Enhancing the Success of Electronics and Mechanical Engineering Technology Students with an Engineering Calculus II Class Utilizing Open-source Mathematical Software

Dr. Erik A. Mayer, Pittsburg State University

Erik Mayer is a Professor at Pittsburg State University in Kansas where he has been instrumental in forming the Electronic Embedded Systems emphasis in the Electronics Engineering Technology program. His research interests are power electronics and embedded systems. He previously taught at Bowling Green State University in Ohio where he worked with the Electric Vehicle Institute . In addition, he worked at Visteon Corporation designing components for hybrid vehicles. He received his Ph.D. in Engineering Science at the University of Toledo in Ohio.

Dr. Yaping Liu, Pittsburg State University

I teach a variety of mathematics classes at Pittsburg State University. In particular, Calculus I, II, III, Engineering Calculus II, and Differential Equations are closely related to engineering.

Dr. David Miller, Pittsburg State University

David Miller completed a B.S. in biological systems engineering at the University of Nebraska, Lincoln, in 2000, a master's of science in bioengineering at Arizona State University in 2002, and a doctorate in biomedical engineering at the University of Nebraska, Lincoln, in 2008. After completing his Ph.D., he worked in the medical industry as a product development engineer for three years before coming to Pittsburg State University, where he is a member of the mechanical engineering technology faculty. He teaches courses in engineering mechanics, heat transfer, engineering graphics, and product design. He has designed products using AutoCAD, CATIA, Pro/E, and SolidWorks and is a certified SolidWorks Professional.

Dr. Jeremy Wade, PacificSource Health Plans

Dr. Wade was an Assistant/Associate Professor of Mathematics at Pittsburg State University from 2009-2016. He is currently an analyst at PacificSource Health Plans.

Enhancing the success of electronics and mechanical engineering technology students with an Engineering Calculus II class utilizing open-source mathematical software

Introduction

Electronics Engineering Technology (EET) and Mechanical Engineering Technology (MECET) students were previously required to take Calculus I and II. However, EET and MECET students typically struggled with Calculus II. In response to this, faculty from the Mathematics and Engineering Technology departments created an Engineering Calculus II course as an alternative for EET and MECET majors. The proposed outcome of the new course was to motivate EET and MECET students to learn the mathematics and to be able to apply them in their field of study. The new Engineering Calculus II course focused on mathematic topics more relevant to EET and MECET. In addition, the course included laboratory projects which utilized the open-source SageMath software and illustrated the applications of the mathematics to EET and MECET. The effectiveness of the new Engineering Calculus II course was assessed using two methods. A final laboratory project was assigned at the end of the course and assessed with a rubric. In addition, the effect of the new course on EET students was assessed using the Electrical/Electronics Technology Outcome Assessment that was regularly used for assessment for ETAC/ABET accreditation. Approval to use student data was obtained from the university's Institutional Review Board (IRB) which ensures that the rights of the students were not violated.

Course Development

In 2014, faculty from the Mathematics and Engineering Technology departments began to meet to discuss the creation of the alternative Engineering Calculus II course for EET and MECET majors. It was decided that the new Engineering Calculus II course would differ from the traditional Calculus II class in two fundamental ways: the topics covered and the discussion of potential applications.

The sequence of topics in the standard Calculus II course began with integration techniques of a single variable, proceeded to sequences and series, and ended with parametric equations, polar equations and vectors. It was decided that topics such as conics and series would not be included in the Engineering Calculus II course as they would have limited applications in EET and MECET. The topics of differential equations and matrices were added to Engineering Calculus II as they have more applications in EET and MECET. Through subsequent meetings between faculty, a textbook on engineering mathematics was chosen and it was established that the topics in Table 1 would be covered [1], [2].

There is currently a movement within the mathematics community to cover topics more applicable to STEM fields earlier in the calculus sequence [3]. The sequence of course topics in Engineering Calculus II closely aligns with the proposed sequence in this movement because of the removal of the largely theoretical sections on sequences and series and the inclusion of topics in differential equations and matrices.

Table 1: Lectures and lab experiments for Engineering Calculus II

Lecture topics	Labs	
	EET	MECET
Review of differentiation and integration	Introduction to Sage	
Integration: substitution, integration by parts, trigonometric functions, mean and root-mean-square (rms)	Calculating the dc and rms values of periodic waveforms	Centroid of an area
Complex numbers: introduction, operations, polar form, exponential form	Using phasors to analyze ac circuits	Spring-mass-damper behavior
Matrices: introduction, determinants, operations, inverses, linear systems	Branch-current analysis	3D and 2D motion of an object
Vectors: introduction, products, angle	Permanent-magnet dc motor	3D moment about a point
Partial derivatives: rules, rate of change, chain rule, applications	Differential form of Gauss's law	2D motion of a Fluid
Polar coordinates: introduction, polar curves, applications	Phasors in Cartesian and polar coordinate systems	Transverse and radial coordinates of a robot arm
Multiple integrals: double, triple, applications	Integral form of Gauss's law	Volume and surface area of objects
First order differential equations: introduction, separable, homogeneous, linear	Timing applications of RC circuits	Emptying of a water-filled tank
Second order differential equations: homogeneous, inhomogeneous, particular solutions	Series RLC circuit	Spring-mass-damper system
Laplace transforms: introduction	(No lab)	
Review and final lab project	Series RL circuit	Manufacture of heat transfer plugs

In standard calculus courses, the discussion of the applications of the mathematics is limited and often overlooked in order to provide more theoretical depth. This benefits mathematics students who will be focusing more on the theory. However, this is detrimental to engineering technology students who would benefit more from the discussion of applications. Thus, laboratory projects that involved the applications of the mathematics studied were included in Engineering Calculus II. It was hypothesized that the labs would further increase student motivation by illustrating EET and MECET applications of the mathematics covered in the lecture.

A university grant was received in 2016 for Mathematics, EET, and MECET faculty to create a lab manual for Engineering Calculus II. The lab manual consisted of an EET laboratory project and a MECET laboratory project for each topic as shown in Table 1 [2], [4]. The laboratory projects consisted of problems to be solved using the open-source SageMath software [5]. The laboratory projects illustrating the applications of integration are shown in Appendices A and B. The laboratory project in Appendix A was developed for MECET students and covers the centroid of an area. The laboratory project in Appendix B was developed for EET students and covers the calculation of the dc and rms values of periodic waveforms.

Assessment

The effectiveness of the new Engineering Calculus II class was assessed using two methods. A final laboratory project was assigned at the end of the Engineering Calculus II course and then assessed. In addition, the effects of the new course on EET students was assessed using the Electrical/Electronics Technology Outcome Assessment that was regularly used for assessment for ETAC/ABET accreditation [2], [6]. Approval to use student data was obtained from the university's Institutional Review Board (IRB) which ensures that the rights of the students were not violated.

A final laboratory project was used to assess the effectiveness of the Engineering Calculus II course by assessing the students' ability to set-up and solve EET and MECET problems using SageMath and mathematical techniques learned in the course. The final laboratory project was assigned at the end of the semester and was assessed using the rubric in Table 2 [2]. The projects were assigned a rating of 4 (Exceptional) to 0 (No Credit) in four weighted categories: math comprehension (50%), concept comprehension (30%), SAGE technique (10%), and technical communication (10%).

In 2016 and 2017, a total of 16 students took the Engineering Calculus II course. The assessment results of the final laboratory projects are shown in Table 3. The number of projects that received each rating is listed for each category. It can be seen from Table 3 that by using the weights of the categories, the weighted average rating was 2.3. This rating was between the "Meets Expectations" and "Above Expectations" ratings. Therefore, the average student performance on the final laboratory project exceeded expectations.

Table 2: Rubric for final laboratory project

	Exceptional (A=4)	***Above Expectations (B=3)***	***Meets Expectations (C =2)***	***Below Expectations (D=1)***	***No Credit (F=0)***
Math Comprehension (50%)	Elegant solution. Student obtains correct result using correct reasoning, logically connected steps, and precise operations.	Able to provide solid reasoning and perform necessary operations to reach a correct conclusion. May miss some steps or have minor calculation errors.	Able to perform basic operations to reach a conclusion. May have flawed reasoning or missing some key steps, but the work is correct for the most part.	Able to set up mathematical equation and perform some operations, but unable to provide solid reasoning or reach a correct conclusion.	Not able to set up mathematic al equation or do basic operations.

Table 2 (continued): Rubric for final laboratory project

	Exceptional (A=4)	*Above Expectations (B=3)*	*Meets Expectations (C =2)*	*Below Expectations (D=1)*	*No Credit (F=0)*
Concept Comprehension (30%)	In addition to being able to rearrange equations to solve problem, student is able to make inferences to problems outside the scope of the lab activity but relevant to his major.	Able to recognize the connection between mathematical equations and engineering concepts in order to fully resolve the problem at hand.	Able to rearrange equations to suit the needs of Lab problem. May not be able to fully describe rationale.	Able to recognize some equations and engineering concepts, but unable to make the necessary connections so that the problem can be fully understood.	Unable to draw conclusions about engineering content based on math concept.
SAGE Technique (10%)	Elegant code. Streamlined processes. Fully commented with easy-to-follow instructions for use.	Functional code, clear structure, and correctly defined variables. May have redundancy or may lack full comments or explanation.	Functional code. Some unnecessary recursion. Major variables defined in comments. May lack descriptions of variables & processes.	Functional code, but with multiple errors. Unnecessary/few comments. No/few variables defined in comments.	Non-functional code. No comments.
Technical Communication (10%)	Graphics are correct, complete and elegant. Report is presented in a structured format to provide clear understanding and easy read. Adds to discussion of topics.	Able to produce quality graphics or succinctly synthesize results to accompany math concepts. Axis labels with units, properly formatted labels, aligned axes, appropriately scaled to fit available space.	Able to produce graphics or provide results of lab activity to accompany math concepts. May not be optimally scaled for available space. Axis labels may not have units or proper format for audience. Line style/color may not be optimal.	Able to produce graphical content, but it is in the wrong format or scale. Line and marker color/style/weight causes confusion. No/few labels or legends. Text results are difficult to interpret.	Unable to produce any graphical output when required. Text output does not describe solution.

Table 3: Assessment results of final laboratory project for 2016-2017 (Total of 16 students).

	Exceptional (4)	*Above Expectations (3)*	*Meets Expectations (2)*	*Below Expectations (1)*	*No Credit (0)*	*Average score:*
Math Compre-hension (50%)	2	1	12		1	2.2
Concept Compre-hension (30%)		12	3		1	2.6
SAGE Technique (10%)	1	6	8		1	2.4
Technical Commu-nication (10%)	1		13	1	1	1.9
					Weighted average score:	**2.3**

In the EET program at Pittsburg State University, assessment data is regularly collected for ETAC/ABET accreditation using the Electrical/Electronics Technology (EET) Outcome Assessment [6], [7]. This assessment tool was developed by SME, the Electrical/Electronics Department Heads Association (ECETDHA), and IEEE. The EET Outcome Assessment consists of multiple-choice questions designed to assess EET knowledge [8]. EET students participate in the EET Outcome Assessment when they are seniors.

To assess the effect of the Engineering Calculus II class on EET students, the results of the EET Outcome Assessment were used [2], [6]. The EET Outcome Assessment consisted of questions arranged into categories by topic. The question categories that were hypothesized to be most influenced by Engineering Calculus II were identified. These question categories are listed in Fig. 1 and included complex numbers, ac circuits, and RC and RL circuits. The number of questions in each category is listed in parentheses.

In 2016-2018, 36 EET students participated in the EET Outcome Assessment. Out of these 36 students, three had taken the Engineering Calculus II course. Fig. 1 shows the comparison between the average percentage of correct answers for the Engineering Calculus II students and the other students for each question category. It can be seen that Engineering Calculus II students had a higher average percentage of correct answers in the following categories: Phase Relationships, Frequency Response, Maximum Power Transfer, Complex Numbers and Phasors, and Inductance and Inductors.

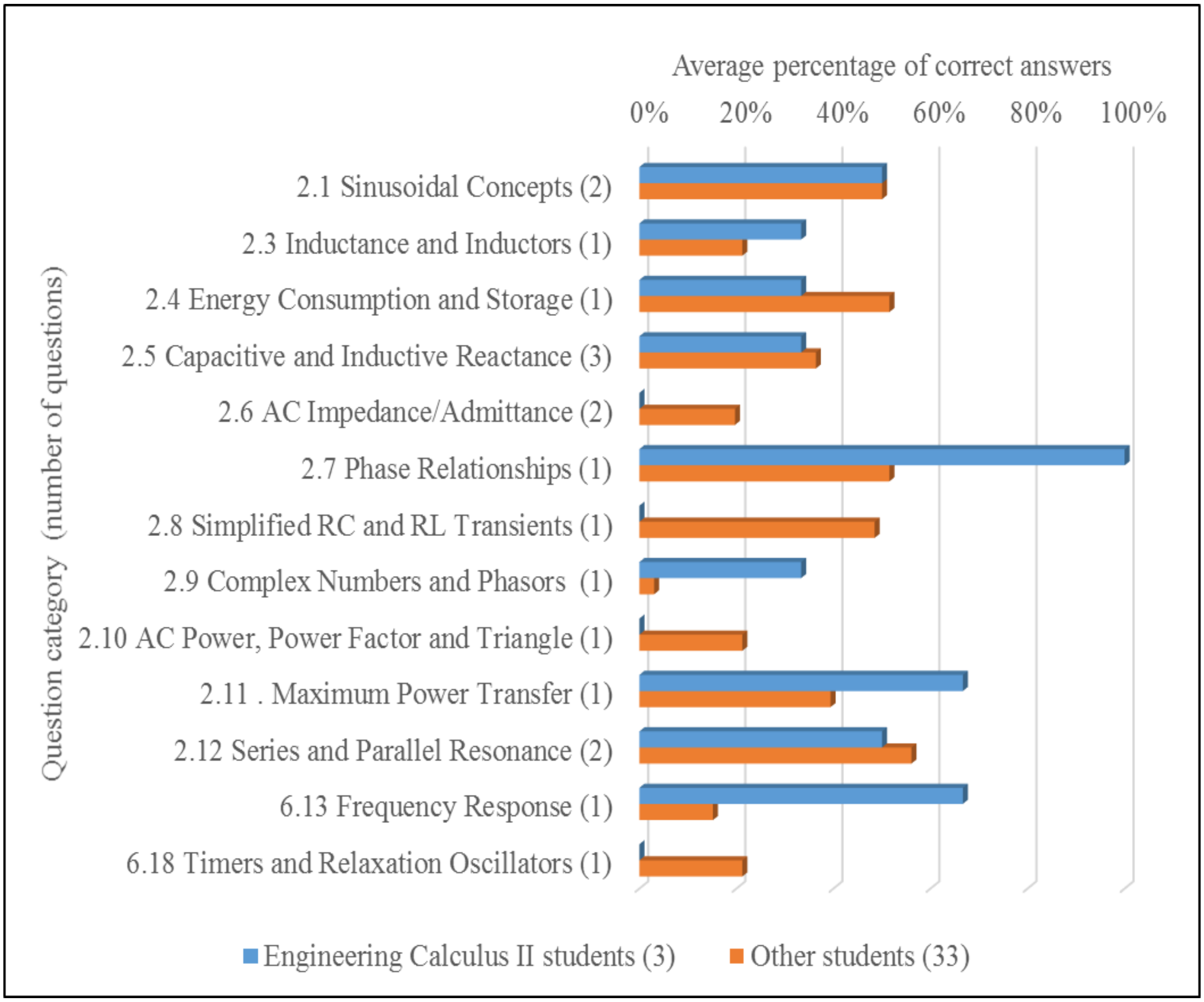

Figure 1: EET Outcome Assessment results for 2016–2018.

While Fig. 1 appears to show that Engineering Calculus II students appear to excel in several categories, it must be taken into account that only three Engineering Calculus II students took the EET Outcome Assessment. In addition, the questions on the EET Outcomes Assessment are multiple-choice and have four possible answers [8]. Thus, the probability that a student could randomly guess the correct answer on a question is 25%.

For the Phase Relationships category, there was one question and all three Engineering Calculus students got the question correct (100%). The probability of all three students guessing the correct answer randomly is 1.56%. So, there is a high probability that the higher average number of correct responses for this category was due to student knowledge gained from Engineering Calculus II as opposed to random guessing of the answers.

For the Maximum Power and Frequency Response categories, there was one question and two out of three Engineering Calculus students got the question correct (66.7%). The probability of two out of three students guessing the correct answer randomly is 14.1%. So, there is a high probability that the higher average number of correct responses for these categories was also due to student knowledge gained from Engineering Calculus II as opposed to random guessing of the answers.

Engineering Calculus II students appeared to perform better in the Complex Numbers and Phasors and Inductance and Inductors categories. However, each of these categories had one question and only one Engineering Calculus student got the question correct (33.3%). If the students randomly guessed the answers, the probability of one out of three students getting the correct answer is 42.2%. So, there is a good chance that the higher percentage of correct answers in these categories was the result of randomly guessing the answers.

There are categories for which it appears that Engineering Calculus II students have done a lot worse than other students. No Engineering Calculus II students got the correct answer (0%) in the following categories: Simplified RL and RC Transients, AC Power, Power Factor and Triangle, Timers and Relaxation Oscillators. However, the probability that all three students guessed an incorrect answer on the question is 42.2%. So, there is a good chance that the 0% of correct answers in these categories was also the result of randomly guessing the answers.

Summary

It was seen how faculty from the Mathematics and Engineering Technology departments created the alternative Engineering Calculus II course for EET and MECET majors. The new course differed from the traditional Calculus II class in two fundamental ways: the topics covered were tailored to EET and MECET and potential applications of the mathematics were included in laboratory projects. The laboratory projects also used the open-source SageMath software.

The assessment of the final laboratory projects showed that students were able to apply what they learned in the new Engineering Calculus II course and apply their knowledge to engineering technology applications. It can be seen from Table 3 that when the projects were evaluated with the rubric in Table 2, the weighted average score was 2.3 which was between the “Meets Expectations” and “Above Expectations” ratings.

Through the use of the EET Outcome Assessment, it appears that Engineering Calculus II is having a positive effect on EET students. Engineering Calculus II students had a higher average percentage of correct answers than other students in five question categories as shown in Fig. 1. However, out of the 36 EET students that participated in the EET Outcome Assessment, only three of them had Engineering Calculus II. This brought up the question of whether the higher percentage in some of the categories was merely due to the probability of student guessing the correct answer. It was determined that the higher average percentage of correct answers in three of the categories had a high probability of being caused by student knowledge gained from Engineering Calculus II.

The collaboration between the Mathematics and Engineering Technology department on the Engineering Calculus II course is anticipated to continue. It is anticipated that many students will benefit from future offerings of this course.

References

[1] K.A. Stroud and D.J. Booth, *Engineering Mathematics,* 7th ed., South Norwalk, CT: Industrial Press, 2013.

[2] Y. Liu, D. Miller, E. Mayer, "2016 Summer Teaching Innovation Grant Summary: Creating Cross-Disciplinary Laboratory Projects in a Calculus Course for Engineering Technology Students," Summer Teaching Innovation Grant Final Report, Pittsburg State University, Pittsburg, KS, March 29, 2017.

[3] "Optimizing the calculus sequence for students in STEM." [Online]. Available: http://resequencingcalculus.com . [Accessed February 4, 2019].

[4] Y. Liu, E. Mayer, D. Miller, *Engineering Calculus II Lab Manual for Sage Projects*, Pittsburg State University, Pittsburg, KS, December 17, 2016.

[5] "SageMath - Open-Source Mathematics Software System." [Online]. Available: http://www.sagemath.org/ . [Accessed February 1, 2019].

[6] "SME – EET Outcome Assessment." [Online]. Available: http://www.sme.org/eet/ . [Accessed February 1, 2019].

[7] "Accreditation." [Online]. Available: https://www.abet.org/accreditation/ . [Accessed February 2, 2019].

[8] "EET Outcome Assessment Sample Questions." [Online]. Available: https://www.sme.org/globalassets/sme.org/training/certifications/eet_sample_questions.pdf . [Accessed March 20, 2019].

Appendix A: Centroid of an Area

For any geometric shape or area, there exists a point at which all of the area can be said to be concentrated. This point is called the *centroid*, and is denoted by a coordinate pair, $(\bar{x}, \bar{y})$. When analyzing a structure for strength or suitability for a given application, the centroid is often used to replace a distributed load – a load that does not occur at a discrete location – by a point load. For many shapes (triangles, squares, circles, etc) the centroid has been readily determined; however, for irregular shapes the centroid must often be calculated using integrals.

Consider the shape in Figure 2.2. The curve represents some function that describes the loading. For problems related to distributed loads, the area under the curve represents the total load on the structure.

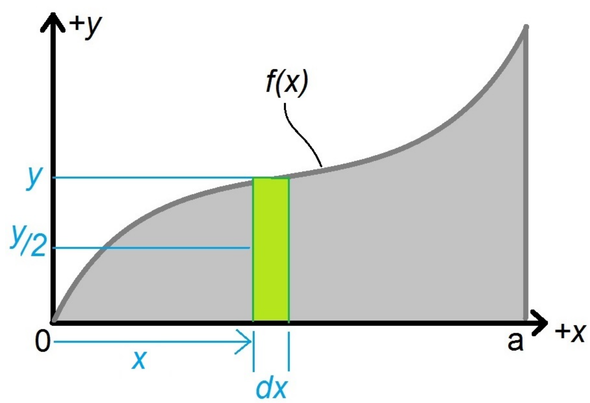

Figure 2.2: For any function $f(x)$ we can identify a rectangular element, shown in green. By taking successively smaller rectangles (i.e. shrinking dx) and adding the areas we approach the true area under this curve.

To find the centroid of the area under the curve, we first divide the region into small slices, the thin rectangle in the shaded area being a typical slice. The area of the rectangle, with width dx and height $y = f(x)$, is given by $dA = ydx$. In general, we can see that the centroid of a rectangle is described by the coordinates $(x, y) = \left(\frac{\text{base}}{2}, \frac{\text{height}}{2}\right)$. In the case of the rectangular element in Figure 2.2, the y-coordinate of the centroid would obviously be $\bar{y}_{el} = \frac{y}{2} = \frac{f(x)}{2}$. Since dx will be made as small as possible, we can say that $\frac{dx}{2} \simeq dx$, and that the x-coordinate of the centroid of the element would be $\bar{x}_{el} = x + \frac{dx}{2} \simeq x + dx \simeq x$. Equation 2.1 shows the coordinates for the centroid of an integral element.

$$(\bar{x}_{el}, \bar{y}_{el}) = \left(x, \frac{f(x)}{2}\right) \tag{2.1}$$

Once the centroid and area of the rectangular element have been determined, we can calculate the centroid of the whole region. One way is to use moments relative to the two coordinate axes. Refer to Programme 20, Frame 12 (p.856) of our textbook [5]. Also refer to any standard calculus book for more details. For a region described by the area under the function $y = f(x)$ on the interval $[a, b]$, the formula for the centroid is shown below.

$$\bar{x} = \frac{\int_a^b xy dx}{\int_a^b y dx} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx}, \qquad \bar{y} = \frac{\frac{1}{2}\int_a^b y^2 dx}{\int_a^b y dx} = \frac{\frac{1}{2}\int_a^b [f(x)]^2 dx}{\int_a^b f(x) dx}.$$

All integrals are taken over the entire length of the curve. In the case of Figure 2.2, the limits of integration would be from 0 to a. For problems involving loads, the line of action of the concentrated load is vertical, so the y-coordinate of the centroid is not used, and is usually not reported.

Problem 2.1. Let the distributed load (in Newtons/meter) on the beam shown in Figure 2.3 be represented by the function $f(x) = 6x^2 - x^3$ and the length of the beam $L = 4$ m.

1. What is the total magnitude of the load?
2. Where is its centroid located?

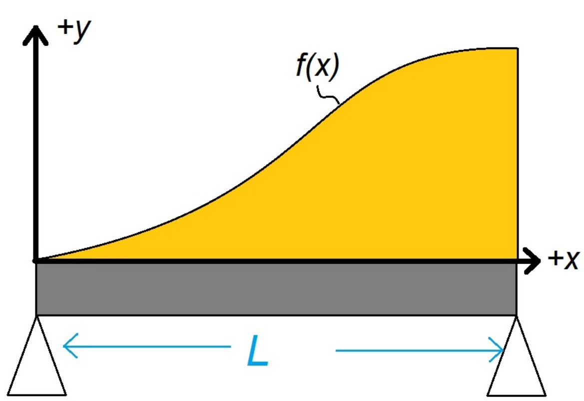

Figure 2.3: If a beam is loaded with a distributed load, the area under the curve can be replaced by a single load, the magnitude being the area under the curve, at a point along its length at the centroid of the area.

Problem 2.2. In order to balance a platform, engineers neet to know where its centroid is located. If the area of the platform is described by the function $g(x) = 4(x-6)^2$ (see Figure 2.4),

1. what is the total amount of material that must be used to construct the platform (*i.e.* the total area)?
2. where should the engineers place the support to keep it from collapsing (*i.e.* the centroid)?
3. Show your centroid on a graph.

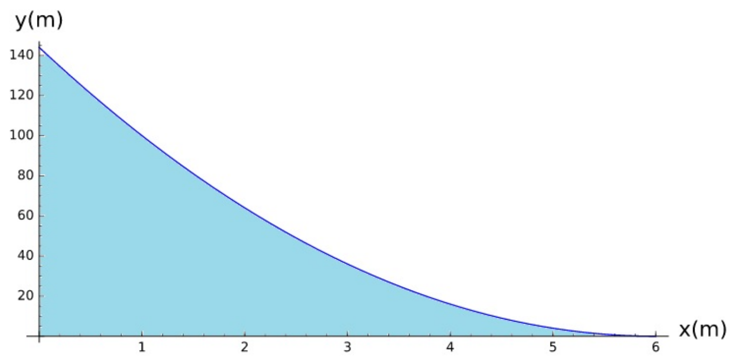

Figure 2.4: A platform represented by the area under the curve $g(x) = 4(x-6)^2$

Appendix B: Calculating the dc and rms Values of Periodic Waveforms

dc Value

An important parameter of periodic waveforms is the dc value of the waveform. The dc value can be found for either voltage or current waveforms. The dc value is the average value of the waveform over one period. Recall that in Calculus I, we defined the average value of a function $f(x)$ on an interval $[a, b]$ to be

$$\text{aver}(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

Programme 19, Frames 23-25 (p.844-845) in [5] provides some numerical examples. Refer to Figure 2.5 for the geometric interpretation of aver(f).

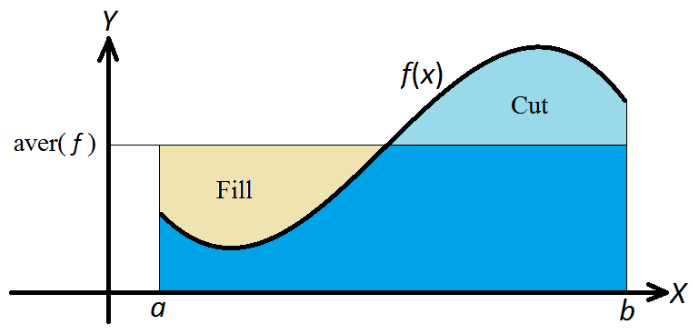

Figure 2.5: The average value of $f(x)$ is such that "Cut" = "Fill" so that the area of the rectangle with length $(b-a)$ and height aver(f) is equal to the area of the region under the curve $y = f(x)$: $(b-a)\text{aver}(f) = \int_a^b f(x)dx$.

If a voltage waveform is given as a function of time $v(t)$, then the dc value of the waveform is given by:

$$V_{dc} = \frac{1}{T}\int_0^T v(t)dt \tag{2.2}$$

where T is the period of the waveform. Likewise, if a current waveform is given as a function of time $i(t)$ with period T, then the dc value of the waveform is given by:

$$I_{dc} = \frac{1}{T}\int_0^T i(t)dt. \tag{2.3}$$

rms Value

Another important parameter of periodic waveforms is the root-mean-square (rms) value of the waveform. Literally, it is the square root of the mean value of the square of the waveform. The rms value can be found for either voltage or current waveforms. If a voltage waveform is given as a function of time $v(t)$, then the rms value is given by:

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}. \tag{2.4}$$

Likewise, if a current waveform is given as a function of time $i(t)$, then the rms value is given by:

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt}. \tag{2.5}$$

Refer to Programme 19, Frames 26-29 (p.846-847) in [5] for some examples on the calculation of rms values.

As have been illustrated earlier, you can find these integrals easily using Sage once the function $v(t)$ or $i(t)$ is given. But sometimes a function contains constants that are not explicitly specified. For example, the function $f(t) = \alpha \sin(\beta t)$ has two parameters α and β that are not given. How do we find the dc value and the rms value of $f(t)$ then? The answer is simple: do the integrals symbolically. First, since $\sin t$ has period 2π, the period of $f(t)$ is given by

$$T = \frac{2\pi}{\beta}.$$

Recall that in the syntax of the integral command there is a variable of interest. In the following code, we declare all the parameters, together with t, as variables. Then in the command `integral(f,t,0,T)` we name t as the variable of integration. This is necessary because otherwise Sage gets confused and will give you an error message. If there is only one variable in the function, you can omit this variable in the command. Now try

```
var('t,alpha,beta')
f(t)=alpha*sin(beta*t); T=2*pi/beta
integral(f,t,0,T)
```

and we get an error message, which ends with the question

```
Is beta positive or negative?
```

Sage wants to make sure that the upper limit of integration T is bigger than the lower limit 0, so β has to be positive (technically a negative value also works). We guarantee this by adding the command `assume(beta>0)`. In case you have made other assumptions about β (this may happen in a long worksheet), we also use the `forget` command to undo all earlier assumptions. Otherwise you may get a `ValueError` message:

```
Assumption is inconsistent.
```

Now we have

```
var('t,alpha,beta')
forget()
assume(beta>0)
f(t)=alpha*sin(beta*t); T=2*pi/beta
dc=integral(f,t,0,T)/T; rms=sqrt(integral(f^2,t,0,T)/T)
dc; rms
|   (t, alpha, beta)
|   0
|   sqrt(1/2)*sqrt(alpha^2)
```

This means $f_{dc} = 0$ and $f_{rms} = \sqrt{\alpha^2/2}$. If we also assume $\alpha > 0$, the rms value will be $f_{rms} = \alpha/\sqrt{2}$.

Carry out these calculations by hand and check that the answers are correct.

Calculating the Power Used by a Resistor

The instantaneous power used by a resistor can be found by:

$$p(t) = \frac{v^2(t)}{R} \tag{2.6}$$

or

$$p(t) = i^2(t)R \tag{2.7}$$

where $v(t)$ is the dc voltage across the resistor, $i(t)$ is the dc current through the resistor, and R is the resistance.

In a direct-current (dc) circuit where the voltage across the resistor and the current through the resistor are constant, the voltage and current can be given as:

$$v(t) = V \tag{2.8}$$

and

$$i(t) = I. \tag{2.9}$$

Using (2.6), (2.7), (2.8), and (2.9), the instantaneous power used by a resistor can be calculated by:

$$p(t) = \frac{V^2}{R} \tag{2.10}$$

or

$$p(t) = I^2R. \tag{2.11}$$

When the voltage and current are changing, it is more practical to rate the resistor using the average power P rather than the instantaneous power. The average power is given by:

$$P = \frac{1}{T}\int_0^T p(t)dt. \tag{2.12}$$

Substituting (2.6) and (2.7) into (2.12) gives:

$$P = \frac{1}{T}\int_0^T \frac{v^2(t)dt}{R} \tag{2.13}$$

and

$$P = \frac{1}{T}\int_0^T i^2(t)Rdt. \tag{2.14}$$

It can be seen that using (2.10), (2.11), and (2.12), that the instantaneous and average power are the same in a dc circuit with constant voltages and currents:

$$P = \frac{V^2}{R} \tag{2.15}$$

or

$$P = I^2R. \tag{2.16}$$

The rms value of the voltage across a resistor will deliver the same average power to the resistor as a constant voltage of the same value. For example, a voltage source with a changing voltage that has an rms value of 5 V will deliver the same power to a resistor as a dc voltage source that is a constant 5 V. The average power used by the resistor can be calculated using V_{rms} in place of V in equation (2.15):

$$P = \frac{V_{rms}^2}{R}. \tag{2.17}$$

The derivation of the equation for the rms voltage can be done by equating (2.17) to (2.13):

$$\frac{V_{rms}^2}{R} = \frac{1}{T}\int_0^T \frac{v^2(t)dt}{R}. \tag{2.18}$$

Solving for V_{rms} yields (2.4).

Likewise, the rms value of the current through a resistor will deliver the same average power to the resistor as a constant current of the same value. For example, a changing current that has an rms value of 2 A will deliver the same power to a resistor as a dc current that is a constant 2 A. The average power used by the resistor can be calculated using I_{rms} in place of I in equation (2.16):

$$P = I_{rms}^2R. \tag{2.19}$$

The derivation of the equation for the rms current can be done by equating (2.19) to (2.14):

$$I_{rms}^2 R = \frac{1}{T}\int_0^T i^2(t)Rdt. \tag{2.20}$$

Solving for I_{rms} yields (2.5).

Problem 2.3. In an electric power grid that delivers an alternating-current (ac) voltage to homes, the voltage is ideally sinusoidal and can be given by the equation:

$$v(t) = V_{peak}\sin(2\pi ft) \tag{2.21}$$

where V_{peak} is the amplitude of the sinusoid. f is the frequency of the sinusoid which is related to the period T by:

$$f = \frac{1}{T}. \tag{2.22}$$

1. Derive the equation for the dc value of the sinusoid given by (2.21).
2. Derive the equation for the rms value of the sinusoid given by (2.21).
3. Using Sage, calculate the dc value of the sinusoid given by (2.21) if $V_{peak} = 169.7V$ and $f = 60Hz$.
4. Using Sage, calculate the rms value of the sinusoid given by (2.21) if $V_{peak} = 169.7V$ and $f = 60Hz$.

Problem 2.4. The voltage of a pulse-width modulated (PWM) voltage source can be given as:

$$v(t) = \begin{cases} V_{peak} & : nT \leq t \leq nT + DT \\ 0 & : nT + DT < t < (n+1)T \end{cases}$$

where $n = 0, 1, 2, \ldots$ and $0 \leq D \leq 1$. D is the duty cycle which can be varied to set the dc value of the PWM voltage source and T is the period.

1. Derive the equation for the dc value of the PWM voltage source.
2. Derive the equation for the rms value of the PWM voltage source.
3. Using Sage, calculate the dc value of the PWM voltage source if $V_{peak} = 12V$ and $D = 0.6$.
4. Using Sage, calculate the rms value of the PWM voltage source if $V_{peak} = 12V$ and $D = 0.6$.