Digital Filter Frequency Response and Eigenfunctions: An Opportunity to Reinforce Linear System Concepts

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Abstract

Undergraduate EE and EET students often master problem-solving techniques at the expense of the understanding of fundamental principles. Furthermore, they often see their education as the study of a set of unrelated topics rather than as the mastery of a single discipline which encompasses several related areas. An eigenfunction-based introduction to digital filter frequency response can help to ameliorate these undesired side effects of undergraduate EE/EET curricula.

Introduction

A critical examination of undergraduate Electrical Engineering and Electrical Engineering Technology programs exposes the following:

- The discipline is artificially partitioned into topics (e.g., Circuit Analysis, Power and Machinery, Computer Architecture, Digital Signal Processing, etc.) which appear to the student to be somewhat unrelated. This is a result of the traditional packaging of study into a set of courses. The effect is that the students often do not make connections between courses and therefore fail to see a single cohesive discipline.
- Students often master and retain problem-solving procedures at the expense of their understanding of the underlying principles. This occurs for several reasons. It is often easier to master the skills necessary to solve a particular class of problem than it is to fully grasp the physical and mathematical principles on which the procedure is based. Engineering/Engineering Technology exams tend to emphasize problem solving; students understand this and react appropriately.

Neither of these situations are fatal flaws in the educational process. They are, however, less than ideal outcomes. Educators should seek techniques that ameliorate these outcomes.

An approach to mitigate these outcomes is to revisit the fundamentals, which were introduced early in the curriculum, in upper level courses. A result is an improved understanding of the principles (students comment, "it made a lot more sense this time"). If the second discussion of the topic occurs in a course that *appears* to be unrelated to the course in which the topic was introduced, then the artificial decomposition of the discipline is also reduced. Crowded undergraduate curricula do not often afford the luxury of discussing, for a second time, topics covered in earlier courses. Therefore, faculty must seek opportunities which both allow a topic to be revisited and require only a minimum of extra time to do so. Such an opportunity occurs when introducing the frequency response of digital filters in a Digital Signal Processing course.

Discussion

Steady State AC Analysis - Sophomores

Charles Steinmetz, in the late nineteenth century, developed a method for the steady state analysis of electric circuits that are excited by sinusoidal signals[i]. Sophomore EE/EET students recognize this as the complex impedance approach to AC circuit analysis. Presentation of this technique frequently begins with the assertion that the forced response of a linear time invariant (LTI) system will often be of the same form as the forcing function (e.g., a sinusoidal input yields a sinusoidal output of the same frequency, with, perhaps, a phase shift). A more detailed examination exposes that the mathematical origin of this assertion is the fact that: (1) the response of a LTI system to the complex exponential, $e^{j\omega t}$, is $H(j\omega)e^{j\omega t}$ (i.e., $e^{j\omega t}$ is an eigenfunction and the corresponding complex valued eigenvalue is $H(j\omega)$), and (2) the Euler identity $\cos(wt) = ½(e^{j\omega t} + e^{-j\omega t})$ relates sinusoids and the complex exponential. The concept of an eigenfunction grows out of the solution of linear differential equations with exponential forcing functions. The Euler identity frequently appears in the form $A_m\cos(\omega t + \phi) = Re(A_m e^{j\omega t+\phi})$ and is directly related to the concept of a phasor, a complex number that represents the magnitude and phase of a sinusoid.

The study of Steinmetz’s method requires that the student deal with several mathematical topics including:

- complex algebra
- Euler's identity
- the application of differential equations (a mathematical area which is still rather new to them) to the solution of circuit problems
- phasors – yet another mathematical notation to master.

In addition, during this first course in circuit analysis, the student is wrestling with several new physical phenomena including:

- electric current, voltage, and power
- resistance, inductance, and capacitance
- Kirchhoff's circuit laws and a variety of circuit theorems.

It is not surprising, in light of the plethora of topics addressed in an introductory circuits course, that most students do not retain an in-depth understanding of all these concepts. Near the end of the sophomore year, when students are asked to summarize this technique, a typical response is, "That's the method that allow us to analyze AC 'just like' DC and avoid (complicated and confusing) differential equations." They are fluent in using phasor notation, the impedance notation, and Kirchhoff's laws to analyze complex networks. Concepts that are less well retained include a comfortable understanding of Euler's identity and how it can be applied to the solution of differential circuit equations with sinusoidal excitation. The connection between impedance-based analysis and the underlying principles (e.g., differential circuit element definitions) is weak.

Steady State AC Analysis - Seniors

The approach promoted by this paper suggests that topics such as Steinmetz's approach to AC analysis be revisited in later terms. The first step is to query the students, now in their senior year, about AC analysis. They are reminded that:

- passive circuit elements, resistance, inductance, and capacitance, are defined by differential relations (e.g., $v_L(t) = L(di/dt)$)
- the common approach to AC analysis represents these elements with impedances (e.g., $Z_L(j\omega) = j\omega L$).

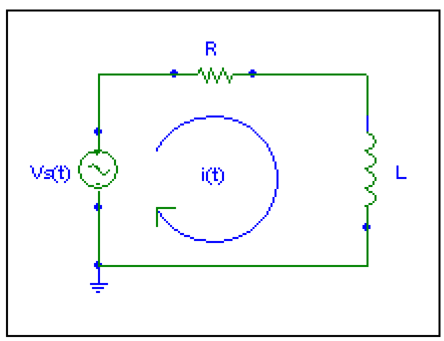
When asked to relate these two models of the circuit elements, the replies include searching looks, confused and groping responses, and expressions of a genuine desire to understand this "lost connection." The seniors are attentive, and ready to revisit this topic.

The main lesson is prefaced with a homework problem. The students are asked to determine the current flowing through a sinusoidally excited series RL circuit at several different frequencies. The problem helps the students to review their circuit analysis skills; they return to class with impedance-based solutions in the forefront of their minds. The lesson continues with an alternate solution of the problem that is based on:

- the circuit differential equation
- Euler's identity
- $e^{j\omega t}$ is an eigenfunction for LTI systems
- the superposition principle.

Given: Series RL circuit shown
Required: i(t)
Solution:

$$\begin{aligned} v_s(t) &= V_m \cos(\omega t) \\ &= \frac{V_m}{2} e^{j\omega t} + \frac{V_m}{2} e^{-j\omega t} \\ &= V e^{j\omega t} + V e^{-j\omega t} \qquad \text{where } V = \frac{V_m}{2} \\ &= v_1(t) + v_2(t) \end{aligned}$$



Recognize each complex exponential voltage source as an eigenfunction. Thus, assume current responses of the form:

$$i_1(t) = I_1 e^{j\omega t}$$
$$i_2(t) = I_2 e^{-j\omega t}$$

Using superposition, decompose the solution into two components one each for:

- the source $v_1(t) = Ve^{j\omega t}$, represented by a counterclockwise (CCW) phasor
- the source $v_2(t) = Ve^{-j\omega t}$ represented by a clockwise (CW) phasor.

CCW Solution	CW Solution	Comments
$v_1(t) = Ri_1(t) + L\dfrac{di_1}{dt}$	$v_2(t) = Ri_2(t) + L\dfrac{di_2}{dt}$	Sbst exp'l form
$Ve^{j\omega t} = RI_1 e^{j\omega t} + L\dfrac{d(I_1 e^{j\omega t})}{dt}$	$Ve^{-j\omega t} = RI_2 e^{-j\omega t} + L\dfrac{d(I_2 e^{-j\omega t})}{dt}$	Differentiate
$= RI_1 e^{j\omega t} + (j\omega)LI_1 e^{j\omega t}$	$= RI_2 e^{-j\omega t} + (-j\omega)LI_2 e^{-j\omega t}$	Divide $e^{+/-j\omega t}$
$V = RI_1 + j\omega LI_1$ $= (R + j\omega L)I_1$	$V = RI_2 - j\omega LI_2$ $= (R - j\omega L)I_2$	Regroup algebraically
$I_1 = \dfrac{V}{R + j\omega L}$	$I_2 = \dfrac{V}{R - j\omega L}$	Sbst I_n in exp'l form:
$i_1(t) = \dfrac{V}{R + j\omega L} e^{j\omega t}$	$i_2(t) = \dfrac{V}{R - j\omega L} e^{-j\omega t}$	$i_n(t) = I_n e^{+/-j\omega t}$

Combining the two solution components yields,

$$
\begin{aligned}
i(t) &= i_1(t) + i_2(t) \\
&= \frac{V}{R + j\omega L} e^{j\omega t} + \frac{V}{R - j\omega L} e^{-j\omega t}
\end{aligned}
$$

Let $Z(j\omega) = R + j\omega L$ the impedance, then

$Z^*(j\omega) = R - j\omega L$ its complex conjugate

or, in polar form

$$Z(j\omega) = |Z(j\omega)| e^{j\theta}$$

$$Z^*(j\omega) = |Z(j\omega)| e^{-j\theta} \quad \text{where}$$

$$|Z(j\omega)| = \sqrt{R^2 + (\omega L)^2} \quad \text{and}$$

$$\theta = \tan^{-1}\left(\omega L / R\right)$$

substituting the polar form in *i(t)* above yields

$$
\begin{aligned}
i(t) &= \frac{V}{|Z(j\omega)| e^{j\theta}} e^{j\omega t} + \frac{V}{|Z(j\omega)| e^{-j\theta}} e^{-j\omega t} \\
&= \frac{V}{|Z(j\omega)|} e^{j(\omega t - \theta)} + \frac{V}{|Z(j\omega)|} e^{-j(\omega t - \theta)} \\
&= \frac{V}{|Z(j\omega)|} \left(e^{j(\omega t - \theta)} + e^{-j(\omega t - \theta)}\right) \\
&= \frac{2V}{|Z(j\omega)|} \frac{1}{2} \left(e^{j(\omega t - \theta)} + e^{-j(\omega t - \theta)}\right) \quad \text{substitute } V = V_m / 2 \text{ and apply Euler's Identity} \\
&= \frac{V_m}{|Z(j\omega)|} \cos(\omega t - \theta) \quad \text{Precisely the same result obtained when solving the}
\end{aligned}
$$

homework problem using impedance based techniques!

The students are now much more comfortable with this approach than they were as sophomores. This is not surprising; they have acquired several years of academic maturity including solutions of differential equations in both the time domain and the frequency (Fourier and Laplace) domain. This particular problem is familiar and elementary; the answer is known almost by inspection. This level of familiarity allows the students to concentrate on the fundamental principles underlying the solution rather than the (now well-understood) procedure and its resulting answer.

Students see that the answer, reassuringly, is identical to the one obtained by their impedance based solution. They quickly realize that the equation $V = (R + j\omega L)I_1$, in the CCW solution above, is *identical in form* to the impedance based loop equation in their homework problem solution. Discussion of the solution leads to the conclusion that Steinmetz's notation can be view as *shorthand* for the above solution. That is:

- Given: Sinusoidally excited RL circuit
- Required: i(t)
- Solution:
  1. Write the differential circuit equation.

2. Decompose sinusoidal source into conjugate complex exponentials. Each component is represented by a phasor – one clockwise (CW) the other counterclockwise (CCW). (Euler identity)
3. Using the superposition principle:
   - Solve CCW problem using the eigenfunction $e^{j\omega t}$.
   - Solve CW problem – solution is *always* the complex conjugate of the CCW problem, therefore the solution is arrived at by inspection.
   - Combine the solution components to yield a sinusoidal result. (Euler identity)

A notable difference in notation is the fact that:

- $V = V_m/2$ (above) is a result of the Euler identity
- $V = V_m /\sqrt{2}$ (impedance solution) is the result of an arbitrary choice. The magnitude of the phasor is chosen to be the root mean square (rms) value of the corresponding sinusoid so that AC and DC power equations are identical.

At the conclusion of this exercise, the following has been established:

- Euler's identity is now a much more concrete principle.
- The concept of an eigenfunction and its associated eigenvalue has been reviewed and strengthened.
- The notion that a sinusoid can be decomposed into two eigenvalues is seen as central to the impedance based approach to AC circuit analysis.
- The frequency dependence of the solution has been noted and quantified. For this particular problem, the magnitude of the response varies inversely with the magnitude of the total impedance ($R + j\omega L$) which is clearly frequency dependent.
- The secret of Steinmetz's analysis method has been revisited, revealed and *understood!*

Armed with this reinvigorated understanding of eigenfunctions and Euler's identity, the lesson continues to address the frequency response of a digital filter.

Digital Filter Frequency Response

The above discussions of steady state AC analysis suggest that the existence of an eigenfunction for digital filters might lead to productive analysis techniques. It is easy to show that the discrete-time complex exponential $e^{j\omega n}$ is such an eigenfunction.[ii,iii,iv] The associated eigenvalue is $H(e^{j\omega n}) = \Sigma h_i e^{j\omega i}$ where h_i is the filter's unit pulse response (for a Finite Impulse Response (FIR) filter $h_i = b_i$ The filter coefficients). This conclusion leads to the following technique to determine the response of a LTI digital filter to a discrete-time sinusoid:

- Given: Digital Filter - h_i
  Input $x(n) = X_m\cos(\omega n)$
- Required: Filter response $y(n)$
- Solution:
  1. There is no need to write the "circuit" equation for any particular filter. At this introductory level, only a single topology of digital filter is under study; for complex exponential excitation the only input-output relation needed is $y(n) =$

$H(e^{j\omega n})\,x(n)$. The only variance between filters is the number and value of the coefficients.

2. Decompose the sinusoidal input into complex conjugate exponentials. Again, each component can be represented by a phasor - one clockwise (CW) the other counterclockwise (CCW). (Euler identity)
   $x(n) = X_m\cos(\omega n) = (X_m/2)e^{j\omega n} + (X_m/2)e^{-j\omega n}$
   $= x_1(n) + x_2(n)$
3. Using superposition:
   - $x(n) = x_1(n) + x_2(n)$ filter excitation, both components are eigenfunctions
   - Solve CCW problem using eigenfunction $e^{j\omega n}$ yielding:
     $y_1(n) = H(e^{j\omega n})\,x_1(n)$
   - Solve CW problem – again, the solution is the complex conjugate of the CCW solution:
     $y_2(n) = H(e^{-j\omega n})\,x_2(n)$
   - Combine the solution components to yield a sinusoidal result. (Euler identity)

   In this case, the solution is *directly* based on the fact that $e^{j\omega n}$ is an eigenfunction and the corresponding eigenvalue is $H(e^{j\omega n}) = \Sigma h_i e^{j\omega I}$. Thus, the CCW solution is:

   $y_1(n) = H(e^{j\omega n})(X_m/2)e^{j\omega n}$

   and the complete solution is:

   $y(n) = |\,H(e^{j\omega n})|\,X_m\cos(\omega n + \Phi)$ where $\Phi = \text{angle}(H(e^{j\omega n}))$

The parallels between this solution and the previous discussion concerning AC analysis are obvious. The differences are primarily in the details.

Conclusions

This approach to discussing the frequency response of digital filters has the following results:

- Student <u>understanding</u> of Steinmetz's impedance based circuit analysis method is greatly improved. Their grasp of the underlying *principles* is now on par with their mastery of the analysis *procedure*.
- Students learn the fundamental of digital filter frequency response from an eigenfunction based analysis approach.
- Students see *AC circuit analysis* and *digital filter frequency response* as two applications of the <u>same underlying analysis techniques</u>. I.e., the use of eigenfunctions and superposition to analyze linear time-invariant systems.
- Minimal additional time has been devoted to this endeavor. The eigenfunction-oriented discussion of AC analysis takes about the same amount of time as would be devoted to an eigenfunction based discussion of digital filter frequency response. Once the AC analysis review is complete, the coverage of digital filter response goes very quickly because the two mathematical presentations are very similar.

It has been the author's experience that the two goals of this approach, (1) improved understanding of fundamental linear analysis techniques and (2) a more unified view of the discipline, have been accomplished.

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[i] Dorf, Richard C. *Introduction to Electric Circuits*, John Wiley and Sons, New York, 1993.
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[iii] Oppenheim, Alan S. and Schafer, Ronald W. *Discrete-Time Signal Processing*, Prentice Hall, Englewood Cliffs, NJ, 1989.
[iv] Proakis, John G. and Manolakis, Dimitris G. *Digital Signal Processing: Principles, Algorithms, and Applications,* Prentice Hall, Englewood Cliffs, NJ, 1996.

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