Paper ID #6297

# DIMENSIONAL ANALYSIS IN MATHEMATICS FOR ENGINEERS

**Dr. Zohra Manseur, Oswego State University College**

Zohra Manseur holds master and doctorate degrees in mathematics from the University of Florida. Her university studies started with a licence-es-science in Mathematics education that led to a Bachelor of Science in Mathematics at the University of Algiers. Her professional activities include over twenty five years as a faculty member at several universities across three countries, three languages, in two continents. She is currently a member of the mathematics faculty at the State University of New York at Oswego. Her research interests are in applied mathematics, image processing with image algebra as well as mathematics education in general and mathematics for engineering in particular. Zohra Manseur is an active member of the scientific society Sigma Xi.

# DIMENSIONAL ANALYSIS IN MATHEMATICS FOR ENGINEERS

**Zohra Manseur**
**Mathematics Department, State University of New York at Oswego**

**Abstract:**

Engineering professionals apply mathematics in the analysis, design, and development of various systems and processes. Common current mathematics preparation for engineering studies consists of several courses that concentrate on a number of important skills mostly related to computational methods and treatment of numerical values. Yet, most of the quantities modeled as variables in mathematical equations represent quantities that attach a physical unit to a scalar. While numerical techniques are important to the computational process, adequate treatment of units within those computations is equally important. However, very little time is spent considering the effect of various commonly used mathematical processes on the physical units associated with variables involved in those computations. This work discusses the integration of physical-units-treatment, dimensional analysis, in engineering preparation courses as part of a newly developed mathematics for engineers course to serve students enrolled in a newly developed electrical and computer engineering degree program

## Introduction

In 1999, NASA launched the Mars Climate Orbiter to study the atmospheric conditions of the red planet. As usual, the design and development work was performed by cooperating research and development teams at the Jet Propulsion laboratory and Lockheed Martin. The investigation following the crash of the orbiter on Mars revealed that two different systems of units were used by the research teams and system integration did not take that fact into consideration[1] resulting in the loss of a multi-million dollars project.

In engineering practice, computations are nearly exclusively performed by computer using a variety of software tools such as Excel, Matlab[2], MathCad[3], Maple[4], Labview[5] or equivalent packages currently available. These computation software packages are actively used in engineering education in large part to train future engineering in their practice. For more advanced computations, dedicated programs are written in one of numerous high-level languages such as Java, VB.Net, C++, Fortran, or Python[6] to name a few. However, none of these tools do incorporate yet physical unit treatment to a sufficient degree although there are some attempts to remedy this situation[7]. An algebra for numbers with physical units, called metrons, is discussed in [8] and an extensive textbook treatment of units in mathematics is given in [9].

As an example, consider a type of vector commonly used to describe the position and orientation of an object in 3-dimensional space. The vector consists of 3 coordinates x, y, and z with units of distance, and 3 orientation angles $\alpha$, $\beta$, and $\gamma$ which are unitless values typically expressed in radians or degrees. The derivative of such a vector is itself non-uniform with units of distance/time combined with units of 1/time. The Euclidian norm of these vectors does not exist. Consequently any process that tends to minimize or maximize these vectors does not make physical sense. Mathematical entities that combine units of length and angles are used throughout the modeling and analysis processes used to study mechanical motions in 3D space. Motion optimization techniques developed to reduce the mean square error between a desired and actual output vector, when an exact solution cannot be determined, are routinely proposed in the literature. In so doing, the Euclidian norm of a vector with different physical elements is sometime used inadvertently violating the basic rule of adding only similar physical quantities. While the complex mathematical processes that lead to these deviations are common to graduate level courses or post-graduate research, this work advocates the usefulness and advantage of bringing these issues to higher levels of attention and awareness in undergraduate mathematics preparation and engineering courses.

Increased consideration of physical units and dimensional interactions in commonly used mathematical processes helps provide better understanding of those processes.

## Common Math Preparation for Engineering

This work proposes to include dimensional analysis in the coverage of preparatory mathematics for engineers. In most undergraduate engineering programs, students have to take a 3-course calculus series followed by a course on differential equations. Other courses are typically included, either as electives or required courses, and vary from one program to another. These consist of courses like Linear Algebra, Numerical Analysis, Discrete Mathematics, Probability and Statistics, Complex Variables, Vector Analysis or others depending on the engineering field. Engineering curricula are typically overcrowded and demanding. The overcrowding provides a strong incentive to reduce the number of required mathematics courses causing many schools to offer combination courses, commonly referred to as "Engineering Mathematics" that cover several advanced topics[10, 11]. As an example, the newly developed Electrical and Computer Engineering curriculum at SUNY Oswego requires a 5 course series consisting of calculus 1, calculus 2, engineering mathematics, multivariable calculus, and discrete math and statistics. Engineering mathematics is a new 4-credit course designed to combine four important topics: differential equations, linear algebra, and complex variables, and numerical analysis. Another new 3-credit-hour course was developed to provide combined coverage of discrete mathematics and applied probabilities and statistics.

It is understood that combined coverage of these topics is not expected to be as extensive as it would be if students were to take all the separate mathematics courses on those topics. However, mathematics departments offer separate courses in each of the topics mentioned above, they are designed to serve many engineering and scientific disciplines. The combined content of these new courses is reduced to topics of direct relevance and application to Electrical and Computer Engineers only.

## Infusing Dimensional Analysis

Dimensional analysis is covered to a large extent in the textbook titled "Multidimensional Analysis" by George Hart[9] and could legitimately be the topic of a whole course. The topic may appear to be as simple as "adding apples to apples" rather than "apples to oranges" but this simplicity does not carry to practice in some important instances. The first step in adequate treatment of physical units is to realize that such special attention is needed. Typically, once an engineering problem or system is modeled mathematically, the physical units are dropped entirely for the purpose of computations that involve only numerical values, to be appended to results post-computation. However, the computational treatments may involve advanced calculus methods, vectors, matrices, and computational processes that include matrix inversion, singular-value-decomposition, eigenvalues, least-square optimization and other processes that completely obscure the computational interaction of units possibly producing completely unreliable and physically inconsistent result[13,14].

While dimensional analysis can be covered as a special separate topic of discussion in one of the mathematics preparation courses for engineers, better preparation is obtained by integrating this topic as part of several courses in mathematics, physics, and engineering.

## Physical Units Awareness and Education

Basic preparation courses that discuss physical units are introductory physics and chemistry courses where the international system of units (SI) is described. In the US, students are more familiar with commonly used units like the foot for length and the pound for mass along with their derived units. The global adoption of SI units in science and technology has not kept the American system of education, including modern textbooks, from continued reliance on English units as well, leading to the added necessity of frequent conversions from one system to the other and the always present possibility of mishaps.

Beyond the basic preparation obtained in physics and chemistry, students also learn to manipulate physical quantities and units through various applications in mathematics and engineering as they progress towards courses that rely on more advanced mathematics such as differential equations, matrix algebra, numerical analysis, and complex analysis. However, those advanced courses become far more focused on numerical computation processes and much less on physical units and their interactions within those processes. It is this precise shortcoming that needs to be addressed. Another tendency that exacerbates the problem is common reliance on increasingly advanced computation and modeling software tools that further obscure the intermingling of physical units. It is only recently that awareness of this problem has been sufficient to warrant attempts at addressing this issue by incorporating physical units treatment in computing tools[6].

Consider the simple case, offered in [9], of the equation expressing the period of oscillation of a pendulum formed by a point mass *m* attached to a pivot point by a massless string as illustrated in Figure 1,

$$T = 2\pi\sqrt{\frac{l}{g}}. \quad (1)$$

where *l* is length of the string and *g* is the acceleration of gravity.

Intuitively, students will perceive that the period of oscillation *T* is affected by the physical parameters involved consisting of the length *l*, the mass *m*, and the effect of gravity. They may be surprised by the fact that Equation 1 does not include the mass. A physical-units based analysis may clear up that confusion and explain why the period is independent of the mass. Thus, starting with *T* as a function of mass, length, and the acceleration of gravity, this means that seconds are obtained from combining meters, grams, and meters/seconds$^2$.

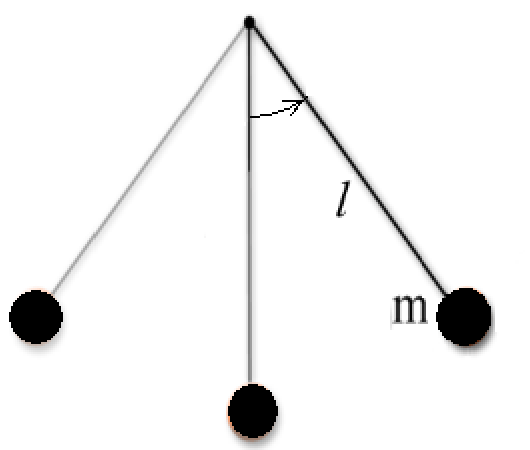


Figure 1. Pendulum

Since the time unit appears only in the acceleration of gravity in squared denominator form, the relation from period to gravity is one of inverse square-root, so

$$T \sim \frac{1}{\sqrt{g}} = \frac{seconds}{\sqrt{meters}} \quad (2)$$

Equation 2 must be treated to remove the square root of meters, from the denominator so the resulting unit of seconds is obtained. This is accomplished by multiplying the right side of Equation 2 by $\sqrt{length}$ to obtain:

$$T \sim \frac{\sqrt{l}}{\sqrt{g}} = \frac{seconds}{\sqrt{meters}}\sqrt{meters} = seconds \quad (3)$$

From a physical-units perspective, the mass *m* cannot play any role as it will introduce another physical unit that will disturb the physical consistency of the equation. However, there is room for a scalar (unitless) proportionality constant, determined experimentally to be 2π resulting in equation 1.

Coverage based on physical units, as illustrated in the above example, provides better understanding of the mathematical relationship between physical quantities as well as the derivation and verification of the validity of physics equations.

## Physical Units in Calculus

Many engineering processes are modeled as differential equations relating inputs to outputs in a system. Common examples include, in mechanics, the mass-spring modeling equation describing the motion of the mass in response to an input stimulus that excites the spring, and in electric circuits, the series or parallel resistor, inductor, capacitor circuit. The equation is of the form:

$$a\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+cy(t)=x(t) \tag{4}$$

where $y(t)$ is the output and $x(t)$ is the input and the coefficients $a$, $b$, and $c$, are system-dependent parameters. From a units consistency point of view, the three additive terms on the right side must have the same physical units as the input function $x(t)$. Physical consistency is achieved by the coefficients $a$, $b$, and $c$, having units that combine with those of $d^2y/dt^2$, $dy/dt$, and $y$, respectively, to achieve a common unit for all three terms that matches the unit of $x(t)$.

## Physical Units in Linear Algebra

Consideration of physical consistency in the treatment of vectors and matrices requires careful attention to the interaction of units in common operations such as multiplication, inversion, norms, and error computations. The description of each element within a vector or matrix in terms of its role within an application example must be considered and verified to lead to a consistent result.

Modeling engineering systems using linear algebra is common. Another example from electric circuits discussed in [9] provides a simple yet adequate illustration of issues that may arise when linear algebra techniques are used in systems where different units are combined. Figure 2 shows a block representation of a circuit with input vector $(v_1,\ i_1)$ and output vector $(v_2,\ i_2)$. In a linear model, the relationship is of the form

$$v_2 = a_1v_1 + b_1i_1 \tag{5}$$

and

$$i_2 = b_2v_1 + a_2i_1. \tag{6}$$

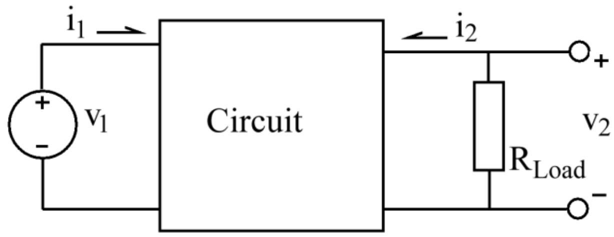


Figure 2. Electric Circuit

Both input and output vectors are non-commensurate[8], meaning they include elements of different physical units. Particular attention must be paid to the physical nature of parameters $a_1$, $b_1$, $a_2$ and $b_2$. In this case, $b_1$= resistance expressed in Ohms, $b_2$ = conductance expressed mhos (1/Ohm), while $a_1$ and $a_2$ are unitless. When equations 5 and 6 are combined in matrix vector form,

$$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ b_2 & a_2 \end{pmatrix} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \mathrm{A} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \qquad (7)$$

The coefficient matrix A is itself non commensurate. When solving the system for the input values that will provide a desired output, when A is invertible, the solution is found as

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = A^{-1} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ b_2 & a_2 \end{pmatrix}^{-1} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}. \qquad (8)$$

In general, no account is made of how the various units in matrix A interact through the inverse matrix computations as the calculations are completed numerically and the units are assumed to fall into place so that the result is correct. However, this example provides yet another possibility to reinforce the teaching of the concepts involved in this simple computation by tracking the step by step interactions of physical units through the process.

In this simple example, it is also worth pointing out that the commonly used Euclidian norm leads to inconsistent results since calculation of the magnitude of either the input or the output vectors will result in taking the square-root of the addition of volts$^2$ and amperes$^2$.

Other teaching topics of relevant interest related to this example are to determine the conditions under which a matrix multiplication is defined since row by column products of elements with different units may lead to inconsistent results.

As another example of embedding a discussion of dimensional issues within coverage of engineering mathematics, consider tracking a vehicle on a planar surface. The vehicle's complete locale is known when its ($a$, $b$) coordinates and its orientation angle $\theta$ (with respect to a reference direction) are known. A common technique used to express the vehicle's location that combines both position and orientation is by use of a specific matrix called the pose matrix in computer vision and in robotics and designed as

$$P = \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \quad (9)$$

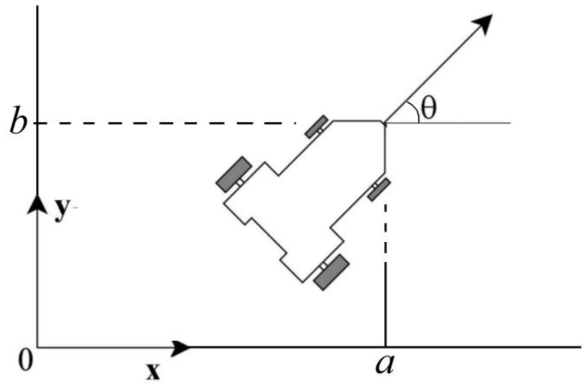


Figure 3. Vehicle Location

by combining both the vehicle orientation in the form of the 2 x 2 orientation matrix

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad (10)$$

and the position vector $\binom{a}{b}$ in one 3 x 3 matrix. The elements of *R* are unitless numbers while *a* and *b* have units of distance. In mechanical engineering, matrices of the form of *P* are commonly multiplied. Students can verify that the elements of the third row of *P* are chosen to insure that physical unit consistency is maintained through matrix multiplication.

## Conclusion

Inconsistent and incorrect results have been reported in the literature that were caused by careless treatment of physical units in complex computations performed by software designed for efficient numerical computations but lacking physical units treatment. This has brought to light the need to improve software as well as the need to better educate engineers and scientists on efficient and correct treatment of physical units in computing.

The development of a new degree program in Electrical and Computer Engineering at SUNY Oswego has provided an opportunity to research ways to leverage best practices in engineering education as well as improved course content in order to offer an innovative, modern, and efficient engineering education. As part of this effort, preparatory courses in mathematics have been reviewed with the objective of streamlining the curriculum and improve its efficiency. In collaboration between engineering and mathematics faculty, a mathematics course for engineers has been developed and its content carefully reviewed to offer a preparation that allows students to better address concepts needed and applied in their engineering studies. The course covers the three important topics of differential equations, linear algebra, and complex analysis. The faculty also recognized that increased awareness and practical training in the processing of physical units in important for engineering. Therefore this course is being updated in terms of content to include dimensional analysis, not as a separate topic, but as an embedded teaching methodology that integrates consideration of physical units in the mathematical processes and concepts covered in the course. This effort is to be accomplished by use of illustrative examples and assignments that directly address the issues of physical consistency discussed in this article. A few examples are discussed in this paper.

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