Elementary Limit Analysis in Mechanical and Civil Engineering Sophomore Curricula

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Abstract: The customary Mechanics of Materials / Strength of Materials courses for Civil and Mechanical Engineering sophomores do not include a unit on *limit analysis* (presumably) due to its absence from textbooks. All the same, background required for limit analysis - uniaxial elastoplastic deformation and plastic hinge - is routinely included in the course. Thus, students are prepared to learn elementary limit analysis and extend their understanding of "plastic hinge" into a usable basis for beam design.

The author has prepared and taught a two lecture unit on limit analysis within a Strength of Materials course. This material is based on and reinforces plasticity topics already addressed in the course and includes examples of increasing complexity involving concentrated and distributed loadings. It is similar in spirit to Chapter 13 of Nash's summary [1]. However, the method of virtual work is employed here as an alternative to static analysis.

Limit analysis offers students a new design methodology and sharpens their geometrical / analytical skills through postulating collapse mechanisms and applying virtual work techniques. They acquire a basis for comparing the relative merits of using limit loads instead of traditional elastic analysis of statically indeterminate beams.

Introduction

It might be worthwhile asking whether exposing sophomores to limit analysis has pedagogical value.

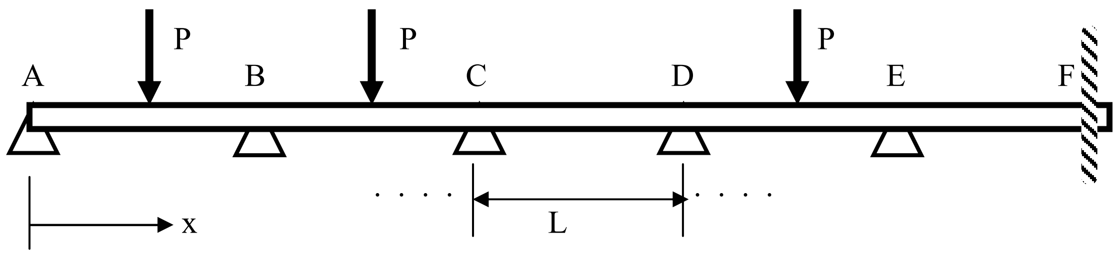
Except for a brief introduction to elastic-plastic material behavior in a mechanics of materials course, our students live in a linearly elastic world. Further, based on the dominant theme of homework problems, this world is also statically determinate. Analysis of statically indeterminate beams is usually accompanied by multiple variables and simultaneous equations[1]. Reactions at supports must be determined. While locating maxima in a moment diagram is straightforward for statically determinate cases, it can be difficult otherwise. Elastic beam design, based on maximum moment, M_{max}, is impeded by static indeterminacy.

[1] The degree of difficulty depends on the method used to handle the indeterminacy.—integration, superposition, or Castigliano's Theorem.

In contrast, limit analysis seems user friendly. It assumes that the beam section's *plastic moment,* M_P, is M_{max}, and requires that the corresponding *ultimate load*, P_U, causes collapse to occur in conjunction with suitably arranged *plastic hinge(s)*[2]. Identifying the smallest load P_U for a prescribed loading defines M_P (or vice versa). This process requires the student to analyze the beam visually and locate possible collapse modes. There is no need to determine reaction forces.

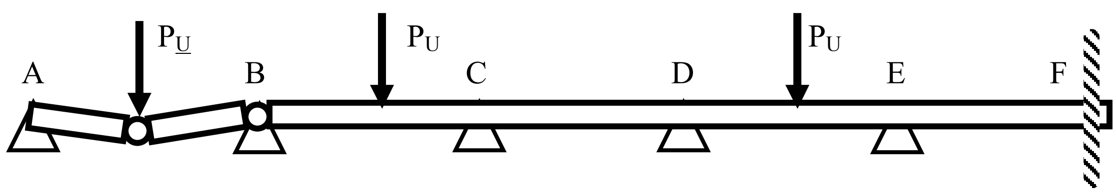
An illustration of elastic and plastic methods is provided by the beam below having equal spacing L between pin supports at A, B, … E and fixed support at F. Identical loads P are applied at mid points of panels AB, BC, and DE. The objective is to determine M_{max}.

Elastic analysis uses the deflection equation, $y'' = M(x)$, subject to boundary conditions $y = 0$ at A,B,…F, and $y' = 0$ at F[3]. *Note that all load / support information is transmitted to the entire elastic curve.*



M_{max} is found from a moment diagram followed by $M_{all} = M_{max} / F.S. = \sigma_{all} S$, where $\sigma_{all} = \sigma_{ULT} / F.S.$; and S is the elastic section modulus. This problem would not be routinely assigned to sophomores for homework.

In limit analysis the plastic moment $M_P = (\sigma_Y Z)$ occurs at hinges that define collapse modes. The plastic section modulus is Z; and σ_Y is the yield stress. For this illustration it is seen that the "easiest" collapse mode (i.e., with the smallest limit load P_U) occurs for a collapse mechanism having plastic hinges ○ at B and under the load between A and B, as shown.



Through virtual work, the collapse load P_U is found to have the value $6M_P / L$.[4] *Note that only local load / support information near the collapse region is required.* The design step is to relate P_U to $P_{all} = P_U / F.S.$ A moment diagram is not needed because M_P is taken to be M_{max}.

The remainder of the paper gives an overview of the lectures, presented within the setting of class notes.

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[2] Limit analysis terms are defined as encountered in the lectures. A useful glossary is given by Nash [1].
[3] Continuity of y and y' at load and support points is automatically met by using singularity < > brackets.
[4] Hinges at B and C and the midpoint of BC (or at D and E and midpoint of DE) would give $P_U = 8M_P/L$

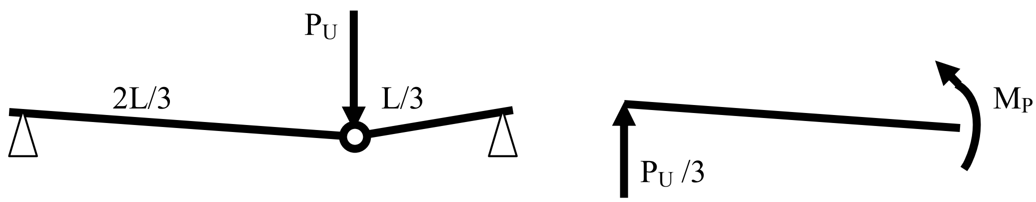
Proceedings of the 2005 American Society for Engineering Education Annual Conference & Exposition

Lecture Highlights

This two-lecture unit is an introduction to limit analysis within a strength of materials course[5]. Text material is a handout of fifteen pages of notes prepared by the author. Limit analysis appears after students have completed minimal plasticity topics, including plastic moment.

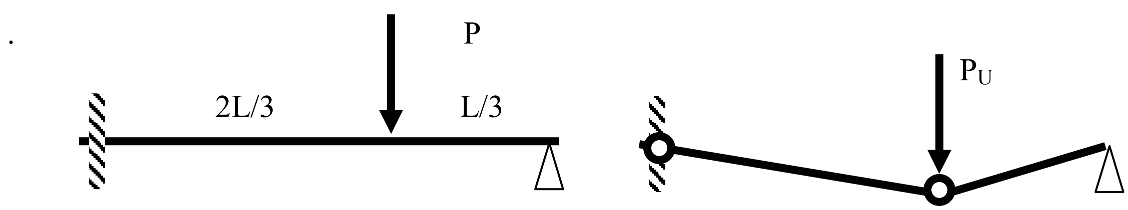
The notes open with an example of a system of elastic-plastic axially loaded rods supporting a rigid bar bearing load P. Collapse is shown to occur at a specific ultimate load, P_U.

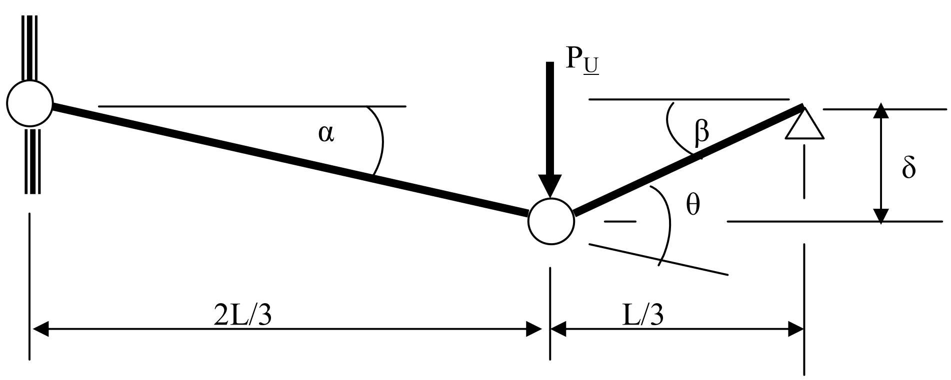
A bending example illustrates collapse in simplest terms. The moment at the hinge is M_P.



Using statics it is found that $P_U = 9\ M_P / 2L$.

Next, a propped cantilever beam is analyzed, first using statics, then by the technique of virtual work; and students learn advantages of the latter.





Angles α, β, θ and deflection δ are used in connection with L to describe the collapse geometry.

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[5] At Lehigh this comes in the fourth semester, following *Elementary Engineering Mechanics,* a "statics" course, half of which contains the early topics of strength of materials, through bending.

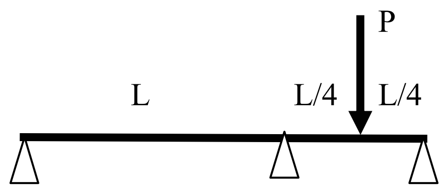
The plastic moment M_P is applied at both hinges; and the virtual works of moments and force are expressed in terms of kinematic parameters as

$$M_P\,\alpha + M_P\,\theta \;=\; P_U\,\delta\,; \quad \theta \;=\; \alpha + \beta\,; \quad \alpha = \; 3\delta / 2L\,; \;\; \beta = 3\delta / L\,;$$

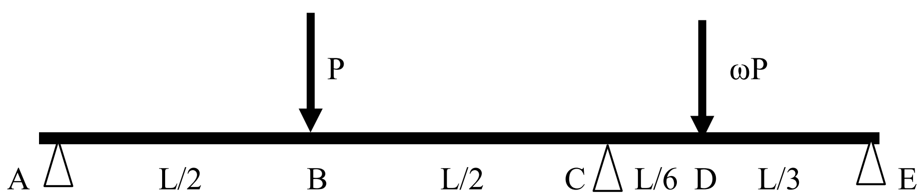
$$M_P\,(\,2\,\alpha + \beta\,) \;=\; P_U\,\delta\,; \qquad 6\,M_P\,\delta / L = \;P_U\,\delta\,; \qquad \boxed{P_U = 6\,M_P/L}$$

An Extended Example

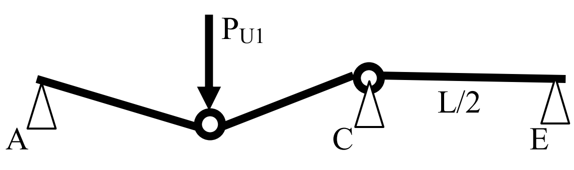
The illustration of a continuous beam

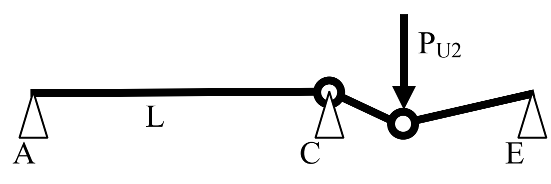


leads to an example having a more complex loading arrangement.



A variety of loadings are possible here by assigning various values to ω. This structure can collapse only for plastic hinges restricted to be at support C and at one or the other load points B and D. For very small, or very large, values of ω, the collapse mode can readily be predicted. Otherwise not. A "limit analysis by parts" is used in which the collapse mechanism for each panel is uncoupled from the other. Thus,



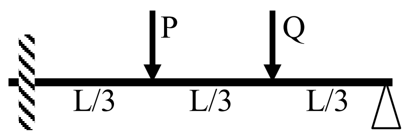


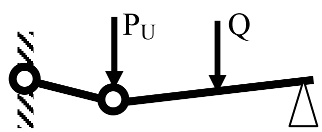
Results for these independent collapse mechanisms, derived in previous illustrations, are

$$P_{U1} = \; 6\,M_P / L \qquad\qquad P_{U2} = 15\,M_P / L$$

To be definite, the parameter ω is assigned values 1/3, 3 and 5/2, sequentially. In the first case the ultimate load for panel AC is $6\,M_P / L$ while the load applied to CE is 1/3 of this, or $2M_P/ L$, a value well below the collapse load for CE. A similar analysis for ω = 3 shows that panel CE collapses at the limit load, 15 M_P / L , while AC remains safe at the load 5 M_P / L . For $\omega = 5/2$ both panels collapse simultaneously at their respective limit loads; and three plastic hinges are required.

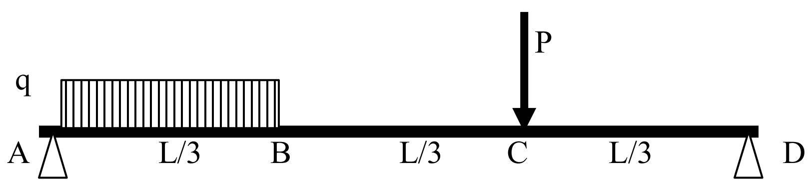
The final example with concentrated loads considers the effect of a fixed load on the limit load. Here the propped cantilever supports a constant load Q and variable load P.





The result, $P_U = 15M_P / L - Q / 2$, shows the effect of the fixed load is to reduce P_U.

The notes conclude with an example of a simply supported beam subject to a distributed load and a concentrated load.

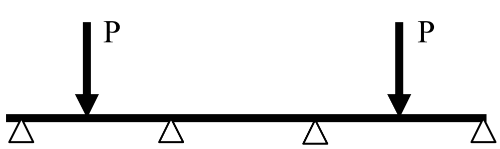


The location of the plastic hinge is at C for P large relative to qL/3, and under the distributed load for the reverse case.

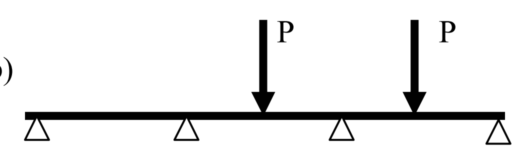
Typical Homework Problems

Generally, homework problems are variations on examples in the notes, stated with increasing difficulty, For example, determine the collapse loads P_U for (a) and (b) in each set:

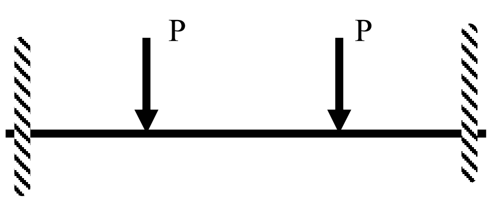
(a)



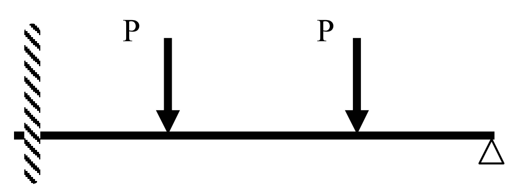
(b)



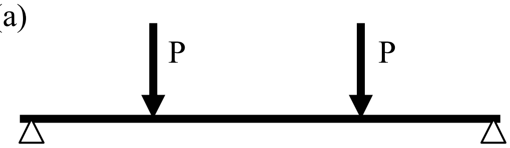
(a)



(b)



(a)



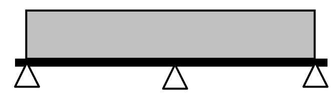
(b)

For the uniformly distributed load p , determine the collapse loads p_U for (a), (b) and (c)

(a)

(b)

(c)



A problem requiring statically indeterminate elastic analysis is this:

For the propped cantilever beam considered in the notes, plastic hinges occur at the fixed support and under the load. Which point is first to begin yielding? [Hint: as load P increases, determine which point first reaches M_Y (yield moment)]. What is the value of P_Y? Use results for the propped beam to express the ratio P_U / P_Y in terms of the section shape factor k ($= M_P / M_Y$).

Bibliography

[1] W. Nash, *Strength of Materials*, Schaum's Outline Series, McGraw-Hill, New York, 1998.

Biographical Information

KENNETH N. SAWYERS, Professor in the Department of Mechanical Engineering and Mechanics at Lehigh University, received the PhD. degree from Brown University in 1966. He joined Lehigh in 1969, following three years on the staff at Stanford Research Institute. He serves as Faculty Liaison to the Co-Op program. He is a member of the Cooperative Education, Mechanical Engineering, Mechanics, and Entrepreneurship Divisions.