Developing Mathematical Intuition by Building Estimation Skills

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Abstract

Open-ended problems are challenging for many students because they often have little sense of what a “correct” answer would be and struggle with evaluating the quality of an answer derived from a calculator or computer model. It is difficult for them to see patterns or associate one type of problem with another and they have few intuitive skills to use to judge the completeness of their answers. These can be significant obstacles for students who don’t define themselves as mathematicians, but whose careers require what we will call “mathematical intuition” to support the use of technology in solving problems and to anticipate a correct solution.

The goal of this paper is to describe a project-based learning experience that has the potential to help students build their mathematical intuition by requiring them to formally estimate within the solution process. By requiring estimations, the project becomes open-ended; students understand that their answers are not exact, or ‘right’, but are still valid. Framing the project as one that corresponds well to students’ sense of what one ‘does’ in their discipline provides a greater degree of student autonomy in completing the project because they understand what a completed project should look like. Finally, allowing students to work in teams encourages the dialogs that often help establish the ‘reasonableness’ of the results. This project was assigned to two groups of lower division students in media arts and engineering, taught, respectively, by the two authors.

Introduction

As humans we have an innate or intuitive sense of ‘one, two, many, more than or less than’; this is what is referred to as non-symbolic mathematics[1], and is distinct from the knowledge of symbolic mathematics which is acquired through formal teaching. While much of this kind of intuitive analysis is perhaps more accurately studied in neuroscience[2], there is also a very applied perspective that suggests that some of this can be taught. There are parallel studies,[3] also based on mathematical skill, that suggest that intuition can extend to an assessment of the reasonableness of numbers associated with physical entities rather than just a numeric count.

With respect to engineering students, and potentially valid for applied mathematics in general, the engineering education literature clearly points to a lack of math preparedness in students due in part to their heavy reliance on calculators and relative inability to detect 'nonsense' answers[3]. One approach to correcting this lack of intuition lies in developing estimation skills. There are two aspects to estimation; first is developing the particular skills and techniques that are used to make good estimates, and second is convincing students of the validity of making these estimates. The first can be done with order of magnitude estimations based on unit conversions[4,5]. Exercises that build estimation skills have been effectively used to help enhance basic arithmetic skills in the early grades. For example, the question of "how many times will your heart beat in your lifetime", which clearly has no single right answer, can be addressed by taking knowledge of a normal heart rate (or quickly counting your own), and converting from beats per minute to the approximate number of beats in the years of a normal projected lifespan. For more advanced students, Fermi questions[6] such as "How many drops of water are in Lake Erie?" where there could be a right answer but it is impossible to obtain, can be posed to strengthen the use of unit conversion sequences, dimensional analysis and to connect math to the physical world. The common ground in Fermi questions is their goal of combining common sense understanding of physical things with readily available benchmark facts to develop estimates that give a good estimate to the solution.

When these exercises are tied to the physical world (see, for example, the popular book <u>Guesstimation</u>[7], where there are a host of these types of problems with engineering and physics applications), it becomes increasingly apparent to students that the only 'right' answer may be an estimate. So, while on the surface these Fermi-type problems may seem like little more than good party conversation, they actually serve to open the mind to abstract thought, to an awareness of what makes sense and to the validity of estimates as solutions. In particular, learning to estimate can help bridge the gap between "doing math" and "solving problems in the real world," help encourage critical thinking, and help students learn better approaches to solving open-ended problems. By thinking critically about the validity and purpose of particular numbers and applying them outside the bounds of their standard usage, students show mastery of comprehension and move on to the pedagogical realms of application and creation.

Motivation

The formal mathematical theory and estimation approaches have limited success, however, when applied to media arts students, whose goals are technology-based careers in the film, music, and game industries, and often to beginning mechanical engineering students as well, who envision careers in engineering design. Both groups prefer math that is immediately and transparently applicable to their career paths. With powerful and inexpensive calculators, estimation-based math seems like a waste of time, and they trust, often without question[7], the output of user-friendly software. In both communities, then, a significant goal is to develop the habits of good mathematicians, and all good problem solvers, based on the problem solving approach suggested by Polya in How to Solve It[8]. Polya boils problem solving down to four simple steps that provide an algorithm to approaching any type of complex problem. These are: 1) understand the problem; 2) devise a plan; 3) carry out the plan; and 4) look back and evaluate your results and process. The emphasis on evaluating progress against goal is helpful, in particular, for less experienced students when dealing with larger-scale problems. However, students still have issues with evaluating the correctness, or reasonableness of their answers, often because they have not developed the often estimation- based skills necessary to support the development of mathematical intuition, which would guide their judgment. Consequently, we knew we needed to create a project that made sense as a discipline-based endeavor and had a logical basis for requiring estimation.

The resulting project was an open-ended, data collection driven project for each student population that had an immediate 'real world feel'. It was important that the project description immediately supplied the answer to the question of "when will I ever need to do this?" that it encouraged iteration and refinement of solutions and solution techniques, which are critical math skills for design and innovation, and that it helped students make the connection between exercises solved in their math classes and real math done in their major field.

The Projects

The two student groups were students in a foundations course in applied mathematics at the Tribeca Flashpoint Media Arts Academy in Chicago, IL, including students majoring in film production, recording arts, and game design, and sophomore level mechanical engineering students in a numerical methods class in the College of Engineering and Computing at the University of South Carolina in Columbia, SC. Common to both groups, students can either immediately "see" the right answer, or are stopped cold because they are too intimidated by a multi-step problem to even begin a solution process.

Media Arts:

Media arts students were asked to estimate the number of tickets to sell for a secret concert event designed to take place at a nearby city plaza. They were given complete freedom to arrange the layout of the venue and add unique touches; to maintain the surprise nature of the event they were restricted from using any visual measuring devices. They were also asked to provide the cost for a perimeter fence, and the volume and cost of filling the plaza fountain with their beverage of choice. The mathematical foundation of the project is practice with perimeter, area, and volume calculations for composite shapes, unit conversions, creating scale drawings, and writing technical reports. In addition, they have the responsibility for making and justifying all of their decisions.

The project was conducted entirely through estimation. Before collecting site data, classroom discussions were held about order of magnitude approximations, proper units of measure, and the types of tools available for estimation when formal measuring devices are unavailable. These tools included individual devices such as the length of a foot, arm span, height, and reach, and universal devices such as a dollar bill, a piece of copy paper, and floor and ceiling tiles. Deliverables for the project included a scale drawing, a spreadsheet with calculations, and a technical report describing the data collection process and defending the results. This compilation of process and results bridges the expanse from calculation to creation, which Bloom's Taxonomy[9] defines as the final step in full comprehension of a topic. The students worked together in class to develop an outline for the report and to create spreadsheets that linked calculations. The self-developed spreadsheet technology made it so that a change in the block size had only a moderate effect on the work needed to adjust the crowd estimate.

Engineering:

Sophomore engineering students were asked to provide estimates for the height, volume and floor area for an open roofed, three story, semi-circular courtyard in the main engineering building. They were told that the Associate Dean wanted to get a feel for the cost of adding heating and cooling to the space, but didn't want to advertise that he was thinking in this direction until he had a sense of the cost. As a result students were not allowed to use conventional measuring tools or to seek out engineering drawings. They were told that any estimation technique could be used as long as they described the method and discussed the assumptions they made when using the method. They were also asked to estimate the number of people who could be seated on the sunken portion of the courtyard floor.

A preliminary exercise in estimation was conducted before starting the project. Students from each of three sections were asked to measure the length of a hallway using their feet, their arm-span, and an average stride. Statistical tools were used to estimate averages and standard deviations from the collected data. These were compared to 'exact' measures of the length of the hallway. As with the media arts students, deliverables for the project included a technical report describing the project and goals, the data collection process, underlying assumptions, and the results.

Observations: Expected and Unexpected Outcomes

Media Arts:

The city plaza is paved with 5 foot square granite tiles, making the data collection process fairly straightforward. It was encouraging to see, and later read, the number of students who used their shoes or pieces of paper to measure the tiles, or who simply laid down on the ground! It had been surprising during the preliminary exercises described above, how little awareness students have of their own scale relative to the environment. Similarly, many students were able to physically create a crowd density estimate by standing several students on a single tile to determine how much personal space they required. Others preferred to proceed step by step through the more precise, though more laborious, method of creating an area measure for a single person and extrapolating to the square footage they calculated for the plaza. Some even created high and low density areas as would be consistent with a concert experience.

One clear struggle was in the lack of connection between memorized formulas and their application. This may be evidence of a certain kind of passivity in approaching mathematics. Students have practice in mimicry – learning certain algebra techniques and repeating them to solve the problem type of the day – but they have not made the connection between this algebraic skill and an actual solving technique. Consequently, knowing which skills to choose and when to apply them, e.g., calculating perimeter, area, converting volume from cubic feet to gallons, when used out of the context of a worksheet or book chapter, was intimidating. When the calculations were combined into a project spreadsheet, the connections became more obvious.

The most challenging aspect for the media arts students was the technical report. Even though the paper had been outlined in class, the submitted technical writing was, in general, too brief and lacking in description. They didn't seem quite convinced that the interesting part of the project was the building of the assumptions, rather than the actual number of tickets sold. Writing about what is a creative process, but from a technical view, was often being done for the first time, and led to some intellectual struggle. They clearly had some difficulty embedding a technical process into a narrative. However, although the written narratives lacked creative flow, the student specific content in the reports was very creative. Students were careful to select appropriate bands for the venue; to lay out the concert area to allow for a mosh pit, regular and VIP viewing, and vendors; debated armed versus unarmed security; and even changed their beverage choice when the cost became extreme. The scale drawings were often well done using software from their discipline classes. One student even made three-dimensional scale model, which included depictions of the plaza statue and concert goers.

On another very positive side, in preparation for the project we had administered a pre-activity survey to see if students had any sense of translation of common measures to strange objects, e.g., gallons to the size of a backpack, or could choose a proper estimate based on the order of magnitude of a provided answer, e.g., how many heart beats in a lifetime. Results from these surveys of the media arts students clearly demonstrated that the students do not have much practice in dealing with units of measure outside their standard context.

The post-activity surveys of the media arts students, however, showed evidence of greater ability in using unit conversion chains to create estimates. This is promising evidence that the class discussion and modeling of a large-scale problem helped make this estimation-based approach seem more tenable to the media arts students.

Student reactions to the project predictably runs the full breadth of "this is too easy/too hard/too abstract" to "why can't we just ask the fire marshal". There was a wide range of quality in the projects, and varied level of detail in both the writing and the calculations. Most importantly, though, the project provided a mechanism that transformed them from passive to active learning and gave them a glimpse of problem solving completely removed from the confines of a textbook.

Engineering Students:

On the day of the project, teams of 3-4 students headed out to the courtyard. There was significant wandering around and looking up at the beginning. Almost all of the groups started out trying to pace out, heel to toe, the outer dimensions of the space. On consideration, it occurred to students that the area was a half-circle and they changed to measuring a radius. Because we had emphasized the ensemble statistical validity of the previous hallway estimates, there was a certain amount of repetition of measurement, with different team members contributing to create averages. On subsequent class days, measurements were repeated or modified.

The main challenge was measuring the three-story height. Several innovative approaches were adopted; in several cases a student, whose height was known, was asked to stand against a wall, and the height of a single story was approximated in terms of the student's height; in another, students went up to the third floor and dropped objects off of the balcony, video cameras in phones filmed the falling object and they estimated distance based on the falling time; only a few groups cited an approximate height for a commercial building story; some saw what other groups were doing and copied their approaches.

For the engineering students, a description of the writing part of the project was included in the project description. The requested format was outlined as a series of paragraphs, with specific information to be included in each paragraph. In previous semesters, for other computational projects this approach has rarely resulted in an organized written document. What was distinct for this project was that they were asked to submit a technical memo or report from an engineering firm, of their creation, to the Associate Dean of Engineering. This apparently made sense to them: the reports were well organized, relatively clear and showed significant imagination in the presentation, i.e. company names and logos. Although better organized than usual, and with a clear narrative, the innate quality of the written work was still relatively rough, limited variety in vocabulary, and provided little evidence of editing to create a second draft.

For the engineering students, area and volume calculations were not major challenges, but they did react strongly, and with confusion, to giving up the use of what they considered more exact measurement tools. Throughout this numerical methods class they are asked to visualize the math or the method. In particular, they are asked for upper and lower bounds on integrals, to compare to approximations made using the Trapezoid Rule or Simpson's Rule. For this class bounds on integrals are required to be a single geometric shape that either encloses or lies within the area under the curve. This is to emphasize a method to establish a reasonable answer as well as to stress the 'area' concept for interpreting the output from integration. Usually the response to requests for bounds is along the lines of "is this all you want?" and "it can't be that simple!" After this project, students were much more comfortable with the concepts of estimation and bounds and less impatient with the inexactness.

Conclusions

The similarity of reaction and response of the two student groups to this project was surprising; their individual responses were gratifying and encourage the continued use of this type of project. Three significant conclusions are apparent.

First, while estimation techniques are an important part of an applied mathematics curriculum, it is also important for students to become aware of and have a sense of the magnitude of their surroundings. It is equally important to teach students the validity and applicability of creating and using estimates. It is essential to continue the discussion until students realize that there are assumptions that entail estimation in even the most exact formulas that they are taught.

Second, in many cases the most interesting problems presented in applied mathematics textbooks concern state of the art technology or complex systems. Using new technology is often as challenging to the students as the problem solving techniques they are being taught; as a result their approaches to the solution process are correspondingly tentative. This project leveraged topics that students "know" and tapped into their perceptions of their own disciplines, i.e. what engineers or media artists 'do' to produce realistic calculations and deliverables. The result was much more independence, autonomy and creativity in their work, even though there were new tools, skills or knowledge being applied.

Finally, one of the goals of education is to get students involved in learning, and to have them become active participants in the creation of knowledge. While all word problems have the intention of simulating a real life experience, they sometimes fall short of capturing student interest. This project appears to be a good contribution to a portfolio of real world problems; it is applicable to a variety of mathematical levels, and in concept it is extremely accessible to the students.

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