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Designing Instruction to Promote a Riemann Sum-Based Understanding of the Definite Integral

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Introduction

The definite integral is one of the primary objects of study in first-semester calculus, and it has a wide variety of applications in physics, engineering, statistics, and beyond. Because of its importance, it is vital for STEM students to develop a robust understanding of the definite integral and its uses. Research has shown that students generally have little difficulty computing definite integrals using the Fundamental Theorem of Calculus (FTC).[1,2] However, this ability alone is inadequate for at least two reasons. First, it cannot account for integrals of functions that lack elementary antiderivatives, such as $f(x) = e^{-x^2}$ which is used regularly in statistics. Second, and arguably more importantly, it bears no obvious relation to the structure of the definite integral. The student who can compute a definite integral using the FTC but who does not understand its structure will likely have difficulty interpreting that integral in context; more, the student will likely have difficulty constructing the integral to be evaluated in the first place.

Research has repeatedly shown that student understanding of the definite integral beyond the FTC is dominated by an area-under-the-curve conception.[1,3,4] However, this understanding alone does not imply a comprehension of the underlying summation structure[5], nor is it sufficient for solving many problems involving the definite integral.[4] The area-under-the-curve interpretation is useful for interpreting the definite integral provided it has a quantitative basis. Specifically, the student should see the area as the product, or sum of products, of values of quantities represented on the horizontal and vertical axes, mirroring the multiplicative sum embedded in the Riemann integral.

Research into how students use and interpret Riemann sums is troubling. Often, students make no mention whatsoever of summation in their interpretation of the definite integral.[1,2] Those few students that do often appear hazy in their understanding. Jones[3] found that many students that spoke of a summation in their explanations either did not specify what was being summed, or thought it was the integrand itself that was summed. Wagner[2] found that upper-level physics majors, all of whom were familiar with Riemann sums, nonetheless had issues connecting these sums to definite integrals; they believed them to be different processes to accomplish the same goal, or else that a Riemann sum was *merely* an approximation to a definite integral.

Traditional ways of teaching the definite integral are at least partly to blame. Often Riemann sums are not taught, or are underemphasized.[2,6] The area-under-the-curve conception is typically foregrounded in approaches to the definite integral, though often with little or no motivation. As discussed above, this conception is useful as an interpretation of a definite integral understood quantitatively in a problem's context; it is less useful as a basis for building that quantitative understanding. Tallman et. al., in a study of 254 Calculus 1 final exams from across the United States, found that "items that assess students' quantitative interpretations of components of an integrand" were "notably absent." Further, they stated that

> traditional calculus curricula tend to be uninformed by an awareness of both the cognitive activity entailed in productive conceptions of key mathematical ideas and empirically-grounded learning trajectories by which students might construct them.[6]

Clearly, we must give careful consideration to the way that we teach the definite integral, and pay attention to the development of a Riemann sum conception.

Research in Calculus Education

Fundamental to the development of a Riemann sum-based understanding of the definite integral (and to calculus broadly) is the concept of limit. Oehrtman[7], in a study of 120 university calculus students, identified five "strong metaphors" that arose spontaneously in their discussion of the limit concept. Of these, "approximation metaphors," that is, descriptions of limit as a value that can be approximated with any desired accuracy, occurred most frequently and were the closest in structure to the formal limit definition. Oehrtman argues that such metaphors have a unique potential to lead students to productive limit understandings, and has developed instructional materials based on approximation metaphors.[8]

Sealey[9] developed a framework for the Riemann integral to study how students come to an understanding of the concept. The framework consists of five "layers," based on a decomposition of the integral structure and her observations of students working with accumulation problems in applied settings: *orientation*, wherein students come to terms with the meanings of $f(x)$ and Δx; *product,* the formation of $f(x_i^*)\Delta x$; *summation*; *limit*; and finally, *function*, wherein students come to view the definite integral as a function of its upper limit of integration. In Sealey's study, students had the greatest difficulty during the product layer, in trying to decide *what* should be summed. Thus, their issues were mainly not related to calculus (limits), and were quantitative rather than operational.

Research Question

Jones[3] detailed how even when students perceive that summation is a critical element of the definite integral, they are often unclear as to *what* is being summed. Wagner[2], meanwhile, showed that even students with significant experience working with integrals don't grasp the connection between the integral and Riemann sums. Additionally, both studies showed that most students with only one semester of calculus behind them don't express their understanding of the definite integral in terms of sums.

Sealey[9] divided the Riemann integral concept into five layers and showed that student difficulty with the Riemann integral often arises before the summation layer is even reached. She also showed how difficulties in these layers can be overcome using classroom activities designed with them mind. However, she spends relatively little time on the latter layers of the framework (*limit* and *function*), and she does not address whether students following her approach continue to think of summation with regard to the definite integral. This is a matter of interest since, for both pedagogical and practical reasons, students tend to leave Riemann sums behind once they become acquainted with the Fundamental Theorem of Calculus. Therefore, I pose the following research question:

> Can a Calculus 1 course with a conceptual focus on the Riemann sum definition of the definite integral, following Sealey's integral framework, instill in students a robust and enduring conception of the definite integral based on Riemann sums?

Course Design

The focus of this study is one section of Calculus 1 which I taught at a small, public university in the eastern United States during the fall 2022 semester. This was a typical first-semester calculus course, teaching limits, the derivative, the definite integral, and applications, culminating with the Fundamental Theorem of Calculus. The textbook for this course was *Calculus Volume 1* by Strang and Herman[10]. Homework was assigned through MyOpenMath (https://www.myopenmath.com). Initial enrollment was 32.

I decided to base my presentation of the limit and definite integral concepts on Oehrtman's[8] research on the approximation metaphor and Sealey's[9] Riemann integral framework, respectively. As neither approach was supported by our existing course materials, I wrote class worksheets and interactive Desmos (https://www.desmos.com) graphs to aid in these presentations, and supplemental "paper homework" exercises to develop students' understanding of these concepts. Most of these I created for a six-week Calculus 1 class that I taught during the summer of 2022. I revised and expanded these materials prior to and during the fall 2022 semester.

Course materials for the limit and derivative concepts were strongly influenced by Oehrtman[8] (and some were direct adaptations). In a typical class activity, students would first be tasked with finding upper and lower bounds on the value of a limit (derivative) at $x = a$ by evaluating the expression (difference quotient) at values of x near a. They would then estimate the value of the limit (derivative) by taking the midpoint of these bounds, and determine an error bound for this estimate. Then, they would often be asked how this estimate might be improved. The answer, of course, is by shrinking the interval used to find the upper and lower bounds in the first step. As Oehrtman points out[7], this approach is structurally compatible to the epsilon-delta definition of limit. Thus, it allows students to develop a productive understanding of limit, while being expressed in terms that students are comfortable with and without requiring the rigor of an epsilon-delta proof.

Accompanying many of these activities were online Desmos activities consisting of interactive graphs. Students could manipulate a secant line and see it approach a fixed tangent line or explore the interplay between a range of function inputs and the corresponding set of outputs to strengthen the approximation and bounding conception of limit. In addition, Desmos provided an intuitive environment in which students could learn to perform tedious computations such as computing repeated error bounds.

My approach for the definite integral topic was to foreground the phenomenon of accumulation in various physical settings. I used the same three problems of accumulation as Sealey[9]: displacement from a variable velocity as time increases, work from a variable force as distance increases, and hydrostatic force from a variable pressure as depth increases. Each problem asked students to first approximate the accumulated quantity, which required students to employ a multiplicative relationship (such as velocity times time equals displacement) but also to understand the impact a varying quantity would have on this solution approach. This would lead students to break the interval under consideration into subintervals, use the multiplicative relationship on each subinterval as an approximation, and sum the products to approximate the accumulated quantity over the whole interval. Following this approximation, students would then be asked to find a bound on the error for their approximation and, finally, to find an improved approximation with a smaller error bound. (For further details on these activities, see Engelke and Sealey[11], and Oehrtman[8].)

Implementation

For brevity's sake I will limit this section to the implementation of class activities regarding the definite integral concept. Class activities were designed to be student-led with the instructor serving as a facilitator. However, I would sometimes take the lead on a question based on time constraints and my sense of the class's progress. Students were encouraged to bring their laptops to class for these activities, mainly for computing large Riemann sums.

For the accumulation activities, I divided the class into groups of two or three students apiece and allowed them to begin each problem with no prompting from me. After several minutes, if I noticed a group struggling, I would provide them with only enough help to get them started. For an account of the difficulties students can encounter with these problems and the ways they can overcome them, see Sealey.[9]

Only after completing these three activities did I introduce the concept of "area under the curve", which I consistently referred to it as an "interpretation" of the definite integral. I frequently illustrated the link between the area bounded by a curve and the specific quantity estimated in one of the three previous activities.

Once we had spent several days approximating accumulation and areas, I presented the formal definition of the definite integral. The activities we had done mirrored those on the limit and derivative concepts, particularly (for our discussion) in the way they dealt with error bounds and improvements on those bounds. This was by design, so that students would anticipate taking a limit of the sums they had constructed. Thus, it was no great leap to move from finite Riemann sums to the definite integral.

Discussion

I designed class materials for all major topics (limit, derivative, definite integral) so that learning subsequent topics would be as seamless as possible. Thus the emphasis on bounds and approximations found in limit activities were taken up when we began to discuss the derivative and later when we reached the definite integral; the idea of applying a limit to an object students had constructed (a difference quotient) was mirrored when students later had to do the same with Riemann sums. Indeed, the similar structures and reiterated concepts found throughout these activities made it so that by the time we reached the definite integral students could often anticipate the next step in an activity without being asked.

I considered the approximation metaphor for limit important to a strong understanding of the definite integral. Calculus instructors are often faced with two bad choices: to teach the rigorous definition of limit with its epsilons and deltas, knowing that most of their students will fail to grasp the topic; or to treat limits only informally. In either case, the concept of limit will likely have little productive bearing on how students see the definite integral. To the former, students can't very well apply a concept they don't understand, and to the latter, Oehrtman[7] has shown that popular informal treatments of limit have little impact on how students think about limits. The approximation metaphor, however, is natural to their own way of thinking, meaning it can be grasped by most students *and* impact how they think about limits. Additionally, it is powerful enough that it can be applied throughout the calculus course, imbedding itself in students' minds until they are ready to apply it to the definite integral.

I very deliberately withheld the idea of area under the curve until several days after we began discussing accumulation. Even when I brought it up, I called it an "interpretation" of the definite integral. I did this for two reasons. First, I believe that the area-under-the-curve problem serves as a motivation for the definite integral to no one but devoted mathematicians (despite what most calculus textbook authors seem to think). Second, while it is a useful way of thinking about practical problems (such as the area under a velocity-time curve representing displacement), it is inherently more abstract than any of the phenomena it illustrates. As it is easier and more natural to move from the concrete to the abstract (the specific to the general) rather than the other way around, I chose to focus on three specific applications first, and generalize them later.

I found the use of Desmos invaluable for my activities. One of the reasons instructors might downplay or even omit Riemann sums is the difficulty in computing them, particularly as the number of terms becomes large enough to give any reasonable approximation. Desmos allows such sums to be constructed easily thanks to an intuitive, friendly interface. Having the tools to work with Riemann sums, we were not only able to devote sufficient time to the topic, but students were able to “get their own hands dirty” by constructing sums themselves, thus (in theory) deepening their understanding of the topic.

Planned Research

The question at the heart of this research is whether a calculus course that spends special attention to the development of a Riemann-sum based conception of the definite integral can influence calculus students to understand the definite integral in terms of Riemann sums. To answer this question, I plan to gather data in two stages.

First, I will administer a survey (as in Jones[3]) consisting of four questions relating to students' conceptions of the definite integral. This survey will be given to all students enrolled in Calculus 2 at my institution at the beginning of the spring 2023 semester. I will categorize their expressed understandings according to the coding scheme that Jones used. Then, I will compare the responses from students who took my Calculus 1 course against (a) the results of students at my institution who did not, and (b) the results in Jones' study.

The second stage of investigation will consist of one-on-one interviews. I plan to invite students who participated in the first stage (likely no more than six) to take part in a sequence of two 50-minute interviews. (If possible, I will have an equal number of students who took my Calculus 1 class, and those who did not.) Students in these interviews will be presented with several additional questions designed to probe their conception of the definite integral. As the interviewer, I will then ask them follow-up questions to gain a deeper understanding of their views. Following these interviews (which I plan to video and audio record), I will perform a qualitative analysis of them, as in Wagner[2].

Conclusion

The ways students think about mathematical concepts are often at odds with what we might expect or hope. Students who have completed a calculus course often fail to grasp the role that Riemann sums play in the definite integral. The definite integral is a complex topic with many interrelated layers. By studying the structure of the definite integral and the thinking involved at each layer, we can better understand and address the difficulties students encounter when learning this topic.

In this paper I have discussed the creation and implementation of a series of semester-spanning materials for a Calculus 1 course, based on recent framework for the definite integral related research. I have presented the theoretical basis for this framework, the design of the course, and the classroom implementation of several days' activities and instruction introducing the definite integral concept. I have also outlined a research plan designed to investigate how students who completed this course perceive the definite integral compared to their peers who took another course. I anticipate that this research will, if nothing else, provide insight into ways methodologies such as those I have discussed may better be implemented in the future, and hopefully will add to our growing understanding of how students can develop a fruitful understanding of the definite integral.

Appendix: Selected Class Exercises

1. A gorilla (wearing a parachute) jumped off the top of a building. We were able to record the velocity of the gorilla with respect to time twice each second. The data is shown below. (Here t represents the elapsed time, in seconds, and v represents the gorilla's velocity, in feet per second.) Note that he touched the ground just after 5 seconds.

t	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
v	0.0	5.0	7.0	8.0	11.0	11.5	12.0	13.0	15.5	18.0	19.0

(a) Approximate how far the gorilla fell during each half-second interval and fill in the table below. Here d represents the gorilla's (approximate) distance traveled.

Δt	0.0 - 0.5	0.5 - 1.0	1.0 - 1.5	1.5 - 2.0	2.0 - 2.5
d					

Δt	2.5 - 3.0	3.0 - 3.5	3.5 - 4.0	4.0 - 4.5	4.5 - 5.0
d					

(b) Approximate the total distance the gorilla fell from the time he jumped off the building until the time he landed on the ground.

(c) Is your approximation an overestimate, an underestimate, or is it the exact value? How can you tell? Explain your answer clearly.

(d) Is there a way to bound the error? If so, find a bound and explain your reasoning. If not, explain why it is not possible.

(e) If you were able to find a bound for the error, how small can you make the bound? Explain your reasoning clearly.

Figure 1: The Gorilla Jump problem, taken from Engelke & Sealey[11]. Parts (a) and (b) are designed to encourage students to attend to the multiplicative nature of the integrand. Parts (d) and (e) leverage their understanding of limits in terms of bounded approximations developed earlier in the semester.

1. View the graph of $f(x) = \frac{1}{4}x^2$ in the accompanying Desmos activity. Use both right- and left-endpoint approximations to estimate the area of the region S bounded between the curve and the x-axis over the interval $[2, 6]$.

   (a) Based on these estimates, what do you conjecture the true area to be?

   (b) Devise a procedure for calculating the true area of the region. What difficulties do you foresee in carrying out this procedure?

   (c) Write an expression to represent the exact value of the area of S. It doesn't matter if you can evaluate this expression (though if you think you can, give it a try).

2. Previously we saw how the work done to stretch a spring with stiffness coefficient $k = 2.2$ a distance 5 meters from equilibrium can be approximated by the Riemann sum

$$W \approx \sum_{i=1}^{n} 2.2x_i^* \Delta x.$$

   (a) Use the graph of $F = 2.2x$ in the accompanying Desmos activity to estimate the value of the work done to stretch the spring 5 meters.

   (b) Use the method from the previous example to calculate the *exact* value of the work done.

   (c) Interpret the work done as the area bounded between the graph of $F = 2.2x$ and the x-axis on the interval $[0, 5]$. Use this interpretation to verify that your previous calculation is correct.

   (d) Compare and contrast the quantities found in this question and question 1, and the methods used to calculate them. What can you conclude about the power of the definite integral?

Figure 2: Two exercises used with the lesson on the definite integral. The Desmos activity is available at https://teacher.desmos.com/activitybuilder/custom/62a249440e38f114173d2f4d.

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