia of gear C about its transverse axis, \( k \) is the radius of gyration of the gear about its transverse axis, and \( m_d \) is the mass of the gear D. The force vectors in vertical plane XZ and horizontal plane YZ are:

\[ F(t) = \begin{bmatrix} 60 \\ 0 \\ 45 \end{bmatrix} + \begin{bmatrix} 197 \\ 342 \\ 311 \end{bmatrix} \sin \omega t \quad (h) \]

and

\[ F(t) = \begin{bmatrix} -520 \\ 342 \\ 769 \end{bmatrix} \cos \omega t \quad (i) \]

where the first vector in equation (h) is due to the weight of the gears, while the small couple induced by the eccentricity of gear D about shaft axis is ignored. The stiffness matrix is obtained by first calculating the flexibility matrix \( a \) for the type of loading on the system, and then taking the inverse of \( a \). To populate the flexibility matrix, we consider simply supported beam under the following loading:
where A, C, D, and B refer to bearing A, gear C, gear D, and bearing B, respectively. For case Figure 10(a), the equations for elastic curve and elastic slope for portions AC and CB under the unit load are:

\[ y_{AC} = \frac{b x}{6EI} (x^2 + b^2 - L^2) \]  \hspace{1cm} (j)

\[ \theta_{AC} = \frac{b x}{6EI} (3x^2 + b^2 - L^2) \]  \hspace{1cm} (k)

\[ y_{CB} = \frac{a(L - x)}{6EI} (x^2 + a^2 - 2Lx) \]  \hspace{1cm} (l)

Likewise, the equations for elastic curve and elastic slope for portions AC and CB under the unit moment load of Figure 10(b) are:

\[ y_{AC} = \frac{x}{6EI} (x^2 + 3a^2 - 6aL + 2L^2) \]  \hspace{1cm} (m)

\[ \theta_{AC} = \frac{1}{6EI} (3x^2 + 3a^2 - 6aL + 2L^2) \]  \hspace{1cm} (n)

\[ y_{CB} = \frac{1}{6EI} (x^3 - 3Lx^2 + x(2L^2 + 3a^2) - 3a^2L) \]  \hspace{1cm} (o)

Similarly, for case Figure 10(c), the equations for elastic curve and elastic slope under unit load is the same as Equations (j), (k), and (l) except for the subscripts; that is:

\[ y_{AD} = \frac{bx}{6EI} (x^2 + b^2 - L^2) \]  \hspace{1cm} (p)

\[ \theta_{AD} = \frac{b}{6EI} (3x^2 + b^2 - L^2) \]  \hspace{1cm} (q)
\[ y_{DB} = \frac{bx}{6EI} (x^2 + b - L^2). \]  

(r)

Since the flexibility influence coefficient \( a_{ij} \) is defined as the displacement at \( i \) due to unit load at \( j \), then, for the problem at hand, one obtains from equations (j-r):

\[
\begin{bmatrix}
\frac{bx}{6EI} (x^2 + b^2 - L^2)|_{x=a} & \frac{x}{6EI} (x^2 + 3a^2 - 6aL + 2L^2)|_{x=a} & \frac{bx}{6EI} (x^2 + b^2 - L^2)|_{x=x_C} \\
\frac{bx}{6EI} (3x^2 + b^2 - L^2)|_{x=a} & \frac{1}{6EI} (3x^2 + 3a^2 - 6aL + 2L^2)|_{x=a} & \frac{b}{6EI} (3x^2 + b^2 - L^2)|_{x=x_C} \\
\frac{a(l-x)}{6EI} (x^2 + a^2 - 2Lx)|_{x=x_D} & \frac{1}{6EI} (x^3 - 3Lx^2 + x(2L^2 + 3a^2) - 3a^2L)|_{x=x_D} & \frac{bx}{6EI} (x^2 + b^2 - L^2)|_{x=x_D}
\end{bmatrix}
\]

With \( L = 24 \text{ in.}, x_D = 18 \text{ in.}, x = a = 8 \text{ in.}, b = 16 \text{ in.} \) when evaluating column one and column two of the flexibility matrix, and with \( x_C = 8 \text{ in.}, a = 18 \text{ in.}, b = 6 \text{ in.}, x_D = 18 \text{ in.} \) when evaluating column three of that matrix, then with \( E = 30 \times 10^6 \text{ psi} \) and \( I = \frac{\pi d^4}{32} = \frac{\pi (1.5)^4}{32} \), this (above) matrix is evaluated as:

\[
\begin{bmatrix}
2.543 \times 10^{-6} & 1.59 \times 10^{-7} & 1.773 \times 10^{-6} \\
1.59 \times 10^{-7} & 2.981 \times 10^{-8} & 1.621 \times 10^{-7} \\
1.773 \times 10^{-6} & 1.621 \times 10^{-7} & 1.811 \times 10^{-6}
\end{bmatrix} \text{ ft.}
\]

See Appendix B for details on the above calculations. The stiffness matrix \( K \) is the inverse of the flexibility matrix and is calculated to be:

\[
K = \text{inv}(a) = \begin{bmatrix}
1.24 \times 10^6 & -2.213 \times 10^4 & -1.212 \times 10^6 \\
-2.213 \times 10^4 & 6.536 \times 10^7 & -5.828 \times 10^6 \\
-1.212 \times 10^6 & -5.828 \times 10^6 & 2.261 \times 10^6
\end{bmatrix} \text{ lbf/ft}.
\]

If the radius of gyration for gear C about its transverse axis x and y is assumed to be \( k = 5.657 \text{ in.} \), then the mass matrix is:

\[
M = \begin{bmatrix}
\frac{60}{32.2} & 0 & 0 \\
0 & \frac{60}{32.2} \left( \frac{5.657}{12} \right)^2 & 0 \\
0 & 0 & \frac{45}{32.2}
\end{bmatrix}
\]

Before continuing on with the calculations, it is noteworthy to say a word about choosing this value for the radius of gyration, namely, \( k = 5.657 \text{ in.} \). In this paper, we assumed most of the rotating inertia to be closer to the periphery of the rotating part than to the center of rotation, which is the shaft. In addition, we wanted to show students the significance of the destructive effects of unbalances on the system. For both of these reasons we chose a relatively high value for \( k \). While the chosen value for \( k \) supports our emphasis, certainly one could have assumed different values for \( k \).
With the help of MathCad (See attached MathCad file in Appendix B), the eigenvalues of \((K - M \omega^2)\) are then calculated as:

\[
eigenvals(\text{inv}(M) \, K) = \begin{bmatrix} 1.491 \times 10^5 \\ 1.759 \times 10^6 \\ 1.582 \times 10^8 \end{bmatrix} \text{ 1/s}^2 \quad (s)
\]

The fundamental natural frequency of the shaft gear system is then the square root of the first element of the vector in Equation (s) and is readily found as:

\[
\omega_{n1} = \sqrt{1.491 \times 10^5} = 386.135 \text{ rad/s} \quad (t)
\]

The rotating speed of the shaft is given by the project as \(\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1200}{60} \approx 126 \text{ rad/s}\) which meets the design recommendation of \(\frac{\omega_{n1}}{\omega} = \frac{126}{386} \approx 3\) (see Equation (f1)).

To check the accuracy of the first natural frequency obtained, Rayleigh's method\(^6\) is implemented as:

\[
\omega_{n1} = \sqrt{\frac{g \sum w_i y_i}{\sum w_i y_i^2}} \quad (u)
\]

where \(g\) is the acceleration of the gravity, \(w_i\) is the equivalent weight of the \(i^{th}\) location and \(y_i\) is the deflection at the \(i^{th}\) body on the shaft. For instance:

\[
\sum w_i y_i = w_c g (m_c a_{11} + J_c a_{12} + m_d a_{13}) + J_c g^2 (m_c a_{21} + J_c a_{22} + m_d a_{23}) + w_d g (m_c a_{31} + J_c a_{32} + m_d a_{33})
\]

where \(w_c\) and \(w_d\) are the weight of gear C and D respectively, and the \(a_{ij}\)'s are the elements of the flexibility matrix. The details of Rayleigh's method calculations are implemented in the attached MathCad file. This calculation renders the fundamental frequency of the system via Rayleigh's approach as:

\[
\omega_{n1,\text{Rayleigh}} = 386.4 \text{ rad/s}
\]

This is very close to the value obtained in Equation (t). The Dunkerley's method\(^6\) is further employed to reinforce the values of fundamental frequency obtained in the above. This method states that the first critical speed of the shaft \(\omega_{n1}\) can be approximated as:

\[
\frac{1}{\omega_{n1}^2} \approx \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2}
\]

where \(\frac{1}{\omega_{11}^2} = m_c a_{11}, \frac{1}{\omega_{22}^2} = J_c a_{22}, \frac{1}{\omega_{33}^2} = m_d a_{33}\).

The fundamental natural frequency from this method is always at the lower side and, as expected, the MathCad calculation in Appendix B gives a lower value for first critical speed than what was obtained from the two previous approaches. This value is:
Finally the deflections at gear locations are calculated for both planes XZ and YZ by employing the principle of superposition in the vertical plane to evaluate the deflection for the static part of the load and harmonic load contributions. In the YZ-plane, one only deals with harmonic load contributions as stated in Equations (h) and (i) above. The final deflection of the shaft at the points of interest is then obtained as the vector sum of the deflections in the XZ and YZ-planes.

To find the deflections in the XZ-plane, for example, one starts with:

\[
M \ddot{X} + KX = \begin{bmatrix} 60 \\ 0 \\ 45 \end{bmatrix} + \begin{bmatrix} 197 \\ 342 \\ 311 \end{bmatrix} \sin \omega t.
\]

Then, using principle of superposition for linear systems, we obtain:

\[
X = X_{\text{static}} + X_{\text{dynamic}}
\]

where:

\[
X_{\text{static}} = K^{-1} \begin{bmatrix} 60 \\ 0 \\ 45 \end{bmatrix} \quad \text{and} \quad X_{\text{dynamic}} = [K - M \omega^2]^{-1} \begin{bmatrix} 197 \\ 342 \\ 311 \end{bmatrix} \sin \omega t
\]

, and where \(\omega\) in the previous equation is the speed of the shaft in rad/s; that is, \(\omega = 126\) rad/s.

The MathCad file in Appendix B evaluates the total deflection of the shaft as explained above in both XZ and YZ planes. It then calculates the deflection of the shaft due to the loads \(F_{XZ}\) and \(F_{YZ}\) in planes XZ and YZ, respectively.

\[
F_{XZ} = \begin{bmatrix} 60 \\ 0 \\ 45 \end{bmatrix} + \begin{bmatrix} 197 \\ 0 \\ 262 \end{bmatrix} \sin \omega t \quad (v)
\]

\[
F_{YZ} = \begin{bmatrix} -540 \\ 0 \\ 720 \end{bmatrix} \cos \omega t \quad (w)
\]

Note that, in Equations (v) and (w), the effect of unbalanced dynamic loads due to inclination of the gear C relative to the axis of the shaft and eccentricity of the gear D is missing.

These calculations are carried out in the MathCad file in the Appendix B and are compared and contrasted in the Results’ section below.

Results

As the MathCad file indicates, the total deflection of the shaft, without considering the eccentric load from gear D and rocking moment of gear C, are:
\[ \delta_1 = 0.0158 \text{ in} \quad \theta = 1.2472 \times 10^{-3} \text{rad} \quad \delta_2 = 0.0139 \text{ in}. \]

When the rocking moments and eccentric load in the gears were considered, these values turned out to be:

\[ \delta_{1d} = 0.0177 \text{ in} \quad \theta_d = 1.5482 \times 10^{-3} \text{rad} \quad \delta_{2d} = 0.0164 \text{ in}. \]

As the MathCad file indicates there is significant difference between these two deflections, especially for the slope of the shaft at the location of the gear C. The percentage error for each of these components is found to be roughly 12\%, 24\% and 17.5\%, respectively. This excessive difference could cause problems at the bearings, making them possibly noisy, and will affect the performance of the gears, likely causing backlash and interference in the gears. In general, this indicates the importance of balancing of rotating components of machineries, especially at very high operating speeds since the unbalanced loads are proportional to \( \sim \omega^2 \).

From stress consideration point of view, calculation of the shaft diameter for the case of no eccentric load and no rocking moment effect versus the case when both of these effects are present did not yield much of a difference. As you have seen already, and as it is evident from MathCAD file in the Appendix A, the shaft diameter for the case when the eccentricity effect and rocking moments are present is roughly \( d_{\text{dynamic}} = 1.5 \text{ in} \). The shaft diameter in the absence of these dynamic loads was obtained in a separate run of the MathCad file and was evaluated as \( d = 1.44 \text{ in} \), which provided a percentage difference of 4.3\%. Apparently inclusion of these eccentric and rocking moment effects in the equations are more dramatic in the induced deflections than shaft fatigue life and stresses.

Conclusions

Unbalanced loads in rotating components of machineries have significant effect in the operating condition and life of the equipment. It was seen that even a minute misalignment of the shaft and its gearing components could produce excessive deflection of the shaft, which in turn affects the bearings and meshing of the gears.

It is to be mentioned that, although students were asked to design the bearings for their designed shaft, the aspects of that design were not elaborated upon in this paper. The main thrust of the work presented here was to show the dynamic effects of load in shaft design and compare it with the case where only static loads are considered. Nonetheless, as students started working on the project, topics regarding dynamic load effects on the shaft and bearing designs were elaborated upon and discussed in the classroom. Students still struggled at times while working on this project. The issues ranged from the need to review basic laws of general physics, to vibrations, to linear algebra and systems of linear ODEs, and even to providing assistance with standard formats (ASEE paper procedures format) to cite references on which their work was based to avoid inconsistencies among other things. Overall, it was worthwhile presenting our students to the challenges of such an endeavor as the assignment provided our students with a deep understanding of the intricate process of designing a rotating component in their design-oriented course and with meaningful practices of communications in engineering.
In a broader context, the project also aims to highlight a key purpose of this senior level Machine Design course, which is to point out to students and even to beginner designers some principles and procedures which will prove to be useful in most design cases in such a way as to call their attention to the problems involved and, at the same time, illustrating sound design practice. They can be summarized as follows: 1) Before starting a design project, get thoroughly familiar with its purpose, requirements or limitations; 2) There are three types of vibrational behavior that are encountered with shafts, namely, lateral vibration, shaft whirl, and torsional vibration, all of which are important in terms of computing unbalances due to manufacturing defects. 3) Shaft deflection is often the more demanding constraint in shaft design. 4) Other key factors that influence alignment in shafts, in particular, and in rotating machinery, in general, are the speed of the drive train and the maximum deviation at either flexing point or point of power transmission. 5) While there are no published (universal) standards for alignment tolerances (unlike the case for bearings), it is necessary that dimensional tolerances and deviations of machine parts be kept within a certain accuracy to achieve their correct and reliable functioning.

We believe that it is a challenging task, for students and teachers alike, to insure that students acquire deep understanding of these principles and procedures. It is even a more challenging task, yet equally important, to help students develop a feel for or an intuitive understanding of the relevance of these topics to principal problems in the design and manufacturing of rotating machinery. Towards this end, and in addition to our project, it came to our attention that this could be further accomplished by way of simple demonstrations or through real or virtual case studies involving alignment concepts and other central design issues such as tolerances and deviations in mechanical design of machine parts and machines, in general.

Bibliography

8. www.vibralign.com (Alignment Resource Center)
http://www.Fixturlaser.com
https://www.youtube.com/watch?v=Q6_eLlUC2A
Appendix A
Calculation of the Shaft Diameter

Given: Tensile strength of the shaft: \( S_{ut} := 100 \text{ ksi} \)
Safety factor \( n := 1.8 \)

Alternating, Mean Bending Moment, and Mean Torsion

\[
M_m := 322.5 \text{ in-lbf} \quad T_m := 2160 \text{ in-lbf} \\
M_a := 2865 \text{ in-lbf}
\]

Stress Concentration factors \( K_t \) and \( K_{ts} \) for Bending and Torsion from any Machine Design Text

\[
\text{Doverd} := 1.8; \quad \text{roverd} := 0.05
\]

For Bending \( K_t := 1.8 \)

For Torsion \( K_{ts} := 1.6 \)

Finding Notch Sensitivity Factor
Using Table 6-6 from Machine Design: an Integrated approach by Robert Norton

For Bending use \( S_{ut} = 100 \text{ ksi}; \text{ Table 6-6} \)
\[
r := 0.1 \text{ir} \\
q := \frac{1}{1 + \sqrt{\frac{a_b}{r}}} \\
q = 0.836
\]

For Torsion use \( S_{ut} = 120 \) and table 6-6
\[
a := 0.049^2 \text{ir} \\
q_b := \frac{1}{1 + \sqrt{\frac{a}{r}}} \\
q_b = 0.866
\]

The Fatigue Stress Concentration Factors

\[
K_f := 1 + q \left( K_t - 1 \right) \\
K_{fs} := 1 + q_b \left( K_{ts} - 1 \right)
\]

\[
K_f = 1.72 \\
K_{fs} = 1.52
\]
and Assume the Following Fatigue Stress Concentration Factor

\[ K_{fm} := K_f \quad K_{fsm} := K_{fs} \]

**Unmodified Endurance Limit**

\[ S'_c := 0.5 S_{ut} \quad S'_c = 50 \text{ ksi} \]

Calculate the endurance limit modification factors for a rotating round beam.

**Load**

\[ C_{load} := 1 \quad \text{Combined bending and torsion} \]

**Size**

\[ C_{size}(d) := 0.879 \left( \frac{d}{\text{in}} \right)^{-0.107} \]

**Surface**

\[ A_s := 2.7, \quad b := -0.26; \quad \text{machined} \]

\[ C_{surf} := A_s \left( \frac{S_{ut}}{\text{ksi}} \right)^b \]

\[ C_{surf} = 0.797 \]

**Temperature**

\[ C_{temp} := 1 \]

**Reliability**

\[ C_{reliab} := 1.0 \]

\[ R = 50\% \]

The modified endurance limit.

\[ S_e(d) := C_{load} \cdot C_{size}(d) \cdot C_{surf} \cdot C_{temp} \cdot C_{reliab} \cdot S'_c \]

**solving** for the diameter at the shoulder.

**Guess**

\[ d := 1.00 \text{ in} \]

**Given**

\[ d = \left[ \frac{32n}{\pi} \left( \sqrt{\left( \frac{K_f \cdot M_m}{S_e(d)} \right)^2 + \sqrt{\left( K_{fm} \cdot M_m \right)^2 + \frac{3}{4} \left( K_{fsm} \cdot T_m \right)^2} \right)} S_{ut} \right]^{\frac{1}{3}} \]

\[ d := \text{Find}(d) \quad d = 1.502 \text{ in} \]
Appendix B

Calculation of Flexibility and Stiffness Matrices, Shaft First 3 Critical Speeds and Shaft Deflections

Shaft diameter, Modulus of Elasticity and Geometry of the Shaft from Figure 10(a) in the Main Body of the Paper

\[ b := 16 \text{ in} \quad E := 30 \times 10^6 \text{ psi} \quad d := 1.5 \text{ in} \quad I := \frac{\pi \cdot d^4}{64} \quad L := 24 \text{ in} \]

\[ a := 8 \text{ in} \quad x := a \quad x = 8 \text{ in} \quad I = 0.249 \text{ in}^4 \]

Unit Load \( F \quad F := 1 \text{lbf} \) \quad Unit Moment \( M \quad M := -1 \text{ in \cdot lbf} \)

Calculation of elements of Flexibility Influence Matrix in its First and Second Column due to Unit Load and Moment. Equations (j), (k), (l), (m), (n), (o) in the Body of the Paper

\[ y_{AC} := \frac{F \cdot b \cdot x}{6E \cdot I \cdot L} \left( x^2 + b^2 - L^2 \right) \]

\[ \theta_{AC} := \frac{F \cdot b}{6E \cdot I \cdot L} \left( 3x^2 + b^2 - L^2 \right) \]

\[ y_{AC} = -3.052 \times 10^{-5} \text{ in} \]

\[ \theta_{AC} = -1.908 \times 10^{-6} \]

\[ \delta_{AC} := \frac{M \cdot x}{6E \cdot I \cdot L} \left( x^2 + 3a^2 - 6aL + 2L^2 \right) \]

\[ \delta_{AC} = -1.908 \times 10^{-6} \text{ in} \]

\[ y_{CB} := \frac{F \cdot a \cdot \left( L - x \right)}{6E \cdot I \cdot L} \left( x^2 + a^2 - 2L \cdot x \right) \]

\[ y_{CB} = -2.128 \times 10^{-5} \text{ in} \]

\[ \delta_{CB} := \frac{M}{6E \cdot I \cdot L} \left[ x^3 - 3L \cdot x^2 + x \left( 2L^2 + 3a^2 \right) - 3a^2 \cdot L \right] \]

\[ \delta_{CB} = -1.945 \times 10^{-6} \text{ in} \]

Calculation of elements of Flexibility Influence Matrix in its Third Column. Equations (p), (q), (r) in the Body of the Paper
\[
\begin{align*}
\chi_1 &= 8 \text{ in} \\
\chi_2 &= 6 \text{ in} \\
y_{AD} &= \frac{F \cdot b \cdot x}{6EIL} \left( \frac{x^2 + b^2 - L^2}{x^2 + b^2 - L^2} \right) \\
y_{AD} &= -2.128 \times 10^{-5} \text{ in} \\
\theta_{AD} &= \frac{F \cdot b}{6EIL} \left( 3 \frac{x^2 + b^2 - L^2}{x^2 + b^2 - L^2} \right) \\
\theta_{AD} &= -1.945 \times 10^{-6} \\
\chi_3 &= 18 \text{ in} \\
y_{AD} &= \frac{F \cdot b \cdot x}{6EIL} \left( \frac{x^2 + b^2 - L^2}{x^2 + b^2 - L^2} \right) \\
y_{AD} &= -2.173 \times 10^{-5} \text{ in}
\end{align*}
\]

**Flexibility Influence Matrix in inches**

\[
\begin{bmatrix}
-3.052 \times 10^{-5} & -1.908 \times 10^{-6} & -2.128 \times 10^{-5} \\
-1.908 \times 10^{-6} & -3.577 \times 10^{-7} & -1.945 \times 10^{-6} \\
-2.128 \times 10^{-5} & -1.945 \times 10^{-6} & -2.173 \times 10^{-5}
\end{bmatrix}
\]

**Flexibility Influence Matrix in Feet**

\[
\begin{bmatrix}
3.052 \times 10^{-5} & 1.908 \times 10^{-6} & 2.128 \times 10^{-5} \\
1.908 \times 10^{-6} & 3.577 \times 10^{-7} & 1.945 \times 10^{-6} \\
2.128 \times 10^{-5} & 1.945 \times 10^{-6} & 2.173 \times 10^{-5}
\end{bmatrix}
\]

**Stiffness Matrix in lbf/ft**

\[
\begin{bmatrix}
1.24 \times 10^6 & -2.213 \times 10^4 & -1.212 \times 10^6 \\
-2.213 \times 10^4 & 6.536 \times 10^7 & -5.828 \times 10^6 \\
-1.212 \times 10^6 & -5.828 \times 10^6 & 2.261 \times 10^6
\end{bmatrix}
\]
Mass Matrix

\[ m := \begin{pmatrix} 60 & 0 & 0 \\ 0 & \frac{60 \cdot 5.657^2}{32.2} & 0 \\ 0 & 0 & \frac{45}{32.2} \end{pmatrix} \]

First 3 Natural Frequencies

K \omega^2 := a m^{-1}

\[ D := \text{minv} \cdot K \]

\[ \text{eigenvals}(D) = \begin{pmatrix} 1.491 \times 10^5 \\ 1.759 \times 10^6 \\ 1.582 \times 10^8 \end{pmatrix} \]

First 3 Natural Frequencies

\[ \omega_1 := \sqrt{1.491 \times 10^5} \frac{\text{rad}}{s} \quad \omega_1 = 386.135 \frac{1}{s} \]

\[ \omega_2 := \sqrt{1.759 \times 10^6} \frac{\text{rad}}{s} \quad \omega_2 = 1.326 \times 10^3 \frac{1}{s} \]

\[ \omega_3 := \sqrt{1.582 \times 10^8} \frac{\text{rad}}{s} \quad \omega_3 = 1.258 \times 10^4 \frac{1}{s} \]
Calculation of First critical Speed of the Shaft Using Dunkerley's Method

\[ \omega_{11} := \sqrt{\frac{32.2}{60 \times (2.543 \times 10^{-6})}} \]

\[ \omega_{22} := \sqrt{\frac{32.2}{30.4 \times (2.981 \times 10^{-8})}} \]

\[ \omega_{33} := \sqrt{\frac{32.2}{45 \times (1.811 \times 10^{-6})}} \]

\[ \omega_{2inv} := \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2} \]

\[ \omega_{2inv} = 7.381 \times 10^{-6} \]

\[ \omega_{Dunkerley} := \sqrt{\omega_{2inv}} \]

Calculation of First critical Speed of the Shaft Using Rayleigh's Method

\[ \sigma_1 := 60 \times (2.543 \times 10^{-6}) + \frac{30.4}{9} \times (1.59 \times 10^{-7}) + 45 \times (1.773 \times 10^{-6}) \]

\[ \sigma_2 := 60 \times (1.59 \times 10^{-7}) + \frac{40}{3} \times (2.981 \times 10^{-8}) + 45 \times (1.621 \times 10^{-7}) \]

\[ \sigma_3 := 60 \times (1.773 \times 10^{-6}) + \frac{40}{3} \times (1.621 \times 10^{-7}) + 45 \times (1.811 \times 10^{-6}) \]

\[ \sigma_4 := 60 \sigma_1 + \frac{40}{3} \sigma_2 + 45 \sigma_3 \]

\[ \sigma_5 := 60 \sigma_1^2 + \frac{40}{3} \sigma_2^2 + 45 \sigma_3^2 \]

\[ \omega_{Rayleigh} := \sqrt{\frac{32.2 \sigma_4}{\sigma_5}} \]
Calculation of Deflections

Static Load in the XZ Plane

\[ F_{lx} := \begin{pmatrix} 60 \\ 0 \\ 45 \end{pmatrix} \]

Static Deflections in the XZ Plane

\[ \text{def}1x := \text{am} \cdot F_{lx} \cdot \frac{1}{2} \]
\[ \text{def}1x = \begin{pmatrix} 2.789 \times 10^{-3} \\ 2.02 \times 10^{-4} \\ 2.255 \times 10^{-3} \end{pmatrix} \]

Dynamic Load in the XZ Plane

\[ F_{2x} := \begin{pmatrix} 197 \\ 342 \\ 311 \end{pmatrix} \]

\[ \text{inver} := K - m(40\pi)^2 \]
\[ \text{inver} = \begin{pmatrix} 1.21 \times 10^6 & -2.213 \times 10^4 & -1.212 \times 10^6 \\ -2.213 \times 10^4 & 6.535 \times 10^7 & -5.828 \times 10^6 \\ -1.212 \times 10^6 & -5.828 \times 10^6 & 2.239 \times 10^6 \end{pmatrix} \]

Dynamic Deflections in the XZ Plane

\[ \text{def}2x := \text{inver}^{-1} \cdot F_{2x} \cdot \frac{1}{2} \]
\[ \text{def}2x = \begin{pmatrix} 0.015 \\ 1.219 \times 10^{-3} \\ 0.013 \end{pmatrix} \]

Total Deflection in the XZ Plane for the Worst Scenario

\[ \text{DEF}_{X} := \text{def}1x + \text{def}2x \]
\[ \text{DEF}_{X} = \begin{pmatrix} 0.018 \\ 1.421 \times 10^{-3} \\ 0.015 \end{pmatrix} \]

Dynamic Load in the YZ Plane

\[ F_{2y} := \begin{pmatrix} -540 \\ 342 \\ 769 \end{pmatrix} \]

Dynamic Deflections in the YZ Plane

\[ \text{def}2y := \text{inver}^{-1} \cdot F_{2y} \cdot \frac{1}{2} \]
\[ \text{def}2y = \begin{pmatrix} 8.413 \times 10^{-4} \\ 6.139 \times 10^{-4} \\ 6.176 \times 10^{-3} \end{pmatrix} \]
Total Deflection of the Shaft Including the Unbalanced Loads in Gears C and D

\[
\text{DEFYD} := \text{def2y}
\]

\[
\text{DEFYD} = \begin{pmatrix}
8.413 \times 10^{-4} \\
6.139 \times 10^{-4} \\
6.176 \times 10^{-3}
\end{pmatrix}
\]

Calculation of Total Deflection of the Shaft In the absence of the Unbalanced Loads in Gears C and D

\[
\text{deffd} := \sqrt{\text{DEFX}^2 + \text{DEFYD}^2}
\]

\[
\text{deffd} = \begin{pmatrix}
0.0177 \\
1.5482 \times 10^{-3} \\
0.0164
\end{pmatrix}
\]

Percentage Difference for not including the Unbalanced Load

\[
\text{Differ} := \text{deffd} - \text{deffs}
\]

\[
\text{Differ} = \begin{pmatrix}
1.897 \times 10^{-3} \\
3.009 \times 10^{-4} \\
2.445 \times 10^{-3}
\end{pmatrix}
\]
So the percent error for not including the dynamic loads due to gear unbalances are 12%, 24% and 17.5%. That is a lot!