

2006-1211: DYNAMIC 3D VISUALIZATION OF STRESS TENSORS

Michael Hennessey, University of St. Thomas

Dr. Michael P. Hennessey is an Assistant Professor of Mechanical Engineering within the School of Engineering and is affiliated with the Center for Applied Mathematics. He has taught over 15 different courses in mechanical engineering at the undergraduate and graduate level and has written 30 papers in research or mechanical engineering education.

Luke Hacker, University of St. Thomas

Luke A. Hacker is a 2005 BSME graduate of the University of St. Thomas and currently a mechanical engineer with Remmele Engineering, Inc., Big Lake, MN 55309.

Dynamic 3D Visualization of Stress Tensors

Abstract

Sophomore and junior engineering students in majors such as mechanical, aerospace, civil, and materials engineering learn about the concept of the “state-of-stress” at a point within an object. Many engineering students have some difficulty in thoroughly grasping this concept, especially the more mathematical and visual aspects. To date, the best method we have for visualizing the state-of-stress has been to use Mohr’s circle(s), named after the famous 19th century German civil engineer, Christian Otto Mohr. Mohr’s circle applies to the case where rotations of a differential cube about a principal direction (only) are considered. While the discovery of Mohr’s circle was a brilliant accomplishment, it is somewhat non-intuitive to many students and it can take quite a bit of practice until the student has mastered the technique. Even when the student finally does grasp the concept, they may not necessarily have a complete picture of the state-of-stress at a point because Mohr’s circle only applies to rotations of a differential cube about a principal direction. In that sense it is a 2D method. Of course, in general one would be interested in viewing the stresses associated with all possible 3D orientations of the differential cube. In addition, while in recent years several education researchers have developed custom software to permit dynamic visualization of the state-of-stress as the differential cube rotates, visualization is typically static. What is needed is a true 3D dynamic visualization tool that permits one to visualize an arbitrary state-of-stress from the perspective of continuously varying and arbitrary 3D differential cube orientations, parameterized by a time varying rotation matrix, such as that driven by an Euler matrix with 3 time varying angles.

The objective of this educational research project is to: (1) develop the mathematics that permit one to arbitrarily change the orientation of a differential cube and determine the stresses in the new coordinate system (i.e. 3D tensor change of bases), (2) create a corresponding computer-aided-engineering (CAE) software tool using primarily MATLAB[®] and SolidWorks[®], (3) generate useful simulations using MATLAB[®] and corresponding animations using SolidWorks[®], and (4) attempt to determine their educational value with “mechanics” students. The animations in particular can be used within the engineering curriculum, specifically within the *Mechanics of Materials* and *Machine Design & Synthesis* courses where the 3D state-of-stress at a point is very important for understanding advanced mechanics concepts and failure theories which are inherently 3D in nature. In summary, this paper presents intriguing and very useful results that others, such as mechanics engineering faculty and students should find useful in enhancing their understanding of stress tensors. This has certainly been our classroom experience.

1. Introduction

Christian Otto Mohr (1835-1918), born in Germany in the coastal area by the North Sea, began his career as a civil engineer employed by the German railroad industry. During these years, Mohr began developing his theories of stress and strength of materials. At the age of 32, Mohr left the railroad industry and became a full-time theoretical engineer and a professor of mechanics. Eventually, after much investigation, Dr. Mohr developed a method for describing the state-of-stress at a point, his “Circles of Stress,” which now bear his name¹⁻³. “Mohr’s circles” have been used extensively in modern engineering, playing significant roles in

mechanical, aerospace, civil, and materials engineering fields and commonly appear today in undergraduate mechanics textbooks, e.g. Beer, *et al.*⁴ and Shigley, *et al.*⁵

Since Mohr's seminal work, over the years several researchers and educators have made strides to extend it, either for purposes of visualizing entire 2D or 3D fields (such as for stress) or for further enhancing the understanding of the state-of-stress at a single point through visualization. Regarding fields, Jermic, *et al.*⁶ present three different approaches to visualizing tensors (such as for stress) in 3D space (hedgehogs, hyperstations, and hyperstream surfaces), primarily for use in geomechanics applications. In another geomechanics application pertaining to oil wells, Zhou, *et al.*⁷ developed an efficient method to visualize symmetric second order tensors along 3D curves. Volume deformation forms the basis of another 3D field visualization technique, as presented in Zheng and Pang⁸. Visualization of the 2D state-of-stress at a point has been studied by several researchers. Almusallam and El-Din Taher⁹ present a brief history of the evolution of Mohr's work in the 20th century in addition to extending the original technique to provide access to the shearing stress components. Ressler¹⁰ developed an educational based animation program for use in understanding Mohr's circle referred to as the *Visual Stress Transformer*. Taking it a step further, Moller and Mokaddem^{11,12} have developed a "learning tool in the form of a system that gives real-time display of the stress state and its corresponding Mohr's circle at a point in a structure that the student manually loads."

The research objectives are to:

1. Develop the applied mathematics used to transform the state-of-stress at a point in a truly 3D sense (i.e. tensor change of bases).
2. Using modern software (i.e. MATLAB[®] and SolidWorks[®] specifically), develop a computer-aided-engineering (CAE) tool, and 3D graphical visualization that facilitates the 3D visual transformation of the state-of-stress at a point.
3. Using the above CAE tool, develop several specific animations (and an associated DVD) corresponding to the 3D state-of-stress at a point whereby the differential cube "tumbles" in a 3D sense.
4. Test the reaction of undergraduate mechanical engineering students first exposed to Mohr's circle(s) to several of these animations to attempt to determine their educational value.

The paper first presents the applied mathematics associated with the 3D transformation of the state-of-stress at a point. This is followed by the development of a CAE model using primarily MATLAB[®] and SolidWorks[®], simulation and animation studies, student feedback, and conclusions.

2. 3D Transformation of the State-of-Stress at a Point

To begin, we note that the state-of-stress at a 3D point can be represented as a symmetric rank 2 tensor $\underline{\underline{\sigma}}$ with 2 directions and 1 magnitude and is given by^{4,13}:

$$\underline{\underline{\sigma}} = \sigma_{xx}\underline{\underline{ii}} + \sigma_{xy}\underline{\underline{ij}} + \sigma_{xz}\underline{\underline{ik}} + \sigma_{yx}\underline{\underline{ji}} + \sigma_{yy}\underline{\underline{jj}} + \sigma_{yz}\underline{\underline{jk}} + \sigma_{zx}\underline{\underline{ki}} + \sigma_{zy}\underline{\underline{kj}} + \sigma_{zz}\underline{\underline{kk}} \quad (1)$$

where a standard stationary *xyz* coordinate frame with associated unit vectors $\underline{\underline{i}}$, $\underline{\underline{j}}$, and $\underline{\underline{k}}$ are utilized. $\sigma_{xx} (\equiv \sigma_x)$, $\sigma_{xy} (\equiv \tau_{xy} = \sigma_{yx})$, $\sigma_{xz} (\equiv \tau_{xz} = \sigma_{zx})$, $\sigma_{yy} (\equiv \sigma_y)$, $\sigma_{yz} (\equiv \tau_{yz} = \sigma_{zy})$, and

$\sigma_{zz} (\equiv \sigma_z)$ represent the 6 scalar stress components. Fig. 1 provides example visual interpretations of several stress components; others follow similarly.

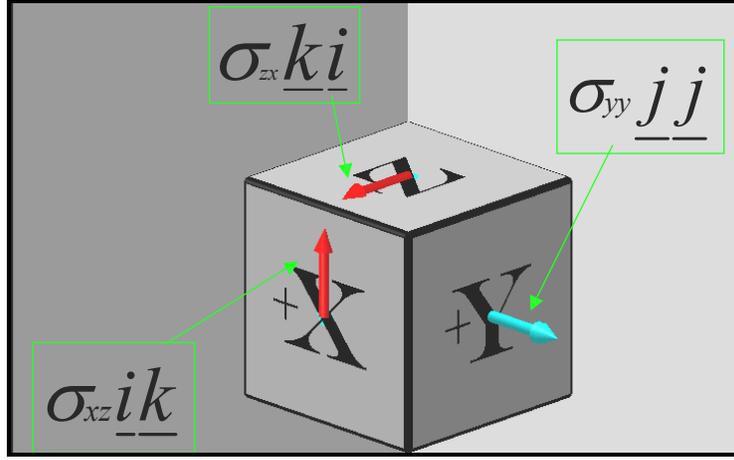


Fig. 1 Differential cube centered about point of interest with several stress components depicted. For example, $\sigma_{xz} \underline{i} \underline{k}$ is to be interpreted as a stress on the $+\underline{i}$ face, in the $+\underline{k}$ direction, with magnitude σ_{xz} .

Next, we would like to be able to change the coordinate system used in an arbitrary manner and represent the state-of-stress in the new coordinate frame. To facilitate this, a 3×3 Euler rotation matrix (R) commonly used in robotics with 3 sequential angular inputs $\phi(t)$, $\theta(t)$, and $\psi(t)$ is utilized and given by¹⁴:

$$R = [R_{ij}] = \begin{bmatrix} \cos \phi \cos \theta \cos \psi - \sin \phi \sin \psi & -\cos \phi \cos \theta \sin \psi - \sin \phi \cos \psi & \cos \phi \sin \theta \\ \sin \phi \cos \theta \cos \psi + \cos \phi \sin \psi & -\sin \phi \cos \theta \sin \psi + \cos \phi \cos \psi & \sin \phi \sin \theta \\ -\sin \theta \cos \psi & \sin \theta \sin \psi & \cos \theta \end{bmatrix} \quad (2)$$

which can be applied to transforming geometric data in the xyz coordinate frame into the moving $x'y'z'$ coordinate frame as: $[x \ y \ z]^T = R[x' \ y' \ z']^T$, or $[x' \ y' \ z']^T = R^T[x \ y \ z]^T$.

The stress tensor $\underline{\underline{\sigma}}'$ in the new coordinate frame is given by:

$$\begin{aligned} \underline{\underline{\sigma}}' = & \sigma_{x'x'} \underline{i}' \underline{i}' + \sigma_{x'y'} \underline{i}' \underline{j}' + \sigma_{x'z'} \underline{i}' \underline{k}' + \sigma_{y'x'} \underline{j}' \underline{i}' + \sigma_{y'y'} \underline{j}' \underline{j}' + \sigma_{y'z'} \underline{j}' \underline{k}' + \\ & \sigma_{z'x'} \underline{k}' \underline{i}' + \sigma_{z'y'} \underline{k}' \underline{j}' + \sigma_{z'z'} \underline{k}' \underline{k}' \end{aligned} \quad (3)$$

Recognizing the relationship between unit vectors in the $x'y'z'$ frame and the xyz frame yields:

$$\underline{i}' = R_{11} \underline{i} + R_{21} \underline{j} + R_{31} \underline{k}; \quad \underline{j}' = R_{12} \underline{i} + R_{22} \underline{j} + R_{32} \underline{k}; \quad \underline{k}' = R_{13} \underline{i} + R_{23} \underline{j} + R_{33} \underline{k} \quad (4)$$

Finally, substitution of Eqn. (4) into Eqn. (3), setting $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}'$, and performing the necessary tensor algebra (e.g. expanding $\underline{i}' \underline{i}' = (R_{11} \underline{i} + R_{21} \underline{j} + R_{31} \underline{k})(R_{11} \underline{i} + R_{21} \underline{j} + R_{31} \underline{k})$, etc.) yields a relationship between the 6 components of stress in the xyz coordinate frame and the 6 components of stress in the $x'y'z'$ frame:

$$[\sigma_{xx} \ \sigma_{yy} \ \sigma_{zz} \ \sigma_{xy} \ \sigma_{xz} \ \sigma_{yz}]^T = T[\sigma_{x'x'} \ \sigma_{y'y'} \ \sigma_{z'z'} \ \sigma_{x'y'} \ \sigma_{x'z'} \ \sigma_{y'z'}]^T \quad (5)$$

where $T \equiv T_{6 \times 6}(\phi, \theta, \psi)$, the stress transformation matrix is given by:

$$T = \begin{bmatrix} R_{11}^2 & R_{12}^2 & R_{13}^2 & 2R_{11}R_{12} & 2R_{11}R_{13} & 2R_{12}R_{13} \\ R_{21}^2 & R_{22}^2 & R_{23}^2 & 2R_{21}R_{22} & 2R_{21}R_{23} & 2R_{22}R_{23} \\ R_{31}^2 & R_{32}^2 & R_{33}^2 & 2R_{31}R_{32} & 2R_{31}R_{33} & 2R_{32}R_{33} \\ R_{11}R_{21} & R_{12}R_{22} & R_{13}R_{23} & \begin{pmatrix} R_{11}R_{22} + \\ R_{12}R_{21} \end{pmatrix} & \begin{pmatrix} R_{11}R_{23} + \\ R_{13}R_{21} \end{pmatrix} & \begin{pmatrix} R_{12}R_{23} + \\ R_{13}R_{22} \end{pmatrix} \\ R_{11}R_{31} & R_{12}R_{32} & R_{13}R_{33} & \begin{pmatrix} R_{11}R_{32} + \\ R_{12}R_{31} \end{pmatrix} & \begin{pmatrix} R_{11}R_{33} + \\ R_{13}R_{31} \end{pmatrix} & \begin{pmatrix} R_{12}R_{33} + \\ R_{13}R_{32} \end{pmatrix} \\ R_{21}R_{31} & R_{22}R_{32} & R_{23}R_{33} & \begin{pmatrix} R_{21}R_{32} + \\ R_{22}R_{31} \end{pmatrix} & \begin{pmatrix} R_{21}R_{33} + \\ R_{23}R_{31} \end{pmatrix} & \begin{pmatrix} R_{22}R_{33} + \\ R_{23}R_{32} \end{pmatrix} \end{bmatrix} \quad (6)$$

Through inversion of T (or other more efficient means) the stress components in the $x'y'z'$ can be determined. With time varying Euler angle inputs $\phi(t)$, $\theta(t)$, and $\psi(t)$, time varying stress data can be generated. The above result (i.e. Eqns. (5) and (6)) also applies to other rank 2 tensors such as for strain, inertia, and curvature.

3. Computer-Aided-Engineering (CAE) Tool

Given the ability to transform the state-of-stress at a point in a time-varying cube orientation sense, this can form the basis for animations that depict a cube whose orientation is changing with the proper stress components being displayed. Software tools utilized are MATLAB^{®15}, Excel^{®16}, and SolidWorks^{®17-19} and the details are described in Hacker^{20,21}. MATLAB[®] was used to create arrays of time-varying stress data, given a state-of-stress in the fixed, or xyz coordinate frame. A function file, “DiffCube” was created for this express purpose. Next, Excel[®] was used to further process and format the MATLAB[®] data for use with SolidWorks[®]. Specifically, “design tables,” with time dependent configurations (typically numbering in the hundreds) were created that subsequently form the basis for animations within SolidWorks[®] using the “add-on” Animator[®], in conjunction with a SolidWorks[®] assembly model of a differential cube with parametrically varying (within limits) stress arrow “parts.”

4. Simulation and Animation Studies

Using the CAE tool discussed above, a variety of simulations and corresponding animations were generated for purposes of exercising the equations and the software (for validation purposes) and to create simulations and animations of educational value. Three major cases considered include the following:

- Rotation about a principal axis
- Rotation about a non-principal axis
- Arbitrary “tumbling” with no fixed axis of rotation

The DVD mentioned above²¹ presents animations corresponding to all of these cases.

Rotation About a Principal Axis

This is the standard case that a number of researchers and educators (as mentioned above) have developed and studied already, including Mohr himself, over one-hundred years ago. We also

performed simulation and animation studies for this case for verification. This effort was successful and included verifying that the Von Mises stress is invariant with respect to orientation changes of the differential cube¹. However, due to space limitations and the desire to focus on more interesting general cases, no results will be presented.

Rotation About a Non-Principal Axis

Recall that rotation of the differential cube about a principal axis corresponds to a circular arc on a “ $\tau - \sigma$ ” plot (i.e. shearing stress vs. normal stress), i.e. Mohr’s circle. Suppose however, that the axis of rotation is not about a principal axis, what happens then? Excluding non-degenerative cases such as when 2 or more principal stresses are equal, the locus of points on a $\tau - \sigma$ plot will lie within the “tri-circular” region, i.e. between the 3 Mohr’s circles whose associated principal stresses are determined by the eigenvalues of the matrix components of $\underline{\underline{\sigma}}$. Fig. 2 presents simulation and partial animation results for such a case. Observe that the $\tau(t) - \sigma(t)$ trajectories for all +cube face normal directions (i.e. $+x(t)$, $+y(t)$, and $+z(t)$ ²) lie within the tri-circular region as alluded to above and fairly irregular continuous curves result. Note: since orientation changes do not occur about a principal axis, the notion of an algebraic sign associated with the shearing stress vector on the normal face (τ) does not have applicability. In view of this we have simply elected to plot $\pm \|\tau\|_2$ vs. σ so that the plots appear symmetric about the σ axis.

Mase and Mase²² present similar numerical results in that the coordinates of a single point within the tri-circular region (with $\tau \geq 0$) are determined. The 7-shot sequence of cube orientations with stresses shown (blue – normal stresses, red – shearing stresses) illustrates how the stresses change over time as the differential cube’s orientation changes. Lastly, to be able to generate the proper Euler angle time trajectories a special utility was created within SolidWorks[®] for this purpose – see Hacker²¹ for details.

Arbitrary “Tumbling” with no Fixed Axis of Rotation

To more fully exercise the machinery developed, we consider the totally arbitrary case where the differential cube’s orientation changes in a fairly arbitrary manner. Fig. 3 depicts an example of such a scenario. In particular, notice how the tri-circular regions are fairly well covered by the $x(t)$, $y(t)$, and $z(t)$ axis $\pm \|\tau\|_2(t) - \sigma(t)$ trajectories, certainly a qualitative check on the correctness of the results. Similarly to the previous simulation and animation run discussed above, the 7-shot sequence of cube orientations illustrates how the stresses change as the cube orientation changes.

The above simulation and animation runs are of an academic nature. What about incorporating an entertainment component? The current “video-game generation” of engineering students just might appreciate something along the lines of a “Mohr’s video-game.” Fig. 4 depicts a

¹ Recall that the Von Mises stress σ' is given

$$\text{by } \sigma' = [(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)]^{1/2} / \sqrt{2} .$$

² Note: to simplify the notation and minimize the labeling area, the instantaneous $x'y'z'$ axes will be labeled $x(t)$, $y(t)$, and $z(t)$, or simply $+X$, $+Y$, and $+Z$.

SolidWorks® model of a differential cube with stresses shown on the outer faces of the differential cube with a medieval motif. If desired, the above “academic animations” can also be played in this venue as well.

5. Student Feedback

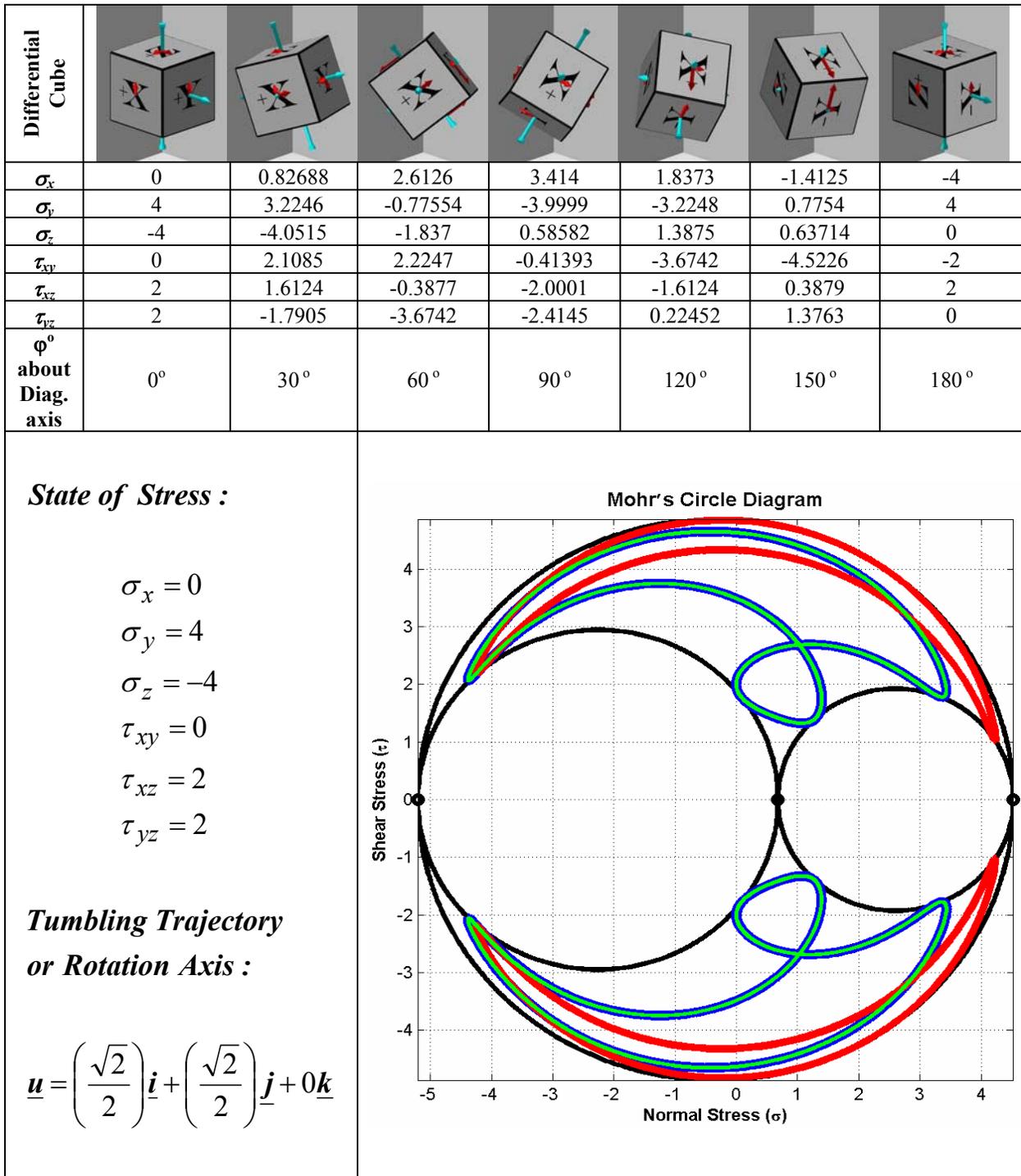
This past fall semester (2005) students in the *Machine Design and Synthesis* (ENGR 320) course were exposed to the Stress Visualization DVD after covering Mohr’s 3 circles in detail^{23,24}. The students viewed the pre-programmed animations on the DVD in their entirety (for about 20 minutes) and the following basic question was posed as part of the written end-of-course evaluation: “Did the Stress Visualization DVD played in class enhance your understanding of the concept of the state-of-stress at a point?” Of the 20 students that responded, 19 responded favorably and 1 did not. A typical favorable comment was “I liked how it showed the gradual change of the stresses.” The one student who did not respond favorably noted: “No, I have fairly good visualization skills. It backed up what I already understood.” In summary, students generally indicated that the DVD enhanced their understanding of the concept of the state-of-stress at a point and the lead author will continue to use this DVD as a teaching aid when teaching this course.

6. Conclusions

A CAE-based simulation / animation tool has been developed for permitting the dynamic 3D visualization of the state-of-stress at a point using an arbitrary time varying coordinate frame attached to the differential cube. It was made possible through a change of tensor bases, numerically computing software (i.e. MATLAB®), data processing software (i.e. Excel®), and 3D visualization software (i.e. SolidWorks®). Several interesting simulation / visualization runs were made and presented, one corresponding to rotation of the differential cube about a non-principal axis, and another corresponding to an arbitrary tumbling of the cube. A DVD of a collection of numerous animations was created and used effectively as a teaching aid when teaching junior mechanical engineering students in a course emphasizing mechanics concepts. In summary, engineering students in majors such as mechanical, aerospace, civil, and materials engineering and mechanics faculty should find this work useful.

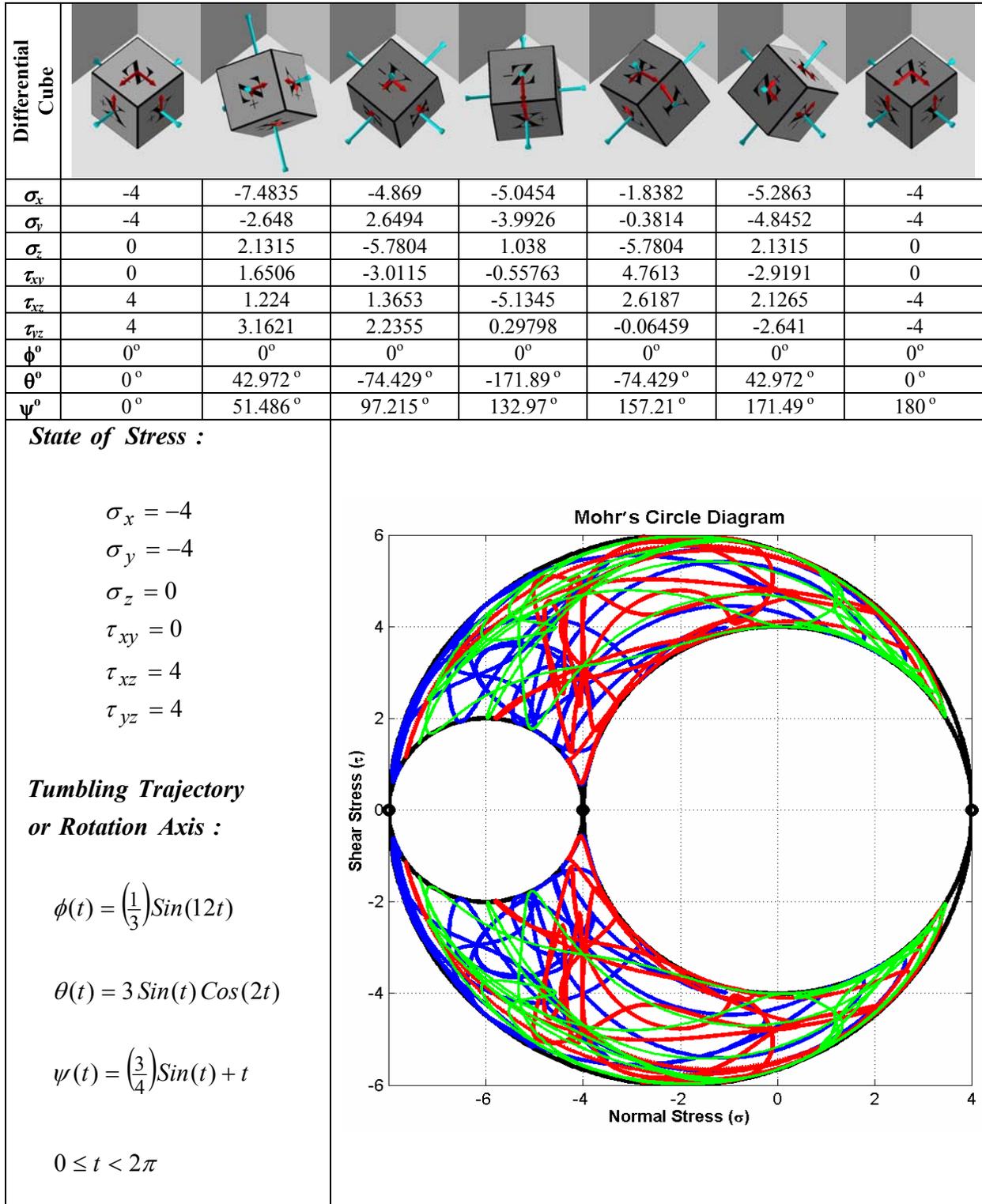
Acknowledgements

The authors acknowledge support of this work from the Center for Applied Mathematics (CAM) at the University of St. Thomas and input from the *Machine Design and Synthesis* (ENGR 320) students during fall semester 2005.



Mohr's Circle Diagram: X face = Blue, Y face = Red, Z face = Green

Fig. 2 Simulation and partial animation run for the case of rotation of the differential cube about a non-principal axis (Von Mises stress $\sigma' = 8.5$).



Mohr's Circle Diagram: X face = Blue, Y face = Red, Z face = Green

Fig. 3 Simulation and partial animation run for the case of arbitrarily tumbling of the differential cube (Von Mises stress $\sigma' = 10.6$).

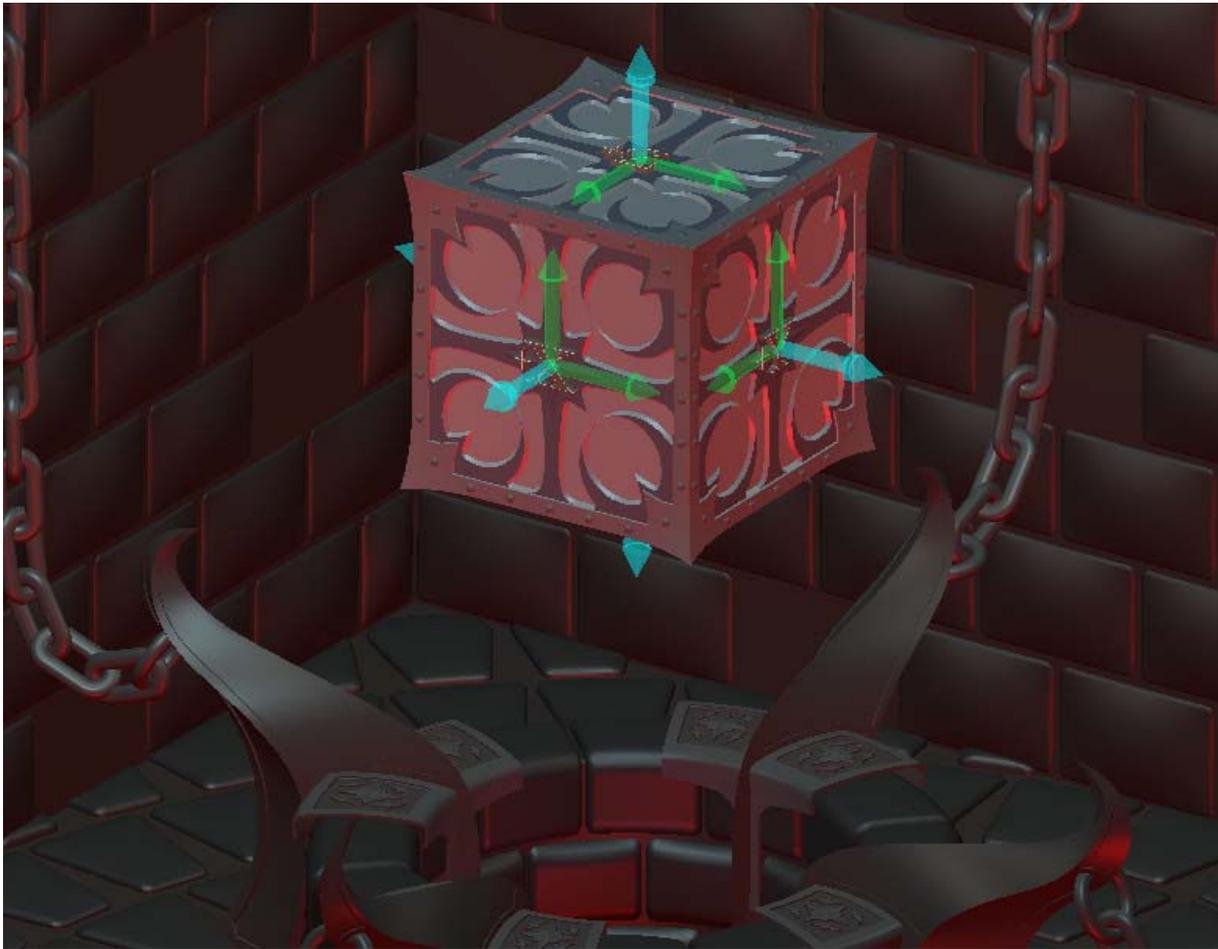


Fig. 4 *Differential cube and the state-of-stress at a point depicted in a medieval motif.*

Bibliography

- [1] Timoshenko, S. P., **History of Strength of Materials**, McGraw-Hill, 1953.
- [2] Mohr, C. O., “uber die Darstellung des Spannungszustandes und des Deformation szustandes eines Korperelementes,” **Zivilingenieur**, 1882.
- [3] Mohr, C. O., **Abhandlungen aus dem Gebiete der technischen Mechanik**, Wilhelm Ernst and Son, Berlin, 1906.
- [4] Beer, F. P., E. R. Johnston, and J. T. DeWolf, **Mechanics of Materials**, 4th Edition, McGraw-Hill, New York, 2006.
- [5] Shigley, J. E., C. R. Mischke, and R. G. Budynas, **Mechanical Engineering Design**, 7th Edition, McGraw-Hill, New York, 2004.
- [6] Jeremic, B., G. Scheuermann, J. Frey, Z. Yang, B. Hamann, K. Joy, and H. Hagen, “Tensor visualizations in computational geomechanics,” **International Journal for Numerical and Analytical Methods in Geomechanics**, 2002; 26: 925-944.
- [7] Zhou, Y. Q.-L. Li, and S.-D. Ma, “Visualization of Geostress Tensor Along Oil Well Trajectory,” *Proceedings of Computer Graphics International 2001*, 3-6 July 2001, Hong Kong.

- [8] Zheng, X. and A. Pang, "Volume Deformation for Tensor Visualization," *IEEE Visualization 2002*, Oct. 27 – Nov. 1, 2002, Boston, MA.
- [9] Almusallam, A. and S. E.-D. Taher, "Three-Dimensional Mohr's Circle for Shear Stress Components," **Journal of Engineering Mechanics**, Vol. 121, No. 3, March 1995.
- [10] Ressler, S. J., "The Visual Stress Transformer: An Animated Computer Graphics Program for Engineering Mechanics Education," *1996 ASEE Annual Conference Proceedings*, Session 1668, pp. 3483-3486.
- [11] Moller, J. and A. Mokaddem, "A Tool for Learning Mohr's Circle," **International Journal of Mechanical Engineering Education**, Vol. 29, No. 1, 2001.
- [12] Moller, J. and A. Mokeaddem, "A Tool for Teaching Stress Transformation by Mohr's Circle," *ASEE Annual Conference*, Session 3266, June 15-18, 1997.
- [13] Lebedev, L. and M. J. Cloud, **Tensor Analysis**, World Scientific, River Edge, New Jersey, 2003.
- [14] Paul, L., **Robot Manipulators: Mathematics, Programming, and Control**, The MIT Press, Cambridge, MA, 1981.
- [15] Hanselman, D. and B. Littlefield, **Mastering MATLAB[®]**, Pearson-Prentice Hall, Upper Saddle River, NJ, 2005.
- [16] Kuncicky, D., **Introduction to Excel 2002**, Third Edition, E-Source: The Prentice Hall Engineering Source, Upper Saddle River, NJ, 2003.
- [17] SolidWorks Corporation, *SolidWorks[®] 2004: SolidWorks Essentials: Parts and Assemblies*, Document Number: PMT0025ENG, SolidWorks Corporation, Concord, MA, 2003.
- [18] SolidWorks Corporation, *SolidWorks[®] 2004: Advanced Assembly Modeling*, Document Number: PMT0027ENG, SolidWorks Corporation, Concord, MA, 2003.
- [19] SolidWorks Corporation, *SolidWorks[®] 2004: SolidWorks Essentials: Drawings*, Document Number: PMT0026ENG, SolidWorks Corporation, Concord, MA, 2003.
- [20] Hacker, L. A., "Visualization of Real, Symmetric, Rank 2 Tensors," Summer Student Research talk given at the Center for Applied Mathematics (CAM), University of St. Thomas, St. Paul, MN, August 25, 2004.
- [21] Hacker, L. A., "Visualization of Real, Symmetric, Rank 2 Tensors," *CAM Summer 2004 Research Report*, University of St. Thomas, St. Paul, MN, October 1, 2004. Accompanying DVD entitled *Stress Visualization* (with M. P. Hennessey) also created.
- [22] Mase, G. E. and G. T. Mase, **Continuum Mechanics for Engineers**, CRC Press, Boca Raton, FL, 1992.
- [23] University of St. Thomas, *Undergraduate Catalog: 2002-2004*, St. Paul, MN, 2002.
- [24] Hennessey, M. P., "A Structured Activity Based Approach to Teaching Machine Design," *ASEE 2001: North Midwest Section Annual Conference*, University of North Dakota, Grand Forks, ND, September 27-29, 2001.