Determining Deflections of Hinge-Connected Beams Using Singularity Functions: Right and Wrong Ways

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Abstract

When the method of double integration is used to determine deflections, as well as statically indeterminate reactions at supports, of a beam in Mechanics of Materials, one has the option of using singularity functions to account for all loads on the entire beam in formulating the solution. This option is an effective way and a *right way* to solve the problem if the beam is a single piece of elastic body. However, this option becomes a *wrong way* to do it if one fails to heed the existence of discontinuity in the slope of the beam under loading. Beginners tend to have a misconception that singularity functions are a powerful mathematical tool, which can allow one to blaze the loads on the entire beam without the need to divide it into segments. It is pointed out in this paper that hinge-connected beams are a pitfall for unsuspecting beginners. The paper reviews the sign conventions for beams and definitions of singularity functions, and it includes illustrations of both right and wrong ways in solving a problem involving a hinge-connected beam. It is aimed at contributing to the better teaching and learning of mechanics of materials.

I. Introduction

There are several established methods for determining deflections of beams in mechanics of materials. They include the following:[1-9] *(a)* method of double integration (*with* or *without* the use of singularity functions), *(b)* method of superposition, *(c)* method using moment-area theorems, *(d)* method using Castigliano's theorem, *(e)* conjugate beam method, and *(f)* method using general formulas. Naturally, there are advantages and disadvantages in using any of the above methods. By and large, the method of double integration is the commonly used method in determining slopes and deflections, as well as statically indeterminate reactions at supports, of beams. Without using singularity functions, the method of double integration has a disadvantage, because it requires division of a beam into segments for individual studies, where the division is dictated by the presence of concentrated forces or moments, or by different flexural rigidities in different segments. Readers, who are familiar with mechanics of materials, may *skip* the *refresher* on the rudiments included in the early part of this paper.

■ **Sign Convention.** In the analysis of beams, it is important to adhere to the generally agreed positive and negative signs for loads, shear forces, bending moments, slopes, and deflections of beams. A segment of beam *ab* having a constant flexural rigidity *EI* is shown in Fig. 1. Note that we adopt the positive directions of the shear forces, moments, and distributed loads as indicated.

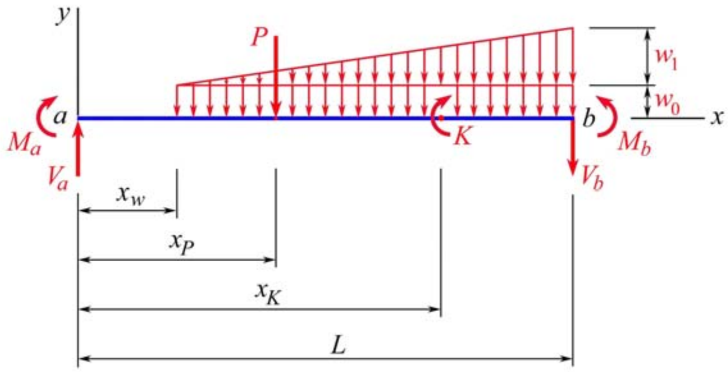

Fig. 1 Positive directions of shear forces, moments, and loads

As in most textbooks for mechanics of materials, notice in Fig. 1 the following conventions:[2-6]

(a) A *positive shear force* is one that tends to rotate the beam segment clockwise (e.g., $\mathbf{V}_a$ at the left end *a*, and $\mathbf{V}_b$ at the right end *b*).

(b) A *positive moment* is one that tends to cause compression in the top fiber of the beam (e.g., $\mathbf{M}_a$ at the left end *a*, $\mathbf{M}_b$ at the right end *b*, and the applied moment **K** tending to cause compression in the top fiber of the beam just to the right of the position where the moment **K** acts).

(c) A *positive concentrated force* applied to the beam is one that is directed downward (e.g., the applied force **P**).

(d) a *positive distributed load* is one that is directed downward (e.g., the uniformly distributed load with intensity w_0, and the linearly varying distributed load with highest intensity w_1).

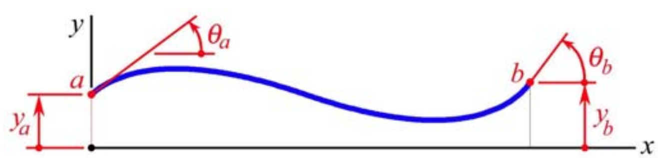

Fig. 2 Positive deflections and positive slopes of beam *ab*

The positive directions of deflections and slopes of the beam are defined as illustrated in Fig. 2. As in most textbooks for mechanics of materials, notice in Fig. 2 the following conventions:[2-6]

(i) A *positive deflection* is an upward displacement (e.g., y_a at position *a*, and y_b at position *b*).

(ii) A *positive slope* is a counterclockwise rotation (e.g., θ_a at position *a*, and θ_b at position *b*).

■ **Singularity functions.** Note that the argument of a singularity function is usually enclosed by angle brackets (i.e., < >), while the argument of a regular function is enclosed by rounded parentheses [i.e., ( )]. The relations between these two functions are defined as follows:[7, 8]

$$< x-a >^n = (x-a)^n \quad \text{if} \quad x-a \ge 0 \text{ and } n \ge 0 \tag{1}$$

$$< x-a >^n = 0 \quad \text{if} \quad x-a < 0 \quad \text{or} \quad n < 0 \tag{2}$$

$$\int_{-\infty}^{x} < x-a >^{n} dx = \frac{1}{n+1} < x-a >^{n+1} \quad \text{if} \quad n \geq 0 \tag{3}$$

$$\int_{-\infty}^{x} < x-a >^{n} dx = < x-a >^{n+1} \quad \text{if} \quad n < 0 \tag{4}$$

Based on the sign conventions and the singularity functions defined above, we may write the loading function *q*, the shear force *V*, and the bending moment *M* for the beam *ab* in Fig. 1 as follows:[1,2]

$$\begin{aligned} q &= V_a < x >^{-1} + M_a < x >^{-2} - P < x - x_P >^{-1} + K < x - x_K >^{-2} \\ &- w_0 < x - x_w >^{0} - \frac{w_1}{L - x_w} < x - x_w >^{1} \end{aligned} \tag{5}$$

$$\begin{aligned} V &= V_a < x >^{0} + M_a < x >^{-1} - P < x - x_P >^{0} + K < x - x_K >^{-1} \\ &- w_0 < x - x_w >^{1} - \frac{w_1}{2(L - x_w)} < x - x_w >^{2} \end{aligned} \tag{6}$$

$$\begin{aligned} M &= V_a < x >^{1} + M_a < x >^{0} - P < x - x_P >^{1} + K < x - x_K >^{0} \\ &- \frac{w_0}{2} < x - x_w >^{2} - \frac{w_1}{6(L - x_w)} < x - x_w >^{3} \end{aligned} \tag{7}$$

II. Analysis of a Hinge-Connected Beam: Right and Wrong Ways

Most textbooks for mechanics of materials or mechanical design do not sufficiently warn their readers that singularity functions can be elegantly used to overcome discontinuities in the various loads acting on the entire beam [such as those shown in Eqs. (5), (6), and (7) for the loads shown in Fig. 1], but they **cannot** blaze the various loads for the **entire** beam when the beam has one or more *discontinuities in its slope* when the loads are applied to act on it. In fact, singularity functions cannot be above the rules of mathematics that require a function to have continuous slopes in a domain if it is to be integrated or differentiated in that domain. Here, the beam is the domain. If a beam is composed of two or more segments that are connected by hinges (e.g., a *Gerber beam*), then the beam has discontinuous slopes at the hinge connections when loads are applied to act on it. In such a case, the deflections and any statically indeterminate reactions must be analyzed by dividing the beam into segments, each of which must have no discontinuity in slope. Otherwise, erroneous results will be reached.

■ **Example 1.** A combined beam (*Gerber beam*) having a constant flexural rigidity *EI* is loaded and supported as shown in Fig. 3. Show a **wrong way** to use singularity functions to attempt a solution for the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at *A*.

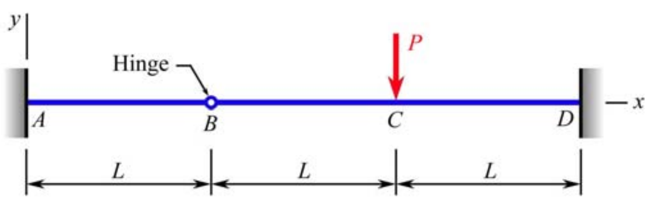

Fig. 3 Fixed-ended beam with a hinge connector

Wrong way. For illustrative purpose, let us first show how a **wrong way** may be used by an unsuspecting person in trying to solve the problem and reaching wrong results as follows:

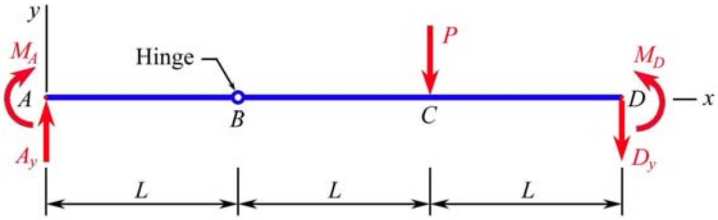

Fig. 4 Free-body diagram with assumption of positive reaction forces and moments

Since this person would use singularity functions to blaze the loading for the entire beam, the loading function q, the shear force V, and the bending moment M for the entire beam would be as follows:

$$q = M_A < x >^{-2} + A_y < x >^{-1} - P < x - 2L >^{-1}$$

$$V = M_A < x >^{-1} + A_y < x >^{0} - P < x - 2L >^{0}$$

$$EI\,y'' = M = M_A < x >^{0} + A_y < x >^{1} - P < x - 2L >^{1}$$

Double integration of the last equation yields

$$EI\,y' = M_A < x >^{1} + \frac{1}{2} A_y < x >^{2} - \frac{1}{2} P < x - 2L >^{2} + C_1$$

$$EI\,y = \frac{1}{2} M_A < x >^{2} + \frac{1}{6} A_y < x >^{3} - \frac{1}{6} P < x - 2L >^{3} + C_1 x + C_2$$

Imposition of boundary conditions yields

$$y'(0) = 0: \qquad 0 = C_1 \qquad (a)$$

$$y(0) = 0: \qquad 0 = C_2 \qquad (b)$$

$$y'(3L) = 0: \qquad 0 = M_A(3L) + \frac{1}{2} A_y(9L^2) - \frac{1}{2} PL^2 + C_1 \qquad (c)$$

$$y(3L) = 0: \qquad 0 = \frac{1}{2} M_A(9L^2) + \frac{1}{6} A_y(27L^3) - \frac{1}{6} PL^3 + C_1(3L) + C_2 \qquad (d)$$

Solution of simultaneous Eqs. *(a)* through *(d)* yields

$$C_1 = 0 \qquad C_2 = 0 \qquad M_A = -\frac{2PL}{9} \qquad A_y = \frac{7P}{27}$$

Consistent with the defined sign conventions, this unsuspecting person would report

$$\mathbf{M}_A = \frac{2PL}{9} \circlearrowleft \qquad \mathbf{A}_y = \frac{7P}{27} \uparrow$$

Note that these two answers are wrong because we can refer to Fig. 4 and show that they do not satisfy the fact that the magnitude of moment $M_B = 0$ at the hinge at B; i.e.,

$$M_B = M_A + A_y L = -\frac{2PL}{9} + \frac{7P}{27}(L) = \frac{PL}{27} \neq 0$$

■ **Example 2.** A combined beam (*Gerber beam*) having a constant flexural rigidity *EI* is loaded and supported as shown in Fig. 3. Show the **right way** to use singularity functions to determine for this beam *(a)* the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at *A*, (*b*) the deflection y_B of the hinge at *B*, *(c)* the slopes θ_{BL} and θ_{BR} just to the left and just to the right of the hinge at *B*, respectively, and *(d)* the slope θ_C and the deflection y_C at *C*.

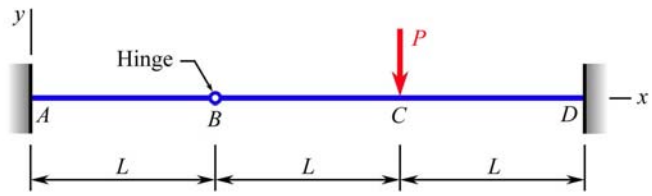

Fig. 3 Fixed-ended beam with a hinge connector (repeated)

Right way. This beam is statically indeterminate to the *first* degree. Nevertheless, because of the discontinuity in slope at the hinge connection *B*, this beam needs to be divided into two segments *AB* and *BD* for analysis in the solution, where no discontinuity in slope exists within each segment.

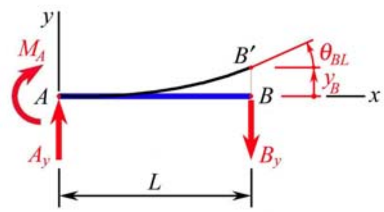

Fig. 5 Free-body diagram for segment *AB* and its deflections

The loading function q_{AB}, the shear force V_{AB}, and the bending moment M_{AB} for the segment *AB*, as shown in Fig. 5, are

$$q_{AB} = M_A < x >^{-2} + A_y < x >^{-1}$$

$$V_{AB} = M_A < x >^{-1} + A_y < x >^{0}$$

$$EI\, y''_{AB} = M_{AB} = M_A < x >^{0} + A_y < x >^{1}$$

Double integration of the last equation yields

$$EI\, y'_{AB} = M_A < x >^{1} + \frac{1}{2} A_y < x >^{2} + C_1$$

$$EI\, y_{AB} = \frac{1}{2} M_A < x >^{2} + \frac{1}{6} A_y < x >^{3} + C_1 x + C_2$$

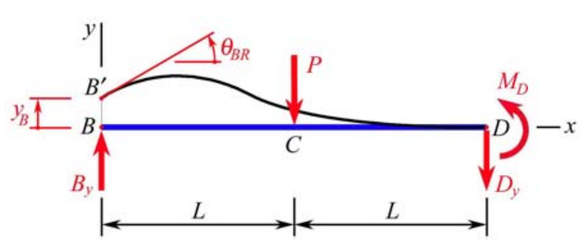

Fig. 6 Free-body diagram for segment *BD* and its deflections

The loading function q_{BD}, the shear force V_{BD}, and the bending moment M_{BD} for the segment *BD*, as shown in Fig. 6, are

$$q_{BD} = B_y < x >^{-1} - P < x - L >^{-1}$$

$$V_{BD} = B_y < x >^{0} - P < x - L >^{0}$$

$$EI\, y''_{BD} = M_{BD} = B_y < x >^{1} - P < x - L >^{1}$$

Double integration of the last equation yields

$$EIy'_{BD} = \frac{1}{2} B_y < x >^2 - \frac{1}{2} P < x - L >^2 + C_3$$

$$EIy_{BD} = \frac{1}{6} B_y < x >^3 - \frac{1}{6} P < x - L >^3 + C_3 x + C_4$$

Imposition of boundary conditions yields

$$y'_{AB}(0) = 0: \qquad 0 = C_1 \qquad (a)$$

$$y_{AB}(0) = 0: \qquad 0 = C_2 \qquad (b)$$

$$y_{AB}(L) = y_{BD}(0): \qquad \frac{1}{2} M_A L^2 + \frac{1}{6} A_y L^3 = C_4 \qquad (c)$$

$$y'_{BD}(2L) = 0: \qquad 0 = \frac{1}{2} B_y (4L^2) - \frac{1}{2} PL^2 + C_3 \qquad (d)$$

$$y_{BD}(2L) = 0: \qquad 0 = \frac{1}{6} B_y (8L^3) - \frac{1}{6} PL^3 + C_3(2L) + C_4 \qquad (e)$$

Imposition of equations of static equilibrium for segment *AB* yields

$$+\circlearrowleft \Sigma M_B = 0: \qquad -M_A - A_y L = 0 \qquad (f)$$

$$+\uparrow \Sigma F_y = 0: \qquad A_y - B_y = 0 \qquad (g)$$

Solution of simultaneous Eqs. *(a)* through *(g)* yields

$$C_1 = 0 \qquad C_2 = 0 \qquad C_3 = -\frac{PL^2}{18} \qquad C_4 = -\frac{5PL^3}{54}$$

$$A_y = \frac{5P}{18} \qquad B_y = \frac{5P}{18} \qquad M_A = -\frac{5PL}{18}$$

Consistent with the defined sign conventions, we report that

$$\mathbf{A}_y = \frac{5P}{18} \uparrow \qquad \mathbf{M}_A = \frac{5PL}{18} \circlearrowleft$$

Substituting the above solutions into foregoing equations for EIy_{BD}, EIy'_{AB}, and EIy'_{BD}, respectively, we write

$$EIy_B = EIy_{BD}(0) = C_4 = -\frac{5PL^3}{54} \qquad y_B = -\frac{5PL^3}{54EI}$$

$$EI\theta_{BL} = EIy'_{AB}(L) = M_A L + \frac{1}{2}A_y L^2 + C_1 = -\frac{5PL^2}{36} \qquad \theta_{BL} = -\frac{5PL^2}{36EI}$$

$$EI\theta_{BR} = EIy'_{BD}(0) = C_3 = -\frac{PL^2}{18} \qquad \theta_{BR} = -\frac{PL^2}{18EI} = -\frac{2PL^2}{36EI}$$

$$EI\theta_C = EIy'_{BD}(L) = \frac{1}{2}B_y L^2 + C_3 = \frac{PL^2}{12} \qquad \theta_C = \frac{PL^2}{12EI}$$

$$EIy_C = EIy_{BD}(L) = \frac{1}{6}B_y L^3 + C_3 L + C_4 = -\frac{11PL^3}{108} \qquad y_C = -\frac{11PL^3}{108EI}$$

Based on the preceding solutions, the deflections of the combined beam *AD* may be illustrated as shown in Fig. 7.

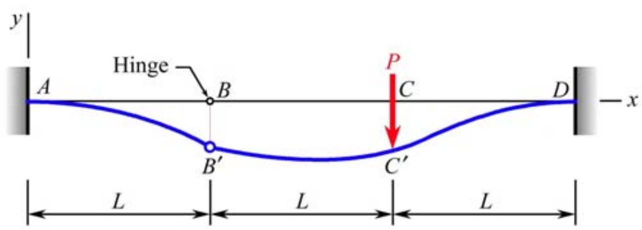

Fig. 7 Deflections of the beam *AD*

Concluding Remarks

This paper provides a refresher on the sign conventions for beams and definitions of singularity functions. Beginners in mechanics of materials are usually not sufficiently warned about the limitations of what singularity functions can do. Students tend to have a misconception that singularity functions are a powerful mathematical tool, which can allow them to blaze the loads on the entire beam without the need to divide it into segments for analysis. It is pointed out in this paper that hinge-connected beams are a pitfall for unsuspecting beginners.

The paper includes two illustrative examples to demonstrate both **wrong** and **right** ways in using singularity functions to solve a problem involving a hinge-connected beam. It is emphasized that singularity functions cannot be above the rules of mathematics that require a function to have continuous slopes in a domain if it is to be integrated or differentiated in that domain. In mechanics of materials, the beam is the domain. If a beam is composed of two or more segments that are connected by hinges (e.g., a *Gerber beam*), then the beam has discontinuous slopes at the hinge connections when loads are applied to act on it. In general, the deflections and any statically indeterminate reactions must be analyzed by dividing the beam into segments, as needed, each of which must have no discontinuity in slope. Otherwise, erroneous results will be reached. This paper is aimed at contributing to the better teaching and learning of mechanics of materials.

References

1. Westergaard, H. M., “Deflections of Beams by the Conjugate Beam Method,” *Journal of the Western Society of Engineers*, Volume XXVI, Number 11, 1921, pp. 369-396.
2. Timoshenko, S., and G. H. MacCullough, *Elements of Strength of Materials*, Third Edition, D. Van Nostrand Company, Inc., New York, NY, 1949, pp.179-181.
3. Singer, F. L., and A. Pytel, *Strength of Materials*, Fourth Edition, Harper & Row, Publishers, Inc., New York, NY, 1987, pp. 228-232.
4. Beer, F. P., E. R. Johnston, Jr., and J. T. DeWolf, *Mechanics of Materials*, Fourth Edition, The McGraw-Hill Companies, Inc., New York, NY, 2006.
5. Pytel, A., and J. Kiusalaas, *Mechanics of Materials*, Brooks/Cole, Pacific Grove, CA, 2003.
6. Gere, J. M., *Mechanics of Materials*, Sixth Edition, Brooks/Cole, Pacific Grove, CA, 2004.
7. Shigley, J. E., *Mechanical Engineering Design*, Fourth Edition, McGraw-Hill Company, New York, NY, 1983, pp. 45-48.
8. Crandall, S. H., C. D. Norman, and T. J. Lardner, *An Introduction to the Mechanics of Solids*, Second Edition, McGraw-Hill Company, New York, NY, 1972, pp. 164-172.
9. Jong, I. C., J. J. Rencis, and H. T. Grandin, “A New approach to Analyzing Reactions and Deflections of Beams: Formulation and Examples,” IMECE2006-13902, *Proceedings of 2006 ASME International Mechanical Engineering Congress and Exposition*, November 5-10, 2006, Chicago, IL, U.S.A.

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