AC 2004-1265: EDUCATING STUDENTS TO QUESTION, TEST, AND VERIFY PROBLEM SOLUTIONS

Joseph Rencis, University of Arkansas

Jr., Hartley T. Grandin,

Educating Students to Question, Test and Verify Problem Solutions

Joseph J. Rencis, Hartley T. Grandin, Jr.

Mechanical Engineering Department
Worcester Polytechnic Institute,
Worcester, MA 01609-2280, USA

Abstract

This paper assumes the importance of educating our engineering students to question, test and verify "answers" to all of their problem solutions. It presents an approach currently practiced by the authors in teaching an introductory mechanics of materials course. In problem solving, emphasis is placed on: (1) writing the governing equations in symbolic form with a bare minimum of algebraic manipulation, (2) solving the equations with a commercially available, student choice, computer equation solver and (3) most importantly, developing and implementing test case scenarios to verify the validity of the problem solution. There are three major advantages to this approach. First, the development of the equations in symbolic form requires the students to focus more on the *physics of the problem.* Second, the use of the computer equation solvers eliminates tedious and often error prone algebraic manipulation. Third, the test case scenarios suggested for *verification of the "answer"* force the student to consider limiting, "known result", solutions of the problem. Throughout the course, the students apply this approach to homework and project activities. Initially they are given the test scenarios, but, with practice, they learn to create their own. This paper presents two example problems to demonstrate the approach.

Introduction

In a homework assignment, the ultimate goal for a majority of undergraduate engineering students is simply to obtain the "answer" in the back of the book. A common approach is to search the textbook chapter for the applicable formula or equation and immediately insert numbers and calculate an answer. This approach is often successful with problems that require few equations, especially if the equations can be solved sequentially or are easily manipulated to isolate the unknown variable. The unfortunate aspect of this is that students may spend very little time focusing on the basic fundamental *physics of the problem* and, generally, no time at all on the very important *verification of the "answer"*! As problems become more complex, with increased numbers of simultaneous equations, such as with statically indeterminate problems, this approach is laborious and fraught with opportunities for equation manipulation errors. As a result, introductory course instruction and textbooks do not involve these types of problems. This paper presents the authors' attempt to prepare the student for problems of greater complexity with emphasis not only on the *physics of the problem*, but also on the *verification of the "answer"*. These two issues are stressed in activities in-class (examples, quizzes and tests) and out-of-class (reading, homework problems and projects). The foundation of this approach is the formulation of all of the basic governing equations in symbolic form, with no algebraic manipulation to isolate unknowns, before entering numeric data. The practicality of this approach is possible because of readily available equation solver computer programs.

Advantage and Challenges of a Symbolic Formulation

Formulating a solution in symbolic form based on a general problem statement is a common approach in introductory mechanics of materials textbooks when few equations are involved[1-19]. As problems become more complex, and many simultaneous equations are involved, the textbooks provide little or no guidance on the solution[1-19]. Teaching the student to model a general physical problem with the fundamental equations written in symbolic form, with no variable values specified, causes the student to more fully concentrate on the fundamental principles taught in the course. Introducing the computer equation solver tool to solve the equations removes the necessary manipulation of the equations to isolate the dependent variables. Finally, teaching the student to examine and test the answer becomes a critical goal in the course. The *advantage* of the approach being presented here is the natural inclusion of emphasis on the following:

1. *Physics of the Problem.* The primary goal of a course should be to bring the student to a thorough understanding of the fundamental principles governing the physics of the topic. The ability to define a problem's physical model and to construct the corresponding mathematical formulation of the model should reflect this understanding. Having written the governing equations, any convenient mechanism may be employed to execute a numerical solution.

2. *Engineering Tools.* The students gain a working familiarity with one or more of the available equation solving programs, and it is stressed that the programs, as well as the general approach to problem solving, has a carryover to their other courses. A symbolic formulation can be naturally and easily solved with the modern engineering tools such as Mathcad, MATLAB™ and TK *Solver*. Furthermore, these engineering tools reduce to a minimum the required algebraic manipulation of the equations because there is no need to isolate the dependent variable or to reduce a set of simultaneous equations to one equation. Redefining the role of the dependent and independent variables is a trivial task.

3. *Problem Complexity.* Coupling symbolically derived equations with an engineering computer equation solver tool permits the solution of problems more complex than traditionally encountered in the first mechanics of materials course.

4. *Verification of the Answer.* With equations written in symbolic form, a readily available computer equation solving program permits an effortless examination of the effects of changes in input variables. The ability to easily recalculate a solution allows the student to explore general and limiting cases for the problem. This leads to implementation of tests which may be compared with expected and/or known responses. This provides greater solution reliability. (The students are reminded that it is the responsibly of a professional engineer to verify his/her solution.)

5. *Design.* With a strategy incorporating a computer equation solver with the 'raw' fundamental symbolic equations, design and redesign activities can be naturally introduced in the first mechanics of materials course. The authors' aim in this introductory course is to introduce design through short, simple and well-defined projects. As the student progresses to more advanced courses, i.e., machine design, structural design, etc., projects become lengthier, open-ended and difficult, leading to the capstone design experience.

   The implementation of this approach carries with it the following significant *challenges*:

1. *Symbolic Equations.* The difficulty in requiring a symbolic approach with sophomore and junior engineering students is motivating them to write a complete set of governing equations in symbolic form before substituting numerical values. They just are not familiar with formulating problems this way. Their training in high school and college has primarily involved sequential solutions of the applicable formulas. The good news is that, for many students, after sufficient practice, the wisdom of the procedure becomes quite apparent.

2. *Computer Equation Solvers.* The second challenge is motivating the students to learn a computer equation solver program if this is their first exposure to such a tool. They resist learning how to use the program on 'simple' problems when they can solve it quicker by hand. When suddenly faced with the more complex problems, the unprepared student becomes quite frustrated!!

3. *Testing the Solution.* The third challenge is getting them to test the solution. They are generally quite content with any answer that will get them partial credit. Making the effort to develop test cases 'takes too much time'!!

What About Using an Equation Solver as a Blackbox?

The proposed approach requires that students use an engineering equation solver tool such as Mathcad, MATLAB™ or TK *Solver* to solve the equations. Many instructors feel that a major disadvantage of using any equation solver is the blackbox input/output response and lack of contact with the solution procedure. This is a very justified concern and is one that the authors share. However, with intelligent use of intermediate and final result tests, the computer program is a much more reliable calculation device than a calculator. We must appreciate, however, that our students will be entering an arena filled with computers and easily run software. What must be addressed is the attitude of the program user. Although the authors believe that students must understand how to solve a system of equations, it is impossible, nevertheless, to expect them to know exactly how all programs they will use are coded. ***An important element of a student's education must include a reflex suspicion of program results and an understanding of the need and the ways and means to check results with alternative methods.*** This is what is expected when a student graduates and becomes a professional engineer in industry. Why not expect the student to be a professional engineer during their academic career?

Points Emphasized in a Symbolic Formulation

The authors emphasize the following points when formulating a problem symbolically for in-class and out-of-class exercises:

- *Definition of Variables and their Fundamental Dimensions.* For the given structure, the quantities of interest are defined symbolically. In addition, only the dimensions associated with each variable are established, e.g., stress is force per area (F/L^2).
- *Define Known and Unknown Variables.* For the predefined structure the student is required to determine what is given (known) and what must be determined (unknown). Once the unknown variables are established then the number of independent equations to solve the problem can be established.
- *Free-Body Diagrams (FBD).* A complete and accurate FBD of the structure and each component is required.
- *Sign Convention.* A consistent sign convention is used throughout the entire analysis of the structure. Students do not realize the importance of sign control.
- *Symbolic Derivation of Equations.* Emphasis is placed on deriving all equations symbolically based on fundamental principles without regard to isolating the unknown variable on the left-hand side of the equation and the known variables on the right-hand side.
- *Independent Equations.* Especially when summing moments, the students need to be reminded of the requirement for generating independent equations. A common error occurs with two force summations and multiple moment summations on one FBD.
- *Dimensional Verification of Equations and Unit Conversion.* Due to the symbolic nature of the equations, the dimensions of a variable are checked to make sure that the variable describes quantitatively the physical properties.

Progression of Symbolic Approach in a Mechanics of Materials Course

To encourage the development of good habits throughout the course, a variety of exercises are posed in-class and additional reading and problems are assigned for homework. Each problem includes a list of simple, known results and limiting case scenarios for solution verification. Early in the course, these test scenarios for the homework and project assignments are provided, and the students compare the computer solution for the anticipated results. Later in the course the students design the test cases for homework and project assignments and comment on the results. Solution verification scenarios become a component of the quizzes and tests throughout the course.

Example 1: Statically Indeterminate Beam on Elastic Supports

Consider the statically indeterminate elastic beam loaded by its own weight and supported by three elastic posts as shown in Figure 1. The symbolic formulation of this problem is discussed in the Appendix. The problem formulation is divided into the following six steps:

1. Model
2. Free Body Diagrams
3. Equilibrium Equations
4. Compatibility
5. Material Law
6. Deflection Analysis of Elastic Beam

The first five steps focus on formulating the problem for a rigid beam and the sixth step introduces the elasticity of the beam. The Appendix shows that 20 independent equations are required to solve this problem. The equations are input into an engineering equation solver program, of the student's choice, in the form and order of their derivation from basic principles. No attempt is made to isolate a variable on the left-hand side, and there is no algebraic combining of the equations. This type of complex problem is rather difficult to solve by hand! There are 20 coupled equations since the problem is statically indeterminate.

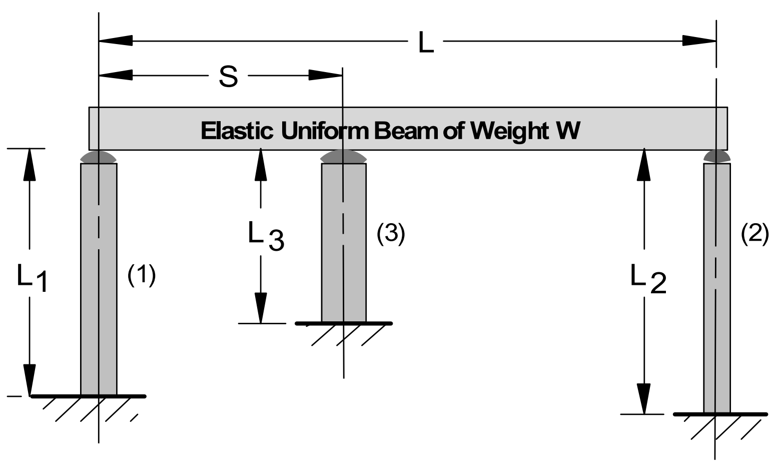


Figure 1. Statically indeterminate elastic beam supported by elastic posts.

The problem in Figure 1 is solved through in-class and out-of-class exercises. First, a simple problem is considered, and then more complexity is added as new concepts are introduced in the course. This problem complexity progression is shown in Figure 2. The following four progressive cases in Figure 2 include:

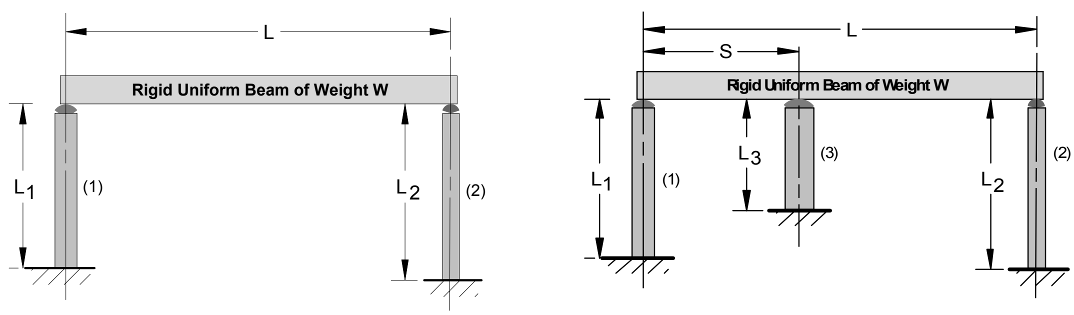
- *Case 1: Statically determinate rigid beam with elastic supports, Figure 2a.* This problem is discussed when statically determinate axial problems are introduced in the course. This statically determinate problem is easy to solve by hand since the equations are uncoupled.

- *Case 2: Statically indeterminate rigid beam with elastic supports, Figure 2b.* This problem is discussed when statically indeterminate axial problems are introduced in the course. This problem can be solved by hand. However, since the equations are now coupled the solution is much easier to solve using a computer.

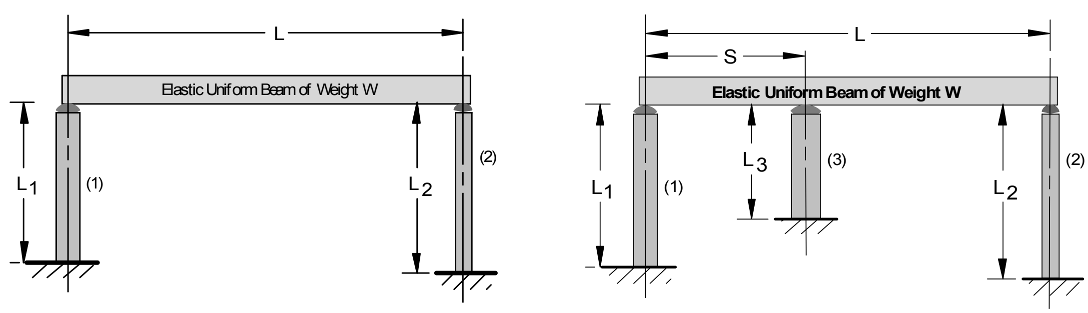
- *Case 3: Statically determinate elastic beam with elastic supports, Figure 2c.* This problem is discussed when statically determinate beam problems are introduced in the course. The addition of the elastic beam makes a computer solution an easy choice.

- *Case 4: Statically indeterminate elastic beam with elastic supports, Figure 2d.* This problem is discussed when statically indeterminate beam problems are introduced in the course. This problem can only be solved using a computer since there are 20 coupled equations (see Appendix).

To limit the discussion, we will focus on Case 4 in Figure 2d and describe how Cases 1 through 3 in Figure 2a through 2c, respectively, can be obtained. In-class examples and out-of-class homework exercises are employed throughout the course so students gain a better understanding of *verification of the "answer"* and the *physics of the problem.*



a. Case 1: Determinate rigid beam. b. Case 2: Indeterminate rigid beam.



c. Case 3: Determinate elastic beam. d. Case 4: Indeterminate elastic beam.

Figure 2. Four cases of a beam supported by elastic posts.

The students gain a better understanding of *verification of the "answer"* from Case 4 through the following exercises:

- *Case 3 as a Limit of Case 4.* Remove the inner post (insert low E and/or A for inner post relative to the outer posts), and let the outer posts have equal stiffness. This problem can be easily checked by hand since it is statically determinate. Each outer post force should approach half the beam weight. The change in lengths of the outer posts can be calculated, ($\Delta L = FL/AE$), and the deflection of the beam mid-span relative to the ends can be calculated from the displacement formula for a uniformly loaded simply supported beam, ($v = 5wL^4/384EI$). By also making the outer posts rigid (insert a large E and/or A for posts relative to the beam) yields the displacement solution for a uniformly loaded simply supported beam.

- *Case 2 as a Limit of Case 4.* Make the posts equally spaced and of identical materials, areas, and lengths (thus same stiffness), and make the beam very stiff (very large E and/or I). The calculated results should yield equal post loads and length changes.

- *Case 1 as a Limit of Case 4.* Case 1 is the same as Case 3 except that the beam is rigid (insert a large E and/or I for the beam relative to the posts).

The authors use qualitative and quantitative exercises in-class and out-of-class so that the students gain a better understanding of the physics of the problem. The *physics of the problem* is introduced in Cases 1 through 4 (Figure 2) through the following exercises:

- *Load Direction.* Change the direction of the distributed load to ensure that the internal forces, stresses and displacements change in direction and not magnitude.

- *Load Magnitude.* Double the distributed load to check for doubling of the support forces and the vertical displacement in the beam and supports.

- *Elastic Support Cross-sectional Area.* Doubling the cross-sectional area of the support will yield half the stress, the same force and half the vertical displacement at each support.

There are many more exercises that could be considered. One should note that the students also obtain a better understanding of the *physics of the problem* through the *verification of the "answer"* exercises.

Example 2: Cantilever Circular Shaft with an Attached Lever

Consider the circular elastic shaft with an attached elastic lever subjected to a concentrated end force as shown in Figure 3. This in-class exercise is presented towards the end of the course when the combined loading topic is covered. In this course, complex deflection analysis is included in addition to the traditional combined stress analysis covered in all textbooks. This problem requires calculating the stiffness of the structure measured at the free end of the lever and determination of the magnitude and location of the maximum von Mises stress in the structure. All equations are written symbolically and then entered by the student into an engineering equation solver tool of their choice.

So that the students gain a better understanding of the *physics of the problem* and *verification of the "answer"*, some of the following exercises are considered:

- *Rigid Shaft and Elastic Lever.* For very large shaft E and G, the solution corresponds to a cantilever beam (lever) of length D and fixed at point B.

- *Elastic Shaft and Rigid Lever.* The solution is the superposition of the shaft in torsion and the cantilever shaft in bending. For this case, the torsion and bending of the shaft are artificially uncoupled by the independent selection of magnitudes for E and G. The solution is based on the following two simpler cases:

    - *Shaft in Torsion.* Assuming a large value for the shaft E and/or I, the solution corresponds to a circular shaft subjected to a concentrated end torque.

    - *Cantilever Beam (Shaft).* When the shaft shear modulus G is assumed very large relative to E, the solution is a cantilever beam with a concentrated force P at the free end of the shaft.

- *Elastic Shaft and Elastic Lever.* This case corresponds to the superposition of the two cases above.

This problem can be increased in complexity by adding to the end of the lever a concentrated force parallel to the longitudinal axis of the shaft. Similar scenarios can then be considered as discussed above.

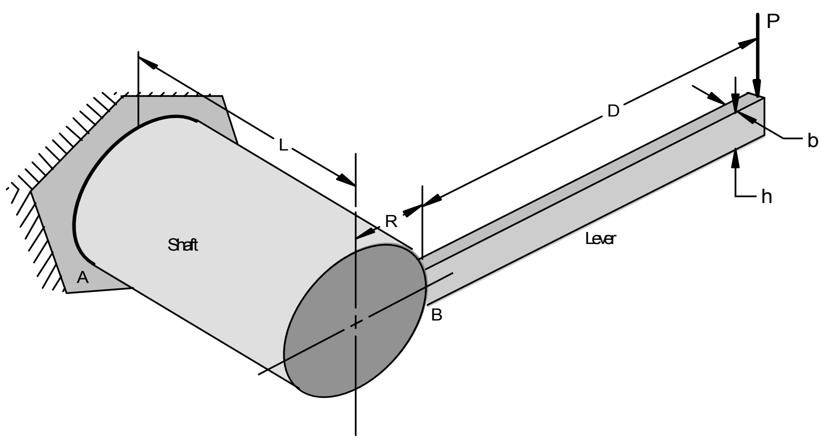


Figure 3. Cantilever circular elastic shaft with an attached elastic lever.

Conclusion

Formulating a mechanics of materials problem in symbolic form and then solving the equations with an engineering equation solver tool has been proposed. This approach allows the students to gain a better understanding of the *physics of the problem* and the *verification of the "answer."* This approach is implemented through in-class examples and out-of-class reading assignments. The students also use this approach in homework, projects, quizzes and tests throughout the course. In the early part of the course the students are given the scenarios for understanding the physics of the problem and verification of the "answer." Later in the course they must define their own scenarios.

There are many advantages in using the proposed approach in a sophomore level mechanics of materials course. One advantage is that solving the symbolic equations with an engineering tool reduces to a minimum the required algebraic manipulation of equations because there is no need to isolate the dependent variable on the left-hand side of the equation. Thus, the student can very easily explore any desired combination of known and unknowns without rewriting the equations. Using this approach students are, for the first time, able to understand what engineering is about since they are allowed to do more than just solve 'one answer' type textbook problem. Another particular advantage is that simple design problems can be easily integrated into the course. The proposed approach can also be used in follow up design and nondesign courses that includes advanced mechanics of materials, machine design, structural analysis, structural design, etc.

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JOSEPH J. RENCIS
Joseph J. Rencis has been a Professor in the Mechanical Engineering Department at Worcester Polytechnic Institute since 1985. His research focuses on the development of boundary and finite element methods for analyzing solid, heat transfer and fluid mechanics problems. He serves on the editorial board of Engineering Analysis with Boundary Elements and is associate editor of the International Series on Advances in Boundary Elements. He is currently writing an introductory mechanics of materials textbook with the co-author. He has been the Chair of the ASEE Mechanics Division, received the 2002 ASEE New England Section Teacher of the Year and is a fellow of the ASME. For the ASEE New England Section he currently serves as Chair of the Awards Committee. He received

his B.S. from the Milwaukee School of Engineering in 1980, a M.S. from Northwestern University in 1982 and a Ph.D. from Case Western Reserve University in 1985. *V-mail:* 508-831-5132; *E-mail:* jjrencis@wpi.edu

HARTLEY T. GRANDIN, JR.
Hartley T. Grandin, Jr. is a Professor Emeritus of Engineering Mechanics and Design in the Mechanical Engineering Department at Worcester Polytechnic Institute. He has authored the textbook Fundamentals of the Finite Element Method that was published by Macmillan in 1986. Since his retirement from WPI in 1996, he teaches a mechanics of materials course each year and is currently writing an introductory textbook with the author. In 1983 he received the WPI Board of Trustees' Award for Outstanding Teaching. He received his B.S. in 1955 and an M.S. in 1960 in Mechanical Engineering from Worcester Polytechnic Institute and a Ph.D. in Engineering Mechanics from the Department of Metallurgy, Mechanics and Materials Science at Michigan State University in 1972. *E-mail:* hgrandin@rcn.com and hgrandin@wpi.edu

Appendix

Example 1: *The uniform elastic beam of, total weight W, (distributed load, $w = \frac{W}{L}$), is now supported on three elastic posts as shown in Fig. A.1. The post spacing, the post cross sectional areas, the lengths and the material properties of beam and posts are all arbitrary. Derive the fundamental equations from which you can obtain the post and beam displacements and the contact forces between the beam and posts.*

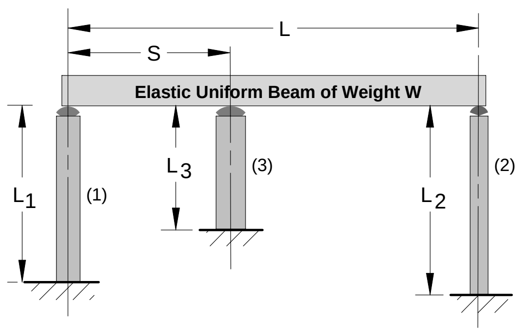

Figure A.1. Statically indeterminate elastic beam supported by elastic posts.

SOLUTION:

1. ***Model.*** *Isolate the posts (bars) and beam elements of the structure as shown in Fig. A.2.*

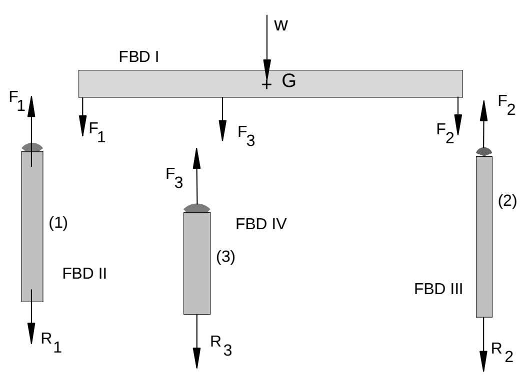

Figure A.2. Beam and posts free body diagrams.

2. ***Free Body Diagrams.*** *The free body diagrams of the posts and the beam are shown in Fig. A.2 Although it may be counter to your intuition, the post unknown forces are assumed in the positive (tensile) direction. The only reason for doing this is to avoid using negative forces in the Hooke's Law equations. The solution will yield a negative if the forces are really compressive.*

3. ***Equilibrium Equations.*** *Applying equilibrium equations to the beam in Fig. 2 will relate the forces exerted by the posts to the applied load which, in this case, is the beam's weight.*

   Summing forces on FBD I:

$$\text{FBD I,} \qquad \sum F_{vertical}, \qquad F_1 + F_2 + F_3 + W = 0 \tag{1}$$

   Summing moments about the left end:

$$\text{FBD I,} \qquad \sum M_{left\_end}, \qquad W(\frac{L}{2}) + F_2(L) + F_3(S) = 0 \tag{2}$$

   We can write only two independent equilibrium equations for the three unknown post forces, F_1, F_2 and F_3. The problem is statically indeterminate

4. ***Compatibility.*** *Compatibility will relate the displacements of the three posts to the displacement of the beam at the respective points of contact. Figure A.3 has been drawn to indicate the deformation of the posts and beam. An XY coordinate system has been arbitrarily placed at the tops of the* ***undeformed*** *posts with the origin at the top of the post (1) ; the displacements v_A, v_C and v_E of the post tops will be referenced to the Y axis and will be positive in the upward $+Y$ direction.*

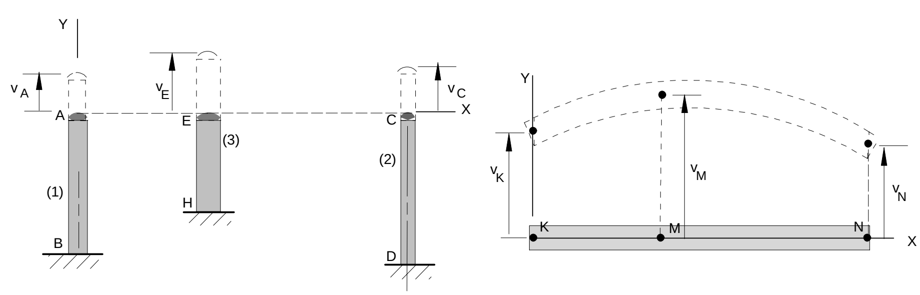

Figure A.3. Compatibility diagrams.

If we make the (reasonable) assumption that the deformation of the beam between the points of contact with the posts and the neutral surface is *negligible, we may write the following compatibility conditions:*

$$v_A = v_K \tag{3}$$
$$v_C = v_N \tag{4}$$
$$v_E = v_M \tag{5}$$

5. ***Material Law.*** *The Material Law for the end loaded bar will be applied to each of the posts (bars). The forces on each post have been assumed in the positive (tensile) sense, and we write the equations with the assumption of a positive change of length; if the bars actually compress, the ΔL answer will be negative. Recalling once again the Hooke's Law relationships for the change of length of a bar (post):*

$$\text{Post (1):} \qquad \Delta L_1 = \frac{F_1 L_1}{A_1 E_1} \tag{6}$$
$$\text{Post (1):} \qquad \Delta L_2 = \frac{F_2 L_2}{A_2 E_2} \tag{7}$$
$$\text{Post (1):} \qquad \Delta L_3 = \frac{F_3 L_3}{A_3 E_3} \tag{8}$$

where the change of length of each post is defined in terms of the displacements of the end points. Recall the following conventions:

- *Displacements are positive in the positive coordinate direction, $+v$ in $+Y$ direction.*
- *The positive change in length of the bars is defined by*

$$\Delta L_1 = v_A \tag{9}$$
$$\Delta L_2 = v_C \tag{10}$$
$$\Delta L_3 = v_E \tag{11}$$

At this point, we have 11 independent equations for 12 unknowns.

$$F_1, F_2, F_3, v_A, v_C, v_E, v_K, v_M, v_N, \Delta L_1, \Delta L_2, \Delta L_3,$$

Since there are more unknowns than equations, we must now obtain the additional equation(s) by considering the deformation of the beam.

6. ***Deflection Analysis of the Elastic Beam.*** *We will elect to use the double integration method to solve for the beam deflection. We will construct free body diagrams of different lengths of the beam and write piecewise continuous internal moment equations valid within spatial constraints. Where the beam length passes though the discontinuity, boundary conditions will be enforced.*

(a) ***Free Body Diagrams.*** *Because of the force applied by the inner post, the internal couple bending moment function is not continuous; it is piecewise continuous over the regions on each side of the inner post. Thus, we construct two free body diagrams, in Fig. A.4, to obtain the bending moment in each region.*

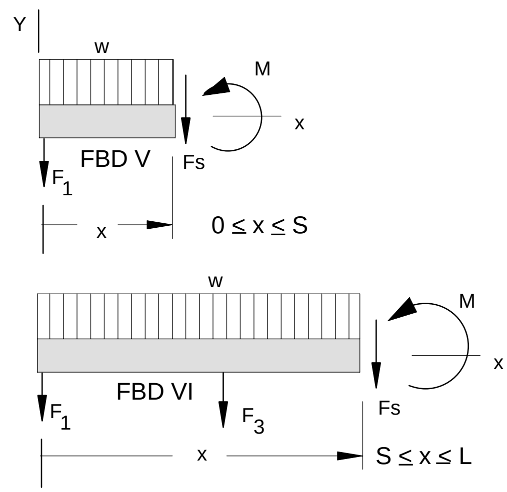

Figure A.4. Beam free body diagrams.

(b) ***Equilibrium Equations.*** *Referring to FBD V and FBD VI in Fig. A.4, we write the moment equations for each section:*

$$M = -F_1 x - \frac{wx^2}{2} \qquad 0 \leq x \leq S:$$

$$M = F_1 x + F_3 (x - S) + \frac{wx^2}{2} \qquad S \leq x \leq L:$$

(c) ***Differential Equation.*** *For each region of x where the internal moment is a continuous function, we substitute the internal moment expression and integrate twice:*

For $0 \le x \le S$ *:*

$$
\begin{aligned}
EI\frac{d^2v}{dx^2} &= M \\
EI\frac{d^2v}{dx^2} &= -F_1 x - \frac{wx^2}{2} \\
EI\theta &= EI\frac{dv}{dx} = -F_1\frac{x^2}{2} - \frac{wx^3}{6} + C_1 && (12) \\
EIv &= -F_1\frac{x^3}{6} - \frac{wx^4}{24} + C_1 x + C_2 && (13)
\end{aligned}
$$

For $S \le x \le L$ *:*

$$
\begin{aligned}
EI\frac{d^2v}{dx^2} &= -F_1 x - F_3(x-S) - \frac{wx^2}{2} \\
EI\frac{dv}{dx} &= -F_1\frac{x^2}{2} - F_3\left(\frac{x^2}{2} - Sx\right) - \frac{wx^3}{6} + C_3 && (14) \\
EIv &= -F_1\frac{x^3}{6} - F_3\left(\frac{x^3}{6} - S\frac{x^2}{2}\right) - \frac{wx^4}{24} + C_3 x + C_4 && (15)
\end{aligned}
$$

(d) ***Boundary and Continuity Conditions.*** *Substitute the following boundary and continuity conditions:*

i. *For* $x = 0$ *in Eq. 13,* $v(0) = v_K$ *(boundary condition at the left end)*

$$EIv_K = C_2 \tag{16}$$

ii. *For* $x = S$ *in Eq. 13,* $v(S) = v_M$ *(boundary condition at inner support)*

$$EIv_M = -F_1\frac{S^3}{6} - \frac{wS^4}{24} + C_1 S + C_2 \tag{17}$$

iii. *For* $x = S$*, Eq. 13 =Eq. 15 (continuity of displacements at the inner support post)*

$$
\begin{aligned}
-F_1\frac{S^3}{6} - \frac{wS^4}{24} + C_1 S + C_2 &= -F_1\frac{S^3}{6} - F_3\left(\frac{S^3}{6} - S\frac{S^2}{2}\right) \\
&\quad - \frac{wS^4}{24} + C_3 S + C_4 \\
C_1 S + C_2 &= F_3\frac{S^3}{3} + C_3 S + C_4 && (18)
\end{aligned}
$$

iv. *For* $x = S$*, Eq. 12 = Eq. 14 (continuity of the slopes at the inner support post)*

$$
\begin{aligned}
-F_1\frac{S^2}{2} - \frac{wS^3}{6} + C_1 &= -F_1\frac{S^2}{2} - F_3\left(\frac{S^2}{2} - S^2\right) - \frac{wS^3}{6} + C_3 \\
C_1 &= F_3\frac{S^2}{2} + C_3 && (19)
\end{aligned}
$$

v. For $x = L$ in Eq. 15, $v(L) = v_N$ (boundary condition at the right end)

$$EIv_N = -F_1\frac{L^3}{6} - F_3\left(\frac{L^3}{6} - S\frac{L^2}{2}\right) - \frac{wL^4}{24} + C_3L + C_4 \quad (20)$$

7. ***Solve.*** *There are 16 unknown variables in the equations as they have been written:*

$$F_1, F_2, F_3, v_A, v_C, v_E, v_K, v_M, v_N, \Delta L_1, \Delta L_2, \Delta L_3, C_1, C_2, C_3, C_4$$

and these will be determined from the simultaneous solution of the eleven independent equations (1) through (11) plus the five equations (16) through (20). These equations have been inserted into an equation solver. Use the equation solver of your choice and compare the answers.

8. ***Verify.*** *Once again, we stress the importance of testing your solution. For example, you should try the following:*

- Make the posts equally spaced and of identical materials, areas, and lengths (thus same stiffness), and make the beam very stiff (very high E and/or I). The calculated result should yield equal post loads and length change.
- Remove the inner post (insert low E and/or A relative to the outer two posts), and let outer posts have equal stiffness. This problem can be checked by hand since it is statically determinate. Each outer post force should approach half the beam weight. The change in lengths of the outer posts can be calculated, ($\Delta L = \frac{FL}{AE}$), and the deflection of the beam mid-span relative to the ends can be calculated from the displacement formula for a uniformly loaded beam, ($\nu = \frac{5wL^4}{384EI}$).
- Make the end posts of very low stiffness relative to the inner post. The inner post force should approach the weight of the beam.